Some New Bounds for the Hadamard Product of a Nonsingular M-matrix and Its Inverse

Zhengge Huang, Ligong Wang and Zhong Xu

***Abstract*—Some new convergent sequences of the lower bounds of the minimum eigenvalue for the Hadamard product of a nonsingular *M*-matrix and an inverse *M*-matrix are given. Numerical examples show that these bounds could reach the true value of the minimum eigenvalue in some cases. These bounds in this paper improve some existing ones.**

***Index Terms*—sequences, nonsingular *M*-matrix, Hadamard product, minimum eigenvalue, lower bounds.**

I. Introduction

The Hadamard product of two nonnegative matrices and the Fan product of two *M*-matrices are two special matrices product. In mathematics, many problems can be transformed into computational problems of the Hadamard product and the Fan product, such as numerical method for solving volterra integral equations with a convolution kernel [1], solving a system of Wiener-Hopf Integral Equations [2] and so on. The upper bounds for the spectral radius $\rho(A\circ B)$ of the Hadamard product of two nonnegative matrices A and B, the lower bounds for the minimum eigenvalue $\tau(A\circ B^{-1})$ of the Hadamard product of a *M*-matrix A and an inverse of *M*-matrix B are research hotspots in matrix theory researching. In this paper, by constructing sequence of iterations and combining with the skills of inequalities scaling, we will conduct further research in the lower bounds for the minimum eigenvalue $\tau(A\circ B^{-1})$ of the Hadamard product of a *M*-matrix A and an inverse of *M*-matrix B. In theory, we prove that the sequences obtained in this paper are more accurate than the existing ones. Some results of the comparison are also considered. To illustrate our results, some examples are given.

For a positive integer n, N denotes the set $N = \{1, 2, \cdots, n\}$ throughout. The set of all $n\times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$, and $\mathbb{C}^{n\times n}$ denotes the set of all $n\times n$ complex matrices.

Let Z_n denote the set of $n\times n$ real matrices all of whose off-diagonal entries are nonpositive. A matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is called an *M*-matrix [3] if there exist a nonnegative matrix B and a nonnegative real number λ such that

$$A=\lambda I-B,\ \lambda\geq\rho(B),$$

where I is the identity matrix, $\rho(B)$ is the spectral radius of the matrix B. If $\lambda=\rho(B)$, then A is a singular *M*-matrix; if $\lambda>\rho(B)$, then A is a nonsingular *M*-matrix. If C is a principal submatrix of A, then C is also a nonsingular *M*-matrix. Denote by M_n the set of all $n\times n$ nonsingular *M*-matrices. Let us denote

$$\tau(A)=\min\{\mathrm{Re}(\lambda):\lambda\in\sigma(A)\},$$

and $\sigma(A)$ denotes the spectrum of A. It is known that [4] $\tau(A)=\frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A\in M_n$.

The Hadamard product of two matrices $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ is defined as the matrix $A\circ B=(a_{ij}b_{ij})_{n\times n}$. If A and B are nonsingular *M*-matrices, then it is proved in [5] that $A\circ B^{-1}$ is also a nonsingular *M*-matrix.

A matrix A is irreducible [6], [7], [8] if there does not exist a permutation matrix P such that

$$PAP^T=\begin{pmatrix} A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix},$$

where A_{11} and A_{22} are square matrices.

For $\alpha\subseteq N$, denote by $|\alpha|$ the cardinality of α and $\alpha'=N-\alpha$. If $\alpha,\beta\subseteq N$, then $A(\alpha,\beta)$ is the submatrix of A lying in the rows indicated by α and the columns indicated by β. In particular, $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$. Assume that $A(\alpha)$ is nonsingular. Then

$$A/\alpha=A/A(\alpha)=A(\alpha')-A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha'),$$

is called the Schur complement of A respect to $A(\alpha)$ [9].

Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant matrix, for any $i,j,k\in N$, $i\neq j$,$t=1,2,\cdots$, denote

$$R_i=\sum_{k\neq i}|a_{ik}|,\quad d_i=\frac{R_i}{|a_{ii}|},$$

$$s_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|d_k}{a_{jj}},\quad s_i=\max_{j\neq i}\{s_{ij}\};$$

$$m_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|s_{ki}}{|a_{jj}|},\quad m_i=\max_{j\neq i}\{m_{ij}\};$$

$$r_{ji}=\frac{|a_{ji}|}{|a_{jj}|-\sum\limits_{k\neq j,i}|a_{jk}|},\quad r_i=\max_{j\neq i}\{r_{ji}\};$$

$$t_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|},\quad t_i=\max_{j\neq i}\{t_{ij}\};$$

$$u_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|t_{ki}}{|a_{jj}|},\quad u_i=\max_{j\neq i}\{u_{ij}\};$$

$$l_i=\max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|t_{ji}-\sum\limits_{k\neq j,i}|a_{jk}|t_{ki}}\right\},$$

$$w_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|t_{ki}l_i}{|a_{jj}|},$$

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$$w_i = \max_{j\neq i}\{w_{ij}\}, \quad q_{ji} = \min\{s_{ji}, t_{ji}\},$$

$$h_i = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum\limits_{k\neq j,i}|a_{jk}|q_{ki}}\right\},$$

$$v_{ji}^{(0)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|q_{ki}h_i}{|a_{jj}|},$$

$$p_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|v_{ki}^{(t-1)}}{|a_{jj}|}, \quad p_i^{(t)} = \max_{j\neq i}\{p_{ij}^{(t)}\},$$

$$h_i^{(t)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(t)} - \sum\limits_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}}\right\},$$

$$v_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}h_i^{(t)}}{|a_{jj}|}.$$

Let A be a nonsingular *M*-matrix and $A^{-1} = (\alpha_{ij})_{n\times n}$ be a doubly stochastic matrix. Before presenting our results, we review the existing results that relate to lower bounds of the minimum eigenvalue for the Hadamard product of a nonsingular *M*-matrix and an inverse *M*-matrix as follows.

It was proved in [10] that $0 < \tau(A\circ A^{-1}) \leq 1$. Fiedler and Markham [5] gave a lower bound on $\tau(A\circ A^{-1})$: $\tau(A\circ A^{-1}) \geq \frac{1}{n}$ and conjectured that $\tau(A\circ A^{-1}) \geq \frac{2}{n}$.

Chen [11], Song [12] and Yong [13] have independently proved this conjecture.

Li *et al.* in [14] and Li *et al.* in [15] obtained the following results

$$\tau(A\circ A^{-1}) \geq \min_{i\in N}\left\{\frac{a_{ii} - s_iR_i}{1 + \sum\limits_{j\neq i}s_{ji}}\right\},$$

$$\tau(A\circ A^{-1}) \geq \min_{i\in N}\left\{\frac{a_{ii} - t_iR_i}{1 + \sum\limits_{j\neq i}t_{ji}}\right\},$$

respectively.

