Representation of Multiple Integrals via the Generalized Lerch Transcendent

Zhongfeng Sun, Aijuan Li, and Huizeng Qin

***Abstract*—In this paper, we derive several properties of the generalized Lerch transcendent $\Phi(z,\vec{s},\vec{u})$ and the generalized Euler sum with parameters $H(z,\vec{s},\vec{u})$, and establish their connections with some certain types of multiple integrals. In particular, $\Phi(z,\vec{s},\vec{u})$ and $H(z,\vec{s},\vec{u})$ can be expressed as the linear combination of the Lerch transcendent $\Phi(z,s,u)$ and the Euler sum with parameters $H(z,s,u)$, respectively. With the aid of those connections, the closed forms of some special multiple integrals which can be expressed by the special constants and the Riemann zeta functions are established.**

***Index Terms*—Multiple Integrals, Lerch Transcendent, Euler Sum, Closed Form.**

I. Introduction

Multiple integrals which can be expressed as the linear forms of Riemann zeta functions with rational coefficients was first proposed by Beukers [1] in the following form($n=0, z=1, s=0, u,v\in\mathbb{N}_0$)

$$\int_0^1\int_0^1 \frac{x^u y^v\left[\ln(xy)\right]^s}{1-zxy}(1-x)^n dxdy, \tag{1}$$

where $\mathbb{N}_0=\{0,1,2,\ldots\}$. In recent years the issue of Beukers' integrals and its generalization have attracted much attention, see [2]∼[8], [11], [12], [13]. Now, a short list about the generalization of Beukers' integrals are given as follows. Hadjicostas [2] provided a method to generalize Beukers' integrals (1) for $s\in\mathbb{N}_0$, $n=0$ and $z=1$, which was achieved by using an expansion of an infinite geometric series. Sondow [3] derived the criteria for irrationality of Euler's constant. Zlobin [4], [5] considered the expansion of multiple integrals as linear forms in generalized polylogarithms. A conjecture-generalization of Sondow's formula (1) for $u=v=0, n=1, z=1, s\in\mathbb{C}, \Re s>-2$ was proposed by Hadjicostas [6] and proved by Chapman [7], where $\mathbb{C}$ denotes the complex set and $\Re s$ denotes the real part of the complex number s. Salikhov and Frolovichev [8] represented multiple integrals as linear forms in $1,\zeta(3),\zeta(5),...,\zeta(2k-1)$ over $\mathbb{Q}$, where $\zeta(p)(p>1)$ are Riemann zeta functions which can also be applied to establish the closed forms of the Beta function in [9] and [10]. Zudilin [11] represented well-poised hypergeometric series and integrals as the Euler-type multiple integrals. Guillera and Sondow [12] used analytic continuation of the Lerch transcendent to unify and generalize the results of [1], [2], [3] and [6]. Brychkov [13] considered the generalization of the Guillera-Sondow double integrals and its connection with the hypergeometric function.

In [12], Guillera and Sondow showed that (1) for $n=0$ can be expressed by the Lerch transcendent in the following forms,

$$\begin{aligned}&\int_0^1\int_0^1 \frac{(xy)^{u-1}\left[-\ln(xy)\right]^{s-1}}{1-zxy}dxdy\\ &= \Gamma(s+1)\Phi(z,s+1,u),\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 \frac{x^{u-1}y^{v-1}\left[-\ln(xy)\right]^{s-1}}{1-zxy}dxdy\\ &= \Gamma(s)\frac{\Phi(z,s,v)-\Phi(z,s,u)}{u-v},\end{aligned} \tag{3}$$

where $\Gamma(s)$ is the Gamma function defined by

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt,\quad \Re s>0 \tag{4}$$

and $\Phi(z,s,a)$ is the Lerch transcendent([14], section 1.11) defined by

$$\Phi(z,s,a)=\sum_{k=0}^\infty \frac{z^k}{(k+a)^s}, \tag{5}$$

for $\Re a>0$ and $|z|<1$ or $|z|=1, \Re s>1$. Special cases include the Riemann zeta function $\zeta(s)=\Phi(1,s,1)$ and the polylogarithm $Li_s(z)=z\Phi(z,s,1)$. Moreover, the Lerch transcendent can be calculated by the following integral

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{1-ze^{-x}}dx, \tag{6}$$

where $\Re a>0$ and either $|z|<1, \Re s>0$ or $|z|=1, \Re s>1$.