In [16], Cheng *et al.* improved the results in [14] and [15], showing that

$$\tau(A\circ A^{-1}) \geq \min_{i\in N}\left\{\frac{a_{ii} - u_iR_i}{1 + \sum\limits_{j\neq i}u_{ji}}\right\}.$$

Recently, Chen in [17] improved the results in [14] and gave the following result:

$$\tau(A\circ A^{-1}) \geq \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}\alpha_{ii} + a_{jj}\alpha_{jj} - \Bigg[(a_{ii}\alpha_{ii} - a_{jj}\alpha_{jj})^2 + 4\Bigg(m_i\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Bigg)\Bigg(m_j\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\}.$$

In [18], Cheng *et al.* proposed $\tau(A\circ A^{-1}) \geq 1 - \rho^2(J_A)$, and in [19], Li *et al.* presented the following result:

$$\tau(A\circ A^{-1}) \geq \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}\alpha_{ii} + a_{jj}\alpha_{jj} - \Bigg[(a_{ii}\alpha_{ii} - a_{jj}\alpha_{jj})^2 + 4\Bigg(w_i\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Bigg)\Bigg(w_j\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\}.$$

In the sequel, Zhao *et al.* [20] improved the results in [14], [15], [16] and arrived at

$$\tau(A\circ A^{-1}) \geq \max\left\{\min_{i\in N}\left\{\frac{a_{ii} - p_i^{(t)}R_i}{1 + \sum\limits_{j\neq i}p_{ji}^{(t)}}\right\}, \min_{i\in N}\left\{\frac{a_{ii} - s_i\sum\limits_{j\neq i}\frac{|a_{ji}|p_{ji}^{(t)}}{s_j}}{1 + \sum\limits_{j\neq i}p_{ji}^{(t)}}\right\}\right\}.$$

In [21], Zhou *et al.* proved the following result: If $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$ are nonsingular *M*-matrices and $A^{-1} = (\alpha_{ij})_{n\times n}$, then

$$\tau(B\circ A^{-1}) \geq \min_{i\in N}\left\{\frac{b_{ii} - t_i\sum\limits_{k\neq i}|b_{ki}|}{a_{ii}}\right\}.$$

In this paper, we exhibit some new lower bounds for $\tau(A\circ A^{-1})$ and $\tau(B\circ A^{-1})$. These bounds improve the results in [14], [15], [16], [17], [18], [19], [20], [21].

The rest of this paper is organized as follows. In Section II, we propose some notations and lemmas which are useful in the following proofs. We propose some new lower bounds for $\tau(B\circ A^{-1})$ and $\tau(A\circ A^{-1})$ in Section III. Section IV is devoted to some numerical experiments to show that the advantages and precise of the new convergent sequences of the lower bounds of the minimum eigenvalue for the Hadamard product of a nonsingular *M*-matrix and an inverse *M*-matrix. Finally, some concluding remarks are given.

II. Preliminaries

In this section, we start with some notations and lemmas that involve inequalities for the entries of A^{-1} and the strictly diagonally dominant matrix. They will be useful in the proofs.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. For $i, j, k \in N$, $j \neq i$, $t = 1, 2, \cdots$, denote

$$g_{ji}^{(0)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|t_{ki}l_i}{|a_{jj}|} = w_{ji},$$

$$f_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|g_{ki}^{(t-1)}}{|a_{jj}|}, \quad f_i^{(t)} = \max_{j\neq i}\{f_{ij}^{(t)}\},$$

$$l_i^{(t)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|f_{ji}^{(t)} - \sum\limits_{k\neq j,i}|a_{jk}|f_{ki}^{(t)}}\right\},$$

$$g_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|f_{ki}^{(t)}l_i^{(t)}}{|a_{jj}|}.$$

Lemma 2.1 [7] Let $A = (a_{ij})_{n\times n}$ be a nonsingular *M*-matrix, $B = (b_{ij})_{n\times n} \in Z_n$ and $A \leq B$. Then B is a nonsingular *M*-matrix and $A^{-1} \geq B^{-1} \geq O$.

Lemma 2.2 [7] Let $A = (a_{ij})_{n\times n}$, $\emptyset \neq \alpha \subseteq N$ and assume that $A(\alpha)$ is nonsingular. Then

$$\det A = \det A(\alpha)\det A/\alpha.$$

Lemma 2.3 [20] If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a strictly row diagonally dominant nonsingular *M*-matrix, then, for all

$i,j\in\mathbb{N},\ j\neq i,\ t=1,2,\cdots,$

$$\begin{aligned}&(a)\ 1>q_{ji}\geq v_{ji}^{(0)}\geq p_{ji}^{(1)}\geq v_{ji}^{(1)}\\&\qquad\geq p_{ji}^{(2)}\geq v_{ji}^{(2)}\geq\cdots\geq p_{ji}^{(t)}\geq v_{ji}^{(t)}\geq\cdots\geq 0;\\&(b)\ 1\geq h_i\geq 0, 1\geq h_i^{(t)}\geq 0.\end{aligned}$$

Using the same technique as the proof of Lemma 1 in [20], we can obtain the Lemma 2.4.

Lemma 2.4 If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is a strictly row diagonally dominant nonsingular *M*-matrix, then, for all $i,j\in\mathbb{N},\ j\neq i,\ t=1,2,\cdots,$

$$\begin{aligned}&(a)\ 1>t_{ji}\geq g_{ji}^{(0)}\geq f_{ji}^{(1)}\geq g_{ji}^{(1)}\\&\qquad\geq f_{ji}^{(2)}\geq g_{ji}^{(2)}\geq\cdots\geq f_{ji}^{(t)}\geq g_{ji}^{(t)}\geq\cdots\geq 0;\\&(b)\ 1\geq l_i\geq 0, 1\geq l_i^{(t)}\geq 0.\end{aligned}$$

Lemma 2.5 [20] Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant nonsingular *M*-matrix. Then, for $A^{-1}=(b_{ij})_{n\times n},\ j\neq i,\forall j\in N, t=0,1,2,\cdots,$ we have

$$\alpha_{ji}\leq\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|v_{ki}^{(t)}}{a_{jj}}\alpha_{ii}=p_{ji}^{(t+1)}\alpha_{ii}.$$

Lemma 2.6 Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant nonsingular *M*-matrix. Then, for $A^{-1}=(b_{ij})_{n\times n}$, we have

$$\alpha_{ji}\leq z_{ji}^{(t+1)}\alpha_{ii}\leq z_j^{(t+1)}\alpha_{ii},\ \forall i,j\in N, i\neq j, t=0,1,2,\cdots,$$

where $z_{ji}^{(t)}=\min\{p_{ji}^{(t)},f_{ji}^{(t)}\}, z_i^{(t)}=\max\limits_{j\neq i}\{z_{ij}^{(t)}\}, t=1,2,\cdots.$

Proof. Using the same techniques as the proof of Lemma 2.2 in [11], for $j\neq i,\forall j\in N, t=0,1,2,\cdots,$ we have

$$\alpha_{ji}\leq\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|g_{ki}^{(t)}}{a_{jj}}\alpha_{ii}=f_{ji}^{(t+1)}\alpha_{ii}.$$

Then it follows from the above inequality and Lemma 2.5 that

$$\alpha_{ji}\leq\min\{p_{ji}^{(t+1)},f_{ji}^{(t+1)}\}\alpha_{ii}=z_{ji}^{(t+1)}\alpha_{ii},$$

for $\forall j\in N, t=0,1,2,\cdots.$

Remark 2.1 By Lemma 2.3, we find that $0\leq h_i\leq 1$, in view of $q_{ji}=\min\{s_{ji},t_{ji}\}\leq s_{ji}$ and by Lemma 2.3, for $i,j\in N, i\neq j, t=1,2,\cdots$ we have

$$\begin{aligned}p_{ji}^{(t)}\leq v_{ji}^{(0)}&=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|q_{ki}h_i}{|a_{jj}|}\\&\leq\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|s_{ki}}{|a_{jj}|}=m_{ji}.\end{aligned}$$

Furthermore, by Lemma 2.4, $0\leq l_i\leq 1$, then, we have

$$\begin{aligned}f_{ji}^{(t)}\leq g_{ji}^{(0)}&=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|t_{ki}l_i}{|a_{jj}|}\\&=w_{ji}\leq\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|t_{ki}}{|a_{jj}|}=u_{ji}.\end{aligned}$$

By Theorem 3.3 in [14], [15] and the above two inequalities, for $i,j\in N, i\neq j, t=1,2,\cdots,$ we infer that

$$z_{ji}^{(t)}\leq v_{ji}^{(0)}\leq m_{ji}\leq s_{ji},\ z_{ji}^{(t)}\leq g_{ji}^{(0)}=w_{ji}\leq u_{ji}\leq t_{ji},$$

which results in

$$z_i^{(t)}\leq v_i^{(0)}\leq m_i\leq s_i,\ z_i^{(t)}\leq g_i^{(0)}=w_i\leq u_i\leq t_i,\ i\in N.$$

Moreover, inasmuch as

$$\begin{aligned}z_i^{(t)}&=\max_{j\neq i}\{z_{ij}^{(t)}\}=\max\{z_{i1}^{(t)},\cdots,z_{i,i-1}^{(t)},z_{i,i+1}^{(t)},\cdots,z_{i,n}^{(t)}\}\\&\leq\max\{p_{i1}^{(t)},\cdots,p_{i,i-1}^{(t)},p_{i,i+1}^{(t)},\cdots,p_{i,n}^{(t)}\}\\&=\max_{j\neq i}\{p_{ij}^{(t)}\}=p_i^{(t)},\\z_i^{(t)}&=\max_{j\neq i}\{z_{ij}^{(t)}\}=\max\{z_{i1}^{(t)},\cdots,z_{i,i-1}^{(t)},z_{i,i+1}^{(t)},\cdots,z_{i,n}^{(t)}\}\\&\leq\max\{f_{i1}^{(t)},\cdots,f_{i,i-1}^{(t)},f_{i,i+1}^{(t)},\cdots,f_{i,n}^{(t)}\}\\&=\max_{j\neq i}\{f_{ij}^{(t)}\}=f_i^{(t)}.\end{aligned}$$

for $i\in N$, the result of Lemma 2.6 is sharper than the results of Theorem 2.1 in [14], Lemma 2.2 in [15], [16], [17] and Lemma 2 in [19], [20].

Lemma 2.7 Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant nonsingular *M*-matrix. Then, for $A^{-1}=(\alpha_{ij})_{n\times n}, i\in N, t=1,2,\cdots,$ we have

$$\frac{1}{a_{ii}-\sum\limits_{k\neq i}\frac{a_{ik}a_{ki}}{a_{kk}}}\leq\alpha_{ii}\leq\frac{1}{a_{ii}-\sum\limits_{j\neq i}|a_{ij}|z_{ji}^{(t)}},\tag{1}$$

where $z_{ji}^{(t)}=\min\{p_{ji}^{(t)},f_{ji}^{(t)}\},\ i,j\in N, j\neq i.$

Proof. Let $B=A^{-1}$. Since A is a nonsingular *M*-matrix, $B\geq 0$. Denote A_i is the submatrix of A obtained by deleting the i-th row and the i-th column of A. Then, A_i is a nonsingular *M*-matrix and

$$A_i\leq\operatorname{diag}(a_{11},\cdots,a_{i-1,i-1},a_{i+1,i+1},\cdots,a_{nn})=A'.$$

Thus, by Lemma 2.1 and A' being a nonsingular *M*-matrix, we have

$$A_i^{-1}\geq\operatorname{diag}(a_{11}^{-1},\cdots,a_{i-1,i-1}^{-1},a_{i+1,i+1}^{-1},\cdots,a_{nn}^{-1}).\tag{2}$$

For $i\in N$, by Lemma 2.3, we have

$$\alpha_{ii}=\frac{\det A_i}{\det A}=\frac{\det A_i}{\det A_i\det A/A_i}=\frac{1}{\det A/A_i}.$$

By Inequality (2), we deduce that

$$\det A/A_i\leq a_{ii}-\sum_{k\neq i}\frac{a_{ik}a_{ki}}{a_{kk}}.$$

Hence

$$\alpha_{ii}=\frac{\det A_i}{\det A}\geq\frac{1}{a_{ii}-\sum\limits_{k\neq i}\frac{a_{ik}a_{ki}}{a_{kk}}},\ i\in N.\tag{3}$$

Combining Lemma 2.7 and $AB=I$ results in

$$\begin{aligned}1=\sum_{j=1}^n a_{ij}\alpha_{ji}&=a_{ii}\alpha_{ii}-\sum_{j\neq i}|a_{ij}|\alpha_{ji}\\&\geq a_{ii}\alpha_{ii}-\sum_{j\neq i}|a_{ij}|z_{ji}^{(t)}\alpha_{ii}\\&=\Big(a_{ii}-\sum_{j\neq i}|a_{ij}|z_{ji}^{(t)}\Big)\alpha_{ii},\end{aligned}$$

i.e.,

$$\alpha_{ii}\leq\frac{1}{a_{ii}-\sum\limits_{j\neq i}|a_{ij}|z_{ji}^{(t)}},\ i\in N.\tag{4}$$

By Inequalities (3) and (4), the result is obtained.

Remark 2.2 According to Remark 2.1, for $j \neq i, j \in N, t = 1, 2, \cdots$, we have

$$s_{ji} \geq m_{ji} \geq z_{ji}^{(t)},\ t_{ji} \geq u_{ji} \geq w_{ji} = g_{ji}^{(0)} \geq z_{ji}^{(t)},\ p_{ji}^{(t)} \geq z_{ji}^{(t)},$$

which implies that

$$\frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| z_{ji}^{(t)}} \leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| m_{ji}} \leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| s_{ji}},$$

$$\frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| z_{ji}^{(t)}} \leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| p_{ji}^{(t)}},$$

$$\begin{aligned}\frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| z_{ji}^{(t)}} &\leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| g_{ji}^{(0)}} = \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| w_{ji}} \\ &\leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| u_{ji}} \leq \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| t_{ji}}.\end{aligned}$$

Moreover, it is easy to get

$$\frac{1}{a_{ii}} \leq \frac{1}{a_{ii} - \sum\limits_{k\neq i} \frac{a_{ik}a_{ki}}{a_{kk}}},\ i \in N.$$

So the bounds of Lemma 2.7 are sharper than the ones of Theorem 2.5 in [14], Lemma 2.3 in [15], [16], [17] and Lemma 3 in [20].

Lemma 2.8 [22] If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = (1, 1, \cdots, 1)^T$.

Lemma 2.9 [17] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. Then $0 \leq \tau(A) \leq a_{ii}$ for all $i \in N$.

Lemma 2.10 [23] Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \cdots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region

$$\bigcup_{i,j=1, i\neq j}^{n} \left\{ z \in \mathbb{C} : \quad |z - a_{ii}||z - a_{jj}| \leq \left(x_i \sum_{k\neq i} \frac{1}{x_k} |a_{ki}| \right) \left(x_j \sum_{k\neq j} \frac{1}{x_k} |a_{kj}| \right) \right\}.$$

Lemma 2.11 [7] Let $A \in \mathrm{Z}_n$. A is a nonsingular *M*-matrix if and only if all its leading principal minors are positive.

III. Main results

In this section, we exhibit some new lower bounds for $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$, which improve the ones in [14], [15], [16], [17], [18], [19], [20], [21].

Theorem 3.1 Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be nonsingular *M*-matrices and $A^{-1} = (\alpha_{ij})_{n\times n}$. Then, for $t = 1, 2, \cdots$,

$$\begin{aligned}\tau(B \circ A^{-1}) \geq \frac{1}{2} \min_{i\neq j} \Bigg\{ &\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\ &+ 4 \left(z_i^{(t)} \sum_{k\neq i} |b_{ki}| \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} |b_{kj}| \alpha_{jj} \right) \Bigg]^{\frac{1}{2}} \Bigg\} \\ = \Delta_t. \end{aligned} \tag{5}$$

Proof. It is evident that the result (5) holds with equality for $n = 1$. Below we assume that $n \geq 2$.