In this paper, we consider the following multiple integrals which are the extension of the double integrals (2) and (3),

$$\int_{[0,1]^m}\prod_{j=1}^m\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right)\frac{dx_1dx_2\cdots dx_m}{1-zx_1x_2\cdots x_m}, \tag{7}$$

and

$$\begin{aligned}&\int_{[0,1]^m}\prod_{j=1}^m\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right)\\ &\times\ln\frac{1}{1-zx_1x_2\cdots x_m}\frac{dx_1dx_2\cdots dx_m}{1-zx_1x_2\cdots x_m},\end{aligned} \tag{8}$$

where $m\in\mathbb{N}=\{1,2,\ldots\}$ and $|z|\le 1, \Re u_j, \Re s_j>0$.

For $\Re u_j>0 (j=1,2,\ldots,m)$ and either $|z|<1$ or $|z|=1, \Re(\sum_{j=1}^m s_j)>1$, we introduce the generalized Lerch transcendent defined by

$$\Phi(z,\vec{s},\vec{u})=\sum_{k=0}^\infty z^k\prod_{j=1}^m\frac{1}{(u_j+k)^{s_j}} \tag{9}$$

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and the generalized Euler sum with parameters defined by

$$\begin{aligned} H(z,\vec{s},\vec{u}) &= \sum_{k=1}^{\infty} z^k H_k \prod_{j=1}^{m} \frac{1}{(u_j+k)^{s_j}} \\ &= \sum_{l=1}^{\infty} \frac{z^l}{l} \Phi(z,\vec{s},\vec{u}+l\cdot\vec{I}), \end{aligned} \tag{10}$$

where $H_k = \sum_{l=1}^{k} \frac{1}{l}$, $\vec{v} = (v_1, v_2, \ldots, v_m)$ and $\vec{I} = (1,1,\ldots,1)$ is a m-dimensional vector. If $u_1 = \ldots = u_m = u$, then $\Phi(z,\vec{s},\vec{u}) = \Phi(z, \sum_{j=1}^{m} s_j, u)$ reduces to the Lerch transcendent and $H(z,\vec{s},\vec{u}) = H(z, \sum_{j=1}^{m} s_j, u)$ reduces to the Euler sum with parameters [15], which is defined by

$$H(z,s,u) = \sum_{k=1}^{\infty} z^k H_k (u+k)^{-s}.$$

Our aim is to discuss some connections of multiple integrals (7) and (8) with the generalized Lerch transcendent (9) and the generalized Euler sum with parameters (10).

The structure of the paper is given as follows. In Section II, several properties of the generalized Lerch transcendent and the generalized Euler sum with parameters are derived. In particular, their connections with some certain types of multiple integrals are established. In Section III, the closed forms of some special multiple integrals are discussed. The conclusion is given in the last Section of the paper.

II. The generalized Lerch transcendent and multiple integrals

In the following Lemma, the integral representation of the Euler sum with parameters $H(z,s,u)$ is established.

Lemma 2.1 If $\Re u > 0$ and either $|z| < 1, \Re s > 0$ or $|z| = 1, \Re s > 1$, then

$$\begin{aligned} &H(z,s,u) \\ =& -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-ux}}{1-ze^{-x}} \ln(1-ze^{-x})dx \\ =& -\frac{1}{\Gamma(s)} \int_0^{1} \frac{x^{u-1}}{1-zx} \ln^{s-1}\frac{1}{x} \ln(1-zx)dx. \end{aligned} \tag{11}$$

Proof. Combining (10) with (6), we conclude that

$$\begin{aligned} &H(z,s,u) \\ =& \sum_{l=1}^{\infty} \frac{z^l}{l} \Phi(z,s,u+l) \\ =& \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-ux}}{1-ze^{-x}} \sum_{l=1}^{\infty} \frac{z^l e^{-lx}}{l} dx \\ =& -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-ux}}{1-ze^{-x}} \ln(1-ze^{-x})dx \\ =& -\frac{1}{\Gamma(s)} \int_0^{1} \frac{x^{u-1}}{1-zx} \ln^{s-1}\frac{1}{x} \ln(1-zx)dx, \end{aligned} \tag{12}$$

where $\Re u > 0$ and either $|z| < 1, \Re s > 0$ or $|z| = 1, \Re s > 1$. Therefore, (11) holds. ∎

It follows from (6) and (11) that $\Phi(z,s,u)$ and $H(z,s,u)$ have the same parameter range and domain.