Since A is a nonsingular *M*-matrix, there exists a positive diagonal matrix D such that $D^{-1}AD$ is a strictly row diagonally dominant nonsingular *M*-matrix, and

$$\tau(B \circ A^{-1}) = \tau(D^{-1}(B \circ A^{-1})D) = \tau((B \circ (D^{-1}AD)^{-1}).$$

Therefore, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix. Now, let us distinguish two cases:

(i) First, we assume that A and B are irreducible matrices. Then, for $i \in N$, $t = 1, 2, \cdots$, without loss of generality, we assume that

$$\begin{aligned} z_i^{(t)} &= \max\{z_{i1}^{(t)}, \cdots, z_{i,i-1}^{(t)}, z_{i,i+1}^{(t)}, \cdots, z_{in}^{(t)}\} \\ &= \max\{p_{i1}^{(t)}, \cdots, p_{i,i-1}^{(t)}, p_{i,i+1}^{(t)}, \cdots, p_{in}^{(t)}\}. \end{aligned}$$

By the definition of $p_{ji}^{(t)}$, obviously, $0 < z_i^{(t)} < 1$ for $i \in N$, $t = 1, 2, \cdots$. Let $\tau(B \circ A^{-1}) = \lambda$. By Lemma 2.10 and Lemma 2.6, there exists a pair (i, j) of positive integers with $i \neq j$, such that

$$\begin{aligned} &|\lambda - \alpha_{ii}b_{ii}||\lambda - \alpha_{jj}b_{jj}| \\ &\leq \left(z_i^{(t)} \sum_{k\neq i} \frac{1}{z_k^{(t)}} |\alpha_{ki}b_{ki}| \right) \left(z_j^{(t)} \sum_{k\neq j} \frac{1}{z_k^{(t)}} |\alpha_{kj}b_{kj}| \right) \\ &\leq \left(z_i^{(t)} \sum_{k\neq i} \frac{1}{z_k^{(t)}} |b_{ki}| z_k^{(t)} \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} \frac{1}{z_k^{(t)}} |b_{kj}| z_k^{(t)} \alpha_{jj} \right) \\ &= \left(z_i^{(t)} \sum_{k\neq i} |b_{ki}| \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} |b_{kj}| \alpha_{jj} \right). \end{aligned} \tag{6}$$

Since A and B are nonsingular *M*-matrices, $B \circ A^{-1}$ is a nonsingular *M*-matrix as well. By Lemma 2.9, we have $0 \leq \lambda \leq \alpha_{ii}b_{ii}$ for all $i \in N$. It follows from Inequality (6) that

$$\begin{aligned} &(\lambda - \alpha_{ii}b_{ii})(\lambda - \alpha_{jj}b_{jj}) \\ &\leq \left(z_i^{(t)} \sum_{k\neq i} |b_{ki}| \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} |b_{kj}| \alpha_{jj} \right). \end{aligned} \tag{7}$$

Thus, Inequality (7) is equivalent to

$$\begin{aligned}\tau(B \circ A^{-1}) \geq \frac{1}{2} \Bigg\{ &\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\ &+ 4 \left(z_i^{(t)} \sum_{k\neq i} |b_{ki}| \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} |b_{kj}| \alpha_{jj} \right) \Bigg]^{\frac{1}{2}} \Bigg\},\end{aligned}$$

that is,

$$\begin{aligned}\tau(B \circ A^{-1}) \geq \frac{1}{2} \min_{j\neq i} \Bigg\{ &\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\ &+ 4 \left(z_i^{(t)} \sum_{k\neq i} |b_{ki}| \alpha_{ii} \right) \left(z_j^{(t)} \sum_{k\neq j} |b_{kj}| \alpha_{jj} \right) \Bigg]^{\frac{1}{2}} \Bigg\}.\end{aligned} \tag{8}$$

(ii) Now, assume that one of A and B is reducible. By Lemma 2.11, we know that all the leading principal minors of A and B are positive. If we denote by $P = (p_{ij})$ the $n \times n$ permutation matrix with $p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1$, the remaining p_{ij} being zero. Then for any chosen positive real number ε, sufficiently small such that all the leading principal minors of $A - \varepsilon P$ and $B - \varepsilon P$ are positive, it follows that $A - \varepsilon P$ and $B - \varepsilon P$ are irreducible nonsingular *M*-matrix. Now, we substitute $A - \varepsilon P$ for A, $B - \varepsilon P$ for B, respectively in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. This completes our proof of Theorem 3.1.

Remark 3.1 Without loss of generality, for $i \neq j$, $t = 1, 2, \cdots$, assume that

$$\alpha_{ii}b_{ii} - z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii} \leq \alpha_{jj}b_{jj} - z_j^{(t)} \sum_{k\neq j} |b_{kj}|\alpha_{jj}, \quad (9)$$

Thus, Inequality (9) can be written as

$$z_j^{(t)} \sum_{k\neq j} |b_{kj}|\alpha_{jj} \leq \alpha_{jj}b_{jj} - \alpha_{ii}b_{ii} + z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}. (10)$$

From Inequalities (5), (10) and by Lemma 2.10 and Lemma 3 in [20], it follows that

$$\begin{aligned}
&\frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\
&+4\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(z_j^{(t)} \sum_{k\neq j} |b_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
\geq& \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(b_{ii}\alpha_{ii} - b_{jj}\alpha_{jj})^2 + 4 \times \\
&\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(\alpha_{jj}b_{jj} - \alpha_{ii}b_{ii} + z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
=& \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} \\
&-\Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 + 4\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)^2 \\
&+4\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(\alpha_{jj}b_{jj} - \alpha_{ii}b_{ii}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
=& \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \\
&\Bigg[\Bigg(\alpha_{jj}b_{jj} - \alpha_{ii}b_{ii} + 2z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)^2\Bigg]^{\frac{1}{2}}\Bigg\} \\
=& \alpha_{ii}b_{ii} - z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii} = \Bigg(b_{ii} - z_i^{(t)} \sum_{k\neq i} |b_{ki}|\Bigg)\alpha_{ii} \\
\geq& \frac{b_{ii} - z_i^{(t)} \sum\limits_{k\neq i} |b_{ki}|}{a_{ii}}.
\end{aligned}$$

This means that

$$\begin{aligned}
&\min_{i\neq j} \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\
&+4\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(z_j^{(t)} \sum_{k\neq j} |b_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
\geq& \min_{i\in N}\left\{\frac{b_{ii} - z_i^{(t)} \sum\limits_{k\neq i} |b_{ki}|}{a_{ii}}\right\} \geq \min_{i\in N}\left\{\frac{b_{ii} - p_i^{(t)} \sum\limits_{k\neq i} |b_{ki}|}{a_{ii}}\right\} \\
\geq& \min_{i\in N}\left\{\frac{b_{ii} - t_i \sum\limits_{k\neq i} |b_{ki}|}{a_{ii}}\right\}.
\end{aligned}$$

So the lower bound in Theorem 3.1 is better than the lower bounds in Theorem 1 in [20] and Theorem 4.8 in [21]. Moreover, since $z_i^{(t)} \leq w_i$ for $i \in N, t = 1, 2, \cdots$, we have

$$\begin{aligned}
&\min_{i\neq j} \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Bigg[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\
&+4\Bigg(z_i^{(t)} \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(z_j^{(t)} \sum_{k\neq j} |b_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
\geq& \min_{i\neq j} \frac{1}{2}\Bigg\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - [(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 \\
&+4\Bigg(w_i \sum_{k\neq i} |b_{ki}|\alpha_{ii}\Bigg)\Bigg(w_j \sum_{k\neq j} |b_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

Thus the lower bound in Theorem 3.1 is better than that in Theorem 2 in [19].

Theorem 3.2 The sequence $\{\Delta_t\}, t = 1, 2, \cdots$ obtained from Theorem 3.1 is monotone increasing with an upper bound $\tau(B \circ A^{-1})$ and, consequently, is convergent.

Proof. By Lemma 2.3 and Lemma 2.4, we have $p_{ji}^{(t)} \geq p_{ji}^{(t+1)} \geq 0$ and $f_{ji}^{(t)} \geq f_{ji}^{(t+1)} \geq 0$, $t = 1, 2, \cdots$. So by the definitions of $p_i^{(t)}$ and $f_i^{(t)}$, it is easy to see that sequence $\{p_i^{(t)}\}$ and $\{f_i^{(t)}\}$ are monotone decreasing, and since $z_{ji}^{(t)} = \min\{f_{ji}^{(t)}, p_{ji}^{(t)}\}$, for $t = 1, 2, \cdots$, we have

$$z_{ji}^{(t)} = \min\{f_{ji}^{(t)}, p_{ji}^{(t)}\} \geq \min\{f_{ji}^{(t+1)}, p_{ji}^{(t+1)}\} = z_{ji}^{(t+1)},$$

so by the definition of $z_i^{(t)} = \max\limits_{j\neq i}\{z_{ij}^{(t)}\}$, for $i \in N$, for $t = 1, 2, \cdots$, we have

$$\begin{aligned}
z_i^{(t)} =& \max_{i\in \mathbb{N}}\{z_{i1}^{(t)}, \cdots, z_{i,i-1}^{(t)}, z_{i,i+1}^{(t)}, \cdots, z_{in}^{(t)}\} \\
&\geq \max_{i\in \mathbb{N}}\{z_{i1}^{(t+1)}, \cdots, z_{i,i-1}^{(t+1)}, z_{i,i+1}^{(t+1)}, \cdots, z_{in}^{(t+1)}\} = z_i^{(t+1)},
\end{aligned}$$

which implies the sequence $\{z_i^{(t)}\}$ is monotone decreasing sequence. Then Δ_t is a monotonically increasing sequence. Hence, the sequence is convergent.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular *M*-matrix. By Lemma 2.9, we know that if A^{-1} is a doubly stochastic matrix, then $A^Te = e$, $Ae = e$, that is, $a_{ii} = 1 + \sum\limits_{j\neq i} |a_{ij}| = 1 + \sum\limits_{j\neq i} |a_{ji}|$. So A is a strictly diagonally dominant matrix by row and by column. If $B = A$, according to Theorem 3.1, the following corollary is established.

Corollary 3.1 Let $A = (a_{ij})_{n\times n}$ be a nonsingular *M*-matrix, and $A^{-1} = (\alpha_{ij})_{n\times n}$ be doubly stochastic. Then, for $t = 1, 2, \cdots$,

$$\begin{aligned}
\tau(A \circ A^{-1}) \geq& \frac{1}{2} \min_{i\neq j}\Bigg\{\alpha_{ii}a_{ii} + \alpha_{jj}a_{jj} - \Bigg[(\alpha_{ii}a_{ii} - \alpha_{jj}a_{jj})^2 \\
&+4\Bigg(z_i^{(t)} \sum_{k\neq i} |a_{ki}|\alpha_{ii}\Bigg)\Bigg(z_j^{(t)} \sum_{k\neq j} |a_{kj}|\alpha_{jj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\} \\
=& \Gamma_t. \qquad (11)
\end{aligned}$$

Remark 3.2 According to Remark 2.1, for $i \in N, j \in N, j \neq i$ and $t = 1, 2, \cdots$, we have

$$\begin{aligned}
&s_{ji} \geq m_{ji} \geq z_{ji}^{(t)},\ u_{ji} \geq w_{ji} = g_{ji}^{(0)} \geq z_{ji}^{(t)},\ p_{ji}^{(t)} \geq z_{ji}^{(t)}, \\
&s_i \geq m_i \geq z_i^{(t)},\ u_i \geq w_i = g_i^{(0)} \geq z_i^{(t)},\ p_i^{(t)} \geq z_i^{(t)},
\end{aligned}$$

which results in

$$\tau(A\circ A^{-1})\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}\alpha_{ii}+a_{jj}\alpha_{jj}-\Big[(a_{ii}\alpha_{ii}-a_{jj}\alpha_{jj})^2 +4\Big(z_i^{(t)}\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Big)\Big(z_j^{(t)}\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Big)\Big]^{\frac{1}{2}}\Big\}$$
$$\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}\alpha_{ii}+a_{jj}\alpha_{jj}-\Big[(a_{ii}\alpha_{ii}-a_{jj}\alpha_{jj})^2 +4\Big(m_i\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Big)\Big]^{\frac{1}{2}}\Big\},$$

and similar to the proof of Remark 3.1 and by Lemma 3.1 in [16], we have

$$\tau(A\circ A^{-1})\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}\alpha_{ii}+a_{jj}\alpha_{jj}-\Big[(a_{ii}\alpha_{ii}-a_{jj}\alpha_{jj})^2 +4\Big(z_i^{(t)}\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Big)\Big(z_j^{(t)}\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Big)\Big]^{\frac{1}{2}}\Big\}$$
$$\geq \min_{i\neq j}\frac{1}{2}\{a_{ii}\alpha_{ii}+a_{jj}\alpha_{jj}-\Big[(a_{ii}\alpha_{ii}-a_{jj}\alpha_{jj})^2 +4\Big(w_i\sum_{k\neq i}|a_{ki}|\alpha_{ii}\Big)\Big(w_j\sum_{k\neq j}|a_{kj}|\alpha_{jj}\Big)\Big]^{\frac{1}{2}}\}$$
$$\geq \min_{i\in N}\left\{\frac{a_{ii}-u_iR_i}{1+\sum\limits_{j\neq i}u_{ji}}\right\}.$$

Furthermore, by Theorem 3.3 in [16], we have

$$\min_{i\in N}\left\{\frac{a_{ii}-u_iR_i}{1+\sum\limits_{j\neq i}u_{ji}}\right\}\geq \min_{i\in N}\left\{\frac{a_{ii}-t_iR_i}{1+\sum\limits_{j\neq i}t_{ji}}\right\}.$$

So the lower bound in Corollary 3.1 is an improvement on the lower bounds in Theorem 3.2 in [17], Corollary 3 in [19] and Theorem 3.1 in [15], [16]. In addition, by Theorem 3.3 in [17], we have

$$\min_{i\neq j}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 +4\Big(m_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Big]^{\frac{1}{2}}\Big\}$$
$$\geq \min_{i\in N}\left\{\frac{a_{ii}-s_iR_i}{1+\sum\limits_{j\neq i}s_{ji}}\right\}.$$

Hence the result of Corollary 3.1 is sharper than that of Theorem 3.1 in [14].

Theorem 3.3 Let $A=(a_{ij}), B=(b_{ij})\in\mathbb{R}^{n\times n}$ be nonsingular M-matrices and $A^{-1}=(\alpha_{ij})_{n\times n}$. Then, for $t=1,2,\cdots$

$$\tau(B\circ A^{-1}) \geq \frac{1}{2}\min_{i\neq j}\left\{\alpha_{ii}b_{ii}+\alpha_{jj}b_{jj}-\Big[(\alpha_{ii}b_{ii}-\alpha_{jj}b_{jj})^2 +4\Big(s_i\alpha_{ii}\sum_{k\neq i}\frac{|b_{ki}|z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\alpha_{jj}\sum_{k\neq j}\frac{|b_{kj}|z_{kj}^{(t)}}{s_k}\Big)\Big]^{\frac{1}{2}}\right\}=\Omega_t. \tag{12}$$

Proof. It is not difficult to verify that the result holds with equality for $n=1$. We next assume that $n\geq 2$. Since A is a nonsingular M-matrix, there exists a positive diagonal matrix D such that $D^{-1}AD$ is a strictly row diagonally dominant nonsingular M-matrix, and

$$\tau(B\circ A^{-1})=\tau(D^{-1}(B\circ A^{-1})D)=\tau((B\circ(D^{-1}AD)^{-1}).$$

So, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix.