Denote

$$F^{(n)}(z,\vec{s},\vec{u}) = \frac{\partial^n}{\partial z^n} F(z,\vec{s},\vec{u}), \tag{13}$$

for $F = \Phi, H$ and $n \in \mathbb{N}_0$.

In the following Lemma, $\Phi^{(n)}(z,s,u)$ and $H^{(n)}(z,s,u)$ $(n \in \mathbb{N})$ can be expressed by $\Phi(z,s,u)$ and $H(z,s,u)$, respectively.

Lemma 2.2 If $n \in \mathbb{N}$, $\Re u > 0$ and either $|z| < 1$ or $|z| = 1, \Re s > n+1$, then

$$\begin{aligned} \Phi^{(n)}(z,s,u) =& \sum_{j=0}^{n} (-1)^{n-j} \Phi(z,s-j,n+u) \\ & \times \sum_{l=j}^{n} C_l^j s(n,l) (n-1+u)^{l-j} \end{aligned} \tag{14}$$

and

$$\begin{aligned} &H^{(n)}(z,s,u) \\ =& z^{-n} \sum_{j=0}^{n} (-1)^{n-j} \\ & \times \Big[H(z,s-j,u) - G(n,z,s-j,u) \Big] \\ & \times \sum_{l=j}^{n} C_l^j s(n,l) (n-1+u)^{l-j}, \end{aligned} \tag{15}$$

where $s(n,l)$ is the Stirling number of the first kind and

$$G(n,z,s-j,u) = \sum_{k=1}^{n-1} \frac{z^k H_k}{(k+u)^{s-j}}, \tag{16}$$

where an empty sum is understood to be nil.

Proof. Calculating n-order partial derivatives on z for (5), we obtain

$$\Phi^{(n)}(z,s,u) = \sum_{k=0}^{\infty} \frac{(k+1)_n z^k}{(k+n+u)^s}, \tag{17}$$

where $(x)_n$ is a Pochhammer symbol, *i.e.*,

$$\begin{aligned} (x)_n &= x(x+1)\cdots(x+n-1) \\ &= \sum_{l=0}^{n} (-1)^{n+l} s(n,l) x^l. \end{aligned} \tag{18}$$

Inserting $(x)_n$ into (17) and exchanging the order of series summation, we come to

$$\begin{aligned} &\Phi^{(n)}(z,s,u) \\ =& \sum_{k=0}^{\infty} \frac{z^k}{(k+n+u)^s} \sum_{l=0}^{n} (-1)^{n+l} s(n,l) (k+1)^l \\ =& \sum_{j=0}^{n} (-1)^{n-j} \Phi(z,s-j,n+u) \\ & \times \sum_{l=j}^{n} C_l^j s(n,l) (n-1+u)^{l-j}, \end{aligned} \tag{19}$$

so (14) holds.

Moreover, (15) can be proved similarly. ∎

In the following theorem, $\Phi^{(n)}(z,\vec{s},\vec{u})$ and $H^{(n)}(z,\vec{s},\vec{u})$ can be expressed as the linear combination of $\Phi^{(n)}(z,s,u)$ and $H^{(n)}(z,s,u)$ $(n \in \mathbb{N})$, respectively.

Theorem 2.3 Let $n, m, i, s_i (i \le m)$ be positive integers, $\Re u_i > 0$ and $F(z, \vec{s}, \vec{u}) \equiv \Phi(z, \vec{s}, \vec{u})$ or $H(z, \vec{s}, \vec{u})$.