(i) If A and B are irreducible matrices. Then, for any $i\in N$, we derive

$$0<s_i=\max_{i\neq j}\left\{\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|d_k}{a_{jj}}\right\}<1.$$

Let $\tau(B\circ A^{-1})=\lambda$. By Lemma 2.10 and Lemma 2.6, there exists a pair (i,j) of positive integers with $i\neq j$, for $t=1,2,\cdots$, such that

$$|\lambda-\alpha_{ii}b_{ii}||\lambda-\alpha_{jj}b_{jj}| \leq\Big(s_i\sum_{k\neq i}\frac{|b_{ki}|\alpha_{ki}}{s_k}\Big)\Big(s_j\sum_{k\neq j}\frac{|b_{kj}|\alpha_{kj}}{s_k}\Big) \leq\Big(s_i\sum_{k\neq i}\frac{|b_{ki}|\alpha_{ii}z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\sum_{k\neq j}\frac{|b_{kj}|\alpha_{jj}z_{kj}^{(t)}}{s_k}\Big) =\Big(s_i\alpha_{ii}\sum_{k\neq i}\frac{|b_{ki}|z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\alpha_{jj}\sum_{k\neq j}\frac{|b_{kj}|z_{kj}^{(t)}}{s_k}\Big). \tag{13}$$

Since A and B are nonsingular M-matrices, $B\circ A^{-1}$ is a nonsingular M-matrix. Having in mind that $0\leq\lambda\leq\alpha_{ii}b_{ii}$ for all $i\in N$, from Inequality (13) we see that

$$(\lambda-\alpha_{ii}b_{ii})(\lambda-\alpha_{jj}b_{jj}) \leq\Big(s_i\alpha_{ii}\sum_{k\neq i}\frac{|b_{ki}|z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\alpha_{jj}\sum_{k\neq j}\frac{|b_{kj}|z_{kj}^{(t)}}{s_k}\Big), \tag{14}$$

which yields that

$$\tau(B\circ A^{-1}) \geq\frac{1}{2}\Big\{\alpha_{ii}b_{ii}+\alpha_{jj}b_{jj}-\Big[(\alpha_{ii}b_{ii}-\alpha_{jj}b_{jj})^2 +4\Big(s_i\alpha_{ii}\sum_{k\neq i}\frac{|b_{ki}|z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\alpha_{jj}\sum_{k\neq j}\frac{|b_{kj}|z_{kj}^{(t)}}{s_k}\Big)\Big]^{\frac{1}{2}}\Big\},$$

and therefore

$$\tau(B\circ A^{-1}) \geq\frac{1}{2}\min_{i\neq j}\{\alpha_{ii}b_{ii}+\alpha_{jj}b_{jj}-\Big[(\alpha_{ii}b_{ii}-\alpha_{jj}b_{jj})^2 +4\Big(s_i\alpha_{ii}\sum_{k\neq i}\frac{|b_{ki}|z_{ki}^{(t)}}{s_k}\Big)\Big(s_j\alpha_{jj}\sum_{k\neq j}\frac{|b_{kj}|z_{kj}^{(t)}}{s_k}\Big)\Big]^{\frac{1}{2}}\}. \tag{15}$$

If one of A and B is reducible, similar to the proof of Theorem 3.1 (ii), the result is obtained.

Remark 3.3 Similar to the proof of Remark 3.1 and by

Lemma 3 in [15], for $t = 1, 2, \cdots$, is given by

$$\begin{aligned}
&\tau(B \circ A^{-1}) \\
&\geq \frac{1}{2} \min_{i \neq j} \Bigg\{ \alpha_{ii} b_{ii} + \alpha_{jj} b_{jj} - \Bigg[(\alpha_{ii} b_{ii} - \alpha_{jj} b_{jj})^2 \\
&\quad + 4 \Bigg(s_i \alpha_{ii} \sum_{k \neq i} \frac{|b_{ki}| z_{ki}^{(t)}}{s_k} \Bigg) \Bigg(s_j \alpha_{jj} \sum_{k \neq j} \frac{|b_{kj}| z_{kj}^{(t)}}{s_k} \Bigg) \Bigg]^{\frac{1}{2}} \Bigg\} \\
&\geq \min_{i \in \mathbb{N}} \left\{ \frac{b_{ii} - s_i \sum\limits_{j \neq i} \frac{|b_{ji}| p_{ji}^{(t)}}{s_j}}{a_{ii}} \right\}.
\end{aligned}$$

Therefore, the bound of Theorem 3.3 is sharper than that in [20].

Using the same method as the proof of Theorem 3.2, the following theorem can be deduced.

Theorem 3.4 The sequence $\{\Omega_t\}$, $t = 1, 2, \cdots$ obtained from Theorem 3.3 is monotone increasing with an upper bound $\tau(B \circ A^{-1})$ and, consequently, is convergent.

Taking $B = A$ and using Theorem 3.3, we can get the following corollary.

Corollary 3.2 Let $A = (a_{ij})_{n \times n}$ be a nonsingular M-matrix, and let $A^{-1} = (\alpha_{ij})_{n \times n}$ be doubly stochastic. Then, for $t = 1, 2, \cdots$,

$$\begin{aligned}
&\tau(A \circ A^{-1}) \\
&\geq \frac{1}{2} \min_{i \neq j} \Bigg\{ \alpha_{ii} a_{ii} + \alpha_{jj} a_{jj} - \Bigg[(\alpha_{ii} a_{ii} - \alpha_{jj} a_{jj})^2 \\
&\quad + 4 \Bigg(s_i \alpha_{ii} \sum_{k \neq i} \frac{|a_{ki}| z_{ki}^{(t)}}{s_k} \Bigg) \Bigg(s_j \alpha_{jj} \sum_{k \neq j} \frac{|a_{kj}| z_{kj}^{(t)}}{s_k} \Bigg) \Bigg]^{\frac{1}{2}} \Bigg\} = \Upsilon_t.
\end{aligned} \tag{16}$$

Remark 3.4 According to Remark 3.1 and by Lemma 4 in [20], for $t = 1, 2, \cdots$, we can conclude that the bound of Corollary 3.2 is sharper than that in Corollary 4 in [20].

Remark 3.5 The sequence $\{\Upsilon_t\}$, $t = 1, 2, \cdots$ obtained from Corollary 3.2 is monotone increasing with an upper bound $\tau(A \circ A^{-1})$ and, consequently, is convergent.

Let $\Phi_t = \max\{\Delta_t, \Omega_t\}$. Combining Theorems 3.1 and 3.3, Theorem 3.5 is easily obtained.

Theorem 3.5 Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be nonsingular M-matrices and $A^{-1} = (\alpha_{ij})_{n \times n}$. Then, for $t = 1, 2, \cdots$,

$$\tau(B \circ A^{-1}) \geq \Phi_t. \tag{17}$$

Let $\Psi_t = \max\{\Gamma_t, \Upsilon_t\}$. By Corollaries 3.1 and 3.2, Theorem 3.6 is easily derived.

Theorem 3.6 Let $A = (a_{ij})_{n \times n}$ be a nonsingular M-matrix, and let $A^{-1} = (\alpha_{ij})_{n \times n}$ be doubly stochastic. Then, for $t = 1, 2, \cdots$,

$$\tau(A \circ A^{-1}) \geq \Psi_t. \tag{18}$$

IV. Numerical examples

Example 4.1 Consider the following two nonsingular M-matrices [19]:

$$A = \begin{pmatrix}
39 & -16 & -2 & -3 & -2 & -5 & -2 & -3 & -5 & 0 \\
-26 & 44 & -2 & -4 & -2 & -1 & 0 & -2 & -3 & -3 \\
-1 & -9 & 29 & -3 & -4 & 0 & -5 & -4 & -1 & -1 \\
-2 & -3 & -10 & 36 & -12 & 0 & -5 & -1 & -2 & 0 \\
0 & -3 & -1 & -9 & 44 & -16 & -3 & -4 & -4 & -3 \\
-3 & -4 & -3 & -4 & -12 & 48 & -18 & -1 & 0 & -2 \\
-2 & -1 & -4 & -3 & -4 & -16 & 45 & -9 & -4 & -1 \\
-1 & -2 & -2 & -2 & -3 & -1 & -5 & 38 & -20 & -1 \\
-2 & -1 & 0 & -3 & -4 & -5 & -2 & -10 & 47 & -19 \\
-1 & -4 & -4 & -4 & 0 & -3 & -4 & -3 & -7 & 31
\end{pmatrix},$$

$$B = \begin{pmatrix}
90 & -3 & -2 & -7 & -4 & -7 & -6 & -3 & -9 & -3 \\
-4 & 100 & -5 & -4 & -8 & -7 & -1 & -9 & -8 & -8 \\
-5 & -9 & 62 & -4 & -7 & -9 & -9 & -1 & -4 & -8 \\
-8 & -8 & -10 & 99 & 0 & -6 & -8 & -9 & -3 & -6 \\
-3 & -8 & -10 & -6 & 62 & -3 & -6 & -7 & -5 & -1 \\
-2 & -3 & -5 & -10 & -6 & 55 & -5 & -1 & -3 & -10 \\
-8 & -5 & -8 & -8 & -3 & -3 & 52 & -6 & -1 & -4 \\
-4 & -5 & -8 & -4 & -1 & -1 & -6 & 57 & -7 & -7 \\
-2 & -1 & -6 & -10 & -2 & -6 & -5 & -9 & 86 & -5 \\
-5 & -7 & -3 & -9 & -5 & -7 & -9 & -5 & -9 & 72
\end{pmatrix}.$$

Numerical results are given in Table I and Table II for the total number of iterations $T = 15$. In fact, $\tau(B \circ A^{-1}) = 3.4570$.

We define

$$RES = |T(a) - \tau(B \circ A^{-1})|,$$

where $T(a)$ stands for the result derived by using sequence a when the number of iteration is T.

In Table I, for given matrices A and B, we list the lower bounds of $\tau(B \circ A^{-1})$ calculated by Theorem 4.8 in [21], Theorem 1 in [19], Theorem 2 in [19], Theorem 1 in [20] and Theorem 3.1 in this paper. For the sequences obtained by Theorem 1 in [20] and Theorem 3.1 in this paper, the results are displayed for every step of iteration.

In Table II, we present the lower bounds of $\tau(B \circ A^{-1})$ calculated by Theorem 3 in [20] and Theorem 3.3 in this paper, respectively. For the sequences obtained by Theorem 3 in [20] and Theorem 3.3 in this paper, the results are displayed for every step of iteration.

Numerical results in Table I and Table II show that:

(a) Lower bounds obtained from Theorem 3.1 and Theorem 3.3 are greater than the ones in Theorem 2 in [19], Theorem 1 in [20], Theorem 4.8 in [21] and Theorem 3 in [20].

(b) Sequences obtained from Theorems 3.1 and 3.3 both are monotone increasing.

(c) The sequences obtained from Theorems 3.1 and 3.3 are closer to the true value of $\tau(B \circ A^{-1})$ generally when the number of iterations T is increasing.

(d) The sequences obtained from Theorems 3.1 and 3.3 are much closer to the true value of $\tau(B \circ A^{-1})$ than those obtained from Theorems 1 and 3 in [20], respectively.

In Figure 1 and Figure 2, we display the RES generated by Theorem 1 in [20], Theorem 3.1 and Theorem 3 in [20], Theorem 3.3, respectively against number of iterations for $T = 100$. From these two figures, we note that the four

TABLE I
THE LOWER BOUNDS OF $\tau(B \circ A^{-1})$

Method	t	$\tau(B \circ A^{-1})$	Method	t	$\tau(B \circ A^{-1})$
Theorem 4.8 in [21]		0.0027			
Theorem 1 in [19]		0.0435			
Theorem 2 in [19]		0.7212			
Theorem 1 in [20]	1	0.2322	Theorem 3.1	1	0.7630
	2	0.2497		2	0.8703
	3	0.2595		3	0.9286
	4	0.2652		4	0.9639
	5	0.2684		5	0.9856
	6	0.2702		6	0.9989
	7	0.2713		7	1.0071
	8	0.2719		8	1.0123
	9	0.2723		9	1.0155
	10	0.2725		10	1.0175
	11	0.2726		11	1.0188
	12	0.2727		12	1.0196
	13	0.2727		13	1.0201
	14	0.2728		14	1.0204
	15	0.2728		15	1.0206

sequences of $\tau(B \circ A^{-1})$ are convergent, while the RES of the sequence obtained from Theorem 3.1 are much less than that obtained from Theorem 1 in [20]. For the sequences obtained from Theorem 3.3 and Theorem 3 in [20], we can get the same conclusion.

TABLE II
THE LOWER BOUNDS OF $\tau(B \circ A^{-1})$

Method	t	$\tau(B \circ A^{-1})$	Method	t	$\tau(B \circ A^{-1})$
Theorem 3 in [20]	1	0.0764	Theorem 3.3	1	0.9370
	2	0.1074		2	1.0902
	3	0.1254		3	1.1780
	4	0.1360		4	1.2290
	5	0.1423		5	1.2588
	6	0.1461		6	1.2764
	7	0.1484		7	1.2867
	8	0.1498		8	1.2928
	9	0.1506		9	1.2964
	10	0.1511		10	1.2986
	11	0.1514		11	1.2998
	12	0.1515		12	1.3006
	13	0.1516		13	1.3010
	14	0.1517		14	1.3013
	15	0.1517		15	1.3015

Example 4.2 [20] Let

$$A = \begin{pmatrix} 20 & -1 & -2 & -3 & -4 & -1 & -1 & -3 & -2 & -2 \\ -1 & 18 & -3 & -1 & -1 & -4 & -2 & -1 & -3 & -1 \\ -2 & -1 & 10 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -3 & -1 & 0 & 16 & -4 & -2 & -1 & -1 & -1 & -2 \\ -1 & -3 & 0 & -2 & 15 & -1 & -1 & -1 & -2 & -3 \\ -3 & -2 & -1 & -1 & -1 & 12 & -2 & 0 & -1 & 0 \\ -1 & -3 & -1 & -1 & 0 & -1 & 9 & 0 & -1 & 0 \\ -3 & -1 & -1 & -4 & -1 & 0 & 0 & 12 & 0 & -1 \\ -2 & -4 & -1 & -1 & -1 & 0 & -1 & -3 & 14 & 0 \\ -3 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 11 \end{pmatrix}.$$

By $Ae = e$, $A^T e = e$, we know that A is a strictly diagonally dominant by row and column. Base on $A \in Z_n$, it is easy to see that A is a nonsingular M-matrix and A^{-1} is doubly stochastic. Numerical results are given in Table III and Table IV for the total number of iterations $T = 10$. In fact, $\tau(A \circ A^{-1}) = 0.9678$.

In Table III, for given matrix A, we list the lower bounds of $\tau(A \circ A^{-1})$ calculated by Theorem 3.1 in [14], [15], [16], Corollary 2.5 in [18], Theorem 3.2 in [17] and Corollary 3 in [19].

In Table IV, we show the lower bounds of $\tau(A \circ A^{-1})$ calculated by Theorem 5 in [20] and Theorem 3.6 in this paper, respectively. For the sequences obtained by Theorem 5 in [20] and Theorem 3.6 in this paper, the results are displayed for every step of iteration.

Numerical results in Table III and Table IV show that:

(a) Lower bounds obtained from Theorem 3.6 are greater than those in Theorem 3.1 in [14], [15], [16], Corollary 2.5 in [18], Theorem 3.2 in [17], Corollary 3 in [19] and Theorem 5 in [20].

(b) Sequence obtained from Theorem 3.6 is monotone increasing.

(c) Sequence obtained from Theorem 3.6 is convergent to the value 0.9409, which is close to the true value of $\tau(A \circ A^{-1})$.

(d) The sequence obtained from Theorem 3.6 is much closer to the true value of $\tau(A \circ A^{-1})$ than that obtained from Theorem 5 in [20], and the sequence obtained from Theorem 3.6 approximates effectively to the true value of $\tau(A \circ A^{-1})$, so we can estimate $\tau(A \circ A^{-1})$ by Theorem 3.6.

In Figure 3, we present the RES generated by Theorem 5 in [20] and Theorem 3.6, respectively against number of iterations for $T = 20$, where RES is defined as Example 4.1. From this figure, we find that the two sequences of $\tau(A \circ A^{-1})$ are convergent, while the RES of the sequence obtained from Theorem 3.6 are much less than that obtained from Theorem 5 in [20].

TABLE III
THE LOWER BOUNDS OF $\tau(A \circ A^{-1})$

Method	$\tau(A \circ A^{-1})$
Theorem 3.1 in [14]	0.2519
Theorem 3.1 in [15]	0.4125
Theorem 3.1 in [16]	0.4471
Corollary 2.5 in [18]	0.1401
Theorem 3.2 in [17]	0.4732
Corollary 3 in [19]	0.6064

Example 4.3 [20] Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, where $a_{11} = a_{22} = \cdots = a_{nn} = 2$, $a_{12} = a_{23} = \cdots = a_{n-1,n} = a_{n1} = -1$, and $a_{ij} = 0$ elsewhere.