1) If $|z| < 1$, then

$$F(z, \vec{s}, \vec{u}) = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) F(z, s_i - l, u_i) \tag{20}$$

and

$$F^{(n)}(z, \vec{s}, \vec{u}) = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) F^{(n)}(z, s_i - l, u_i), \tag{21}$$

where

$$a_{0,i}(\vec{s}, \vec{u}) = \prod_{j=1, j\ne i}^{m} (u_j - u_i)^{-s_j}, \tag{22}$$

$$\begin{aligned} a_{l,i}(\vec{s}, \vec{u}) = & \sum_{k=0}^{l-1} (-1)^{l-k} \frac{1}{l} a_{k,i}(\vec{s}, \vec{u}) \\ & \times \sum_{j=1, j\ne i}^{m} s_j (u_j - u_i)^{k-l}, \end{aligned} \tag{23}$$

for $l = 1, \ldots, s_i - 1$.

2) If $|z| = 1$, then

$$\begin{aligned} F(z, \vec{s}, \vec{u}) = & \sum_{i=1}^{m} sgn(s_i - 1)^{+} \\ & \times \left[\sum_{l=0}^{s_i-2} a_{l,i}(\vec{s}, \vec{u}) F(z, s_i - l, u_i) \right] \\ & + \lim_{z_0 \to z} \sum_{i=1}^{m} a_{s_i-1,i}(\vec{s}, \vec{u}) F(z_0, 1, u_i) \end{aligned} \tag{24}$$

and

$$\begin{aligned} & F^{(n)}(z, \vec{s}, \vec{u}) = \sum_{i=1}^{m} sgn(l_i) \\ & \times \left[\sum_{l=0}^{s_i-2-n} a_{l,i}(\vec{s}, \vec{u}) F^{(n)}(z, s_i - l, u_i) \right] \\ & + \lim_{z_0 \to z} \sum_{i=1}^{m} \sum_{l=l_i}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) F^{(n)}(z_0, s_i - l, u_i), \end{aligned} \tag{25}$$

for $\sum_{j=1}^{m} s_j > n + 1$, where $|z_0| < 1, (k)^{+} = \max(0, k), l_i = (s_i - 1 - n)^{+}$ and $a_{l,i}(\vec{s}, \vec{u})$ in (24) and (25) are given by (22) and (23), and the function $sgn(x)$ defined by

$$sgn(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{26}$$

Proof. 1) Denote

$$f_i(x) = \prod_{j=1, j\ne i}^{m} \frac{1}{(u_j + x)^{s_j}}, \quad i = 1, 2, \ldots, m. \tag{27}$$

It is worth noting that the notation i satisfies $i \in \mathbb{N}$ and $i \le m$ throughout this part.

Using the following rational function decomposition [16],

$$\prod_{j=1}^{m} \frac{1}{(u_j + x)^{s_j}} = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} \frac{a_{l,i}(\vec{s}, \vec{u})}{(u_i + x)^{s_i - l}}, \tag{28}$$

we deduce that

$$\begin{aligned} f_i(x) = & \sum_{l=0}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) (u_i + x)^{l} \\ & + (u_i + x)^{s_i} \sum_{j=1, j\ne i}^{m} \sum_{l=0}^{s_j-1} \frac{a_{l,j}(\vec{s}, \vec{u})}{(u_j + x)^{s_j - l}}. \end{aligned} \tag{29}$$

Calculating l-order derivatives on x for (29) and setting $x = -u_i$, we have

$$a_{l,i}(\vec{s}, \vec{u}) = \frac{1}{l!} \frac{d^l}{dx^l} f_i(-u_i), \quad l = 0, 1, \ldots, s_i - 1. \tag{30}$$

Combining (27) with (30), we get (22).

Next, we calculate $a_{l,i}(\vec{s}, \vec{u}) (l = 1, \ldots, s_i - 1)$. Differentiating (27) with respect to x, we find

$$\frac{d}{dx} f_i(x) = f_i(x) \times \sum_{j=1, j\ne i}^{m} (-s_j) (u_j + x)^{-1}. \tag{31}$$

It follows that

$$\begin{aligned} & \frac{d^l}{dx^l} f_i(x) \\ = & - \sum_{j=1, j\ne i}^{m} s_j \frac{d^{l-1}}{dx^{l-1}} \left[f_i(x) (u_j + x)^{-1} \right] \\ = & \sum_{k=0}^{l-1} (-1)^{l-k} \frac{(l-1)!}{k!} \frac{d^k}{dx^k} f_i(x) \\ & \times \sum_{j=1, j\ne i}^{m} s_j (u_j + x)^{k-l}. \end{aligned} \tag{32}$$

Setting $x = -u_i$ in (32), we can yield (23) with the help of (30).