It is easy to see that A is a nonsingular M-matrix and A^{-1} is doubly stochastic. The results obtained from Theorem 5 in [20] and Theorem 3.6 for $n = 15$ and $T = 10$ are listed in Table 5, where T is defined in Example 4.1. In fact, $\tau(A \circ A^{-1}) = 0.7500$.

In Table V, we show the lower bounds of $\tau(A \circ A^{-1})$ calculated by Theorem 5 in [20] and Theorem 3.6 in this paper, respectively. For the sequences obtained by Theorem 5 in [20] and Theorem 3.6 in this paper, the results are displayed for every step of iteration.

Numerical results in Table V show that the lower bound obtained from Theorem 3.6 could reach the true value of $\tau(A \circ A^{-1})$ in some cases. Moreover, applying Theorem 3.6 is faster to reach the convergence value than Theorem 5 in [20]. From Table 5, it can be seen that applying Theorem 3.6 reaches the true value of $\tau(A \circ A^{-1})$ after only one iteration, but applying Theorem 5 in [20] needs 9 iterations for $n = 15$ when they reach the true value of $\tau(A \circ A^{-1})$.

TABLE IV
THE LOWER BOUNDS OF $\tau(A \circ A^{-1})$

Method	t	$\tau(A \circ A^{-1})$	Method	t	$\tau(A \circ A^{-1})$
Theorem 5 in [20]	1	0.7359	Theorem 3.6	1	0.8182
	2	0.8441		2	0.8889
	3	0.8976		3	0.9186
	4	0.9233		4	0.9313
	5	0.9328		5	0.9368
	6	0.9350		6	0.9393
	7	0.9359		7	0.9404
	8	0.9363		8	0.9408
	9	0.9364		9	0.9409
	10	0.9365		10	0.9409

TABLE V
THE LOWER UPPER OF $\tau(A \circ A^{-1})$, N=15

Method	t	$\tau(A \circ A^{-1})$	Method	t	$\tau(A \circ A^{-1})$
Theorem 5 in [20]	1	0.1905	Theorem 3.6	1	0.7500
	2	0.4364			
	3	0.6379			
	4	0.7191			
	5	0.7422			
	6	0.7481			
	7	0.7495			
	8	0.7499			
	9	0.7500			

V. CONCLUSIONS

In this paper, base on the constructing sequence of iterations, we have established new convergent sequences $\{\Phi_t\}$ and $\{\Psi_t\}$, $t = 1, 2, \cdots$ in Theorem 3.5 and Theorem 3.6, which are more accurate than the existing ones in [14], [15], [16], [17], [18], [19], [20], [21] to approximate $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$, respectively. Numerical results given in Section IV (Tables I-IV) show that the results obtained by the new convergent sequences $\{\Phi_t\}$ and $\{\Psi_t\}$, $t = 1, 2, \cdots$ are more sharper than the those obtained by the lower bounds in [14], [15], [16], [17], [18], [19], [20], [21]. Numerical results also present the feasibility and effectiveness of the new convergent sequences when they are used to estimate $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$.

Inasmuch as the calculation in experiments are involve in controlling accuracy, so there an interesting problem is how to accurately these bounds can be computed which may improve the accuracy of the of these sequences. At present, it is very difficult for us to give the error analysis. We will continue to study this problem in the future. In addition, although the efficiency convergent sequences for $\{\Phi_t\}$ and $\{\Psi_t\}$ are given, they are almost useless for the implementation of these new sequences for $\{\Phi_t\}$ and $\{\Psi_t\}$ when the size n of matrix is large, since the calculations of the inverse of the matrices is very expensive. How to find easier calculated convergent sequences for $\{\Phi_t\}$ and $\{\Psi_t\}$ is still a tough task, which should be further studied in the further. Finally, we can find that the sequences obtained by Theorem 3.1, Theorem 3.3, Theorem 3.5 and Theorem 3.6 are sharper than the existing ones by theory analyzing and numerical examples also verify that, whereas the sequences are less precise sometimes, we can see that the result 1.0206 in Table I obtained by Theorem 3.1 and the result 1.3015 in Table II obtained by Theorem 3.3, they are not close to the true value 3.4570. Hence, finding more accurate sequences for $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$ is also a further work.

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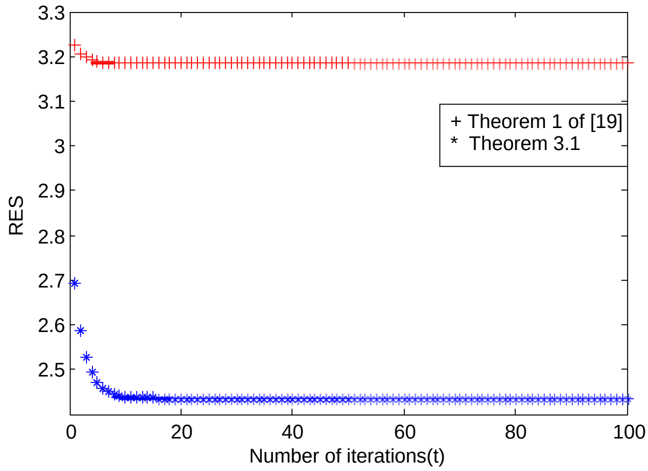

Fig. 1. The iterations curves of two sequences obtained from Theorem 1 in [20] and Theorem 3.1, respectively with $T = 100$.

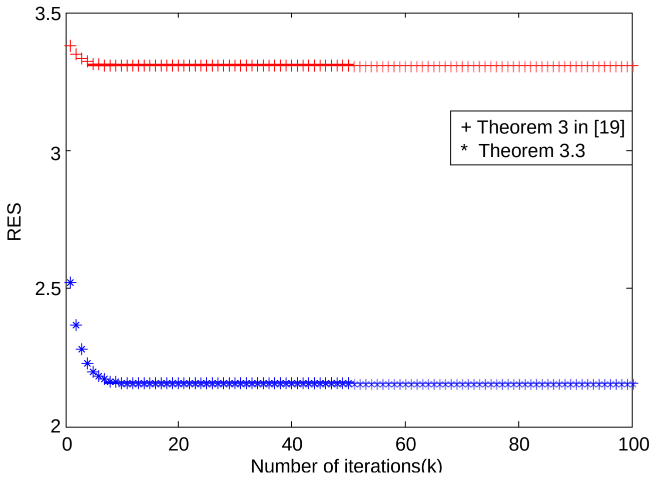

Fig. 2. The iterations curves of two sequences obtained from Theorem 3 in [20] and Theorem 3.3, respectively with $T = 100$.

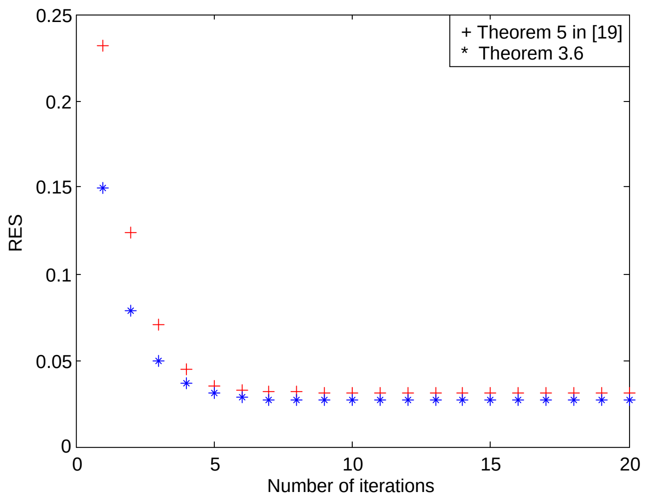

Fig. 3. The iterations curves of two sequences obtained from Theorem 5 in [20] and Theorem 3.6, respectively with $T = 20$.