From (28), (9) and (5), it follows that

$$\begin{aligned} \Phi(z, \vec{s}, \vec{u}) & = \sum_{k=0}^{\infty} z^k \prod_{j=1}^{m} \frac{1}{(u_j + k)^{s_j}} \\ & = \sum_{k=0}^{\infty} z^k \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} \frac{a_{l,i}(\vec{s}, \vec{u})}{(u_i + k)^{s_i - l}} \\ & = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) \sum_{k=0}^{\infty} \frac{z^k}{(u_i + k)^{s_i - l}} \\ & = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} a_{l,i}(\vec{s}, \vec{u}) \Phi(z, s_i - l, u_i). \end{aligned} \tag{33}$$

Therefore, (20) holds for $F(z, \vec{s}, \vec{u}) \equiv \Phi(z, \vec{s}, \vec{u})$ and $|z| < 1$. In the same way, (20) also holds for $F(z, \vec{s}, \vec{u}) \equiv H(z, \vec{s}, \vec{u})$ and $|z| < 1$.

Calculating n-order partial derivatives on z for both sides of (20), we conclude that (21) hold for $|z| < 1$.

2) Attention to $\Phi(1, 1, u_i) = \infty$, so

$$\sum_{i=1}^{m} a_{s_i-1,i}(\vec{s}, \vec{u}) \Phi(1, 1, u_i)$$

does not exist. Setting $x = k$ in (28), we have

$$\begin{aligned} & \sum_{i=1}^{m} \frac{a_{s_i-1,i}(\vec{s}, \vec{u})}{u_i + k} \\ = & \prod_{j=1}^{m} \frac{1}{(u_j + k)^{s_j}} - \sum_{i=1}^{m} \sum_{l=0}^{s_i-2} \frac{a_{l,i}(\vec{s}, \vec{u})}{(u_i + k)^{s_i - l}}, \end{aligned} \tag{34}$$

which implies that

$$\sum_{i=1}^{m} a_{s_i-1,i}(\vec{s},\vec{u})\Phi(z,1,u_i) = \sum_{k=0}^{\infty} z^k \times\left(\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}} - \sum_{i=1}^{m}\sum_{l=0}^{s_i-2}\frac{a_{l,i}(\vec{s},\vec{u})}{(u_i+k)^{s_i-l}}\right). \quad (35)$$

Since the power series $\sum\limits_{k=0}^{\infty} z^k \frac{1}{k^s}(s>1)$ is convergence at $|z|=1$, the following limits

$$\lim_{z_0\to z}\sum_{i=1}^{m} a_{s_i-1,i}(\vec{s},\vec{u})\Phi(z_0,1,u_i) = \lim_{z_0\to z}\sum_{k=0}^{\infty} z_0^k\left(\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}} - \sum_{i=1}^{m}\sum_{l=0}^{s_i-2}\frac{a_{l,i}(\vec{s},\vec{u})}{(u_i+k)^{s_i-l}}\right). \quad (36)$$

exist, where $|z_0|<1, |z|=1, \sum\limits_{i=1}^{m} s_i > 1$ and $s_i - l > 1 (l = 0,1,\ldots,s_i-2)$. It means that (24) holds for $F(z,\vec{s},\vec{u}) \equiv \Phi(z,\vec{s},\vec{u})$ and $|z|=1$.

Similarly, (24) and (25) also hold for $|z| = 1$ and $|z|=1, \sum\limits_{j=1}^{m} s_j > n+1$, respectively. ■

Now, some connections of multiple integrals (7) and (8) with the generalized Lerch transcendent (9) and the generalized Euler sum with parameters (10) are established in the following theorem.

Theorem 2.4 If $\Re u_j > 0, \Re s_j > 0 (j=1,2,\ldots,m)$ and either $|z|<1$ or $|z|=1, \Re(\sum\limits_{j=1}^{m} s_j) > 1$, then

$$\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right) \times\frac{dx_1dx_2\cdots dx_m}{1-zx_1x_2\cdots x_m} = \Phi(z,\vec{s},\vec{u})\prod_{k=1}^{m}\Gamma(s_k) \quad (37)$$

and

$$\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right) \times\ln\frac{1}{1-zx_1x_2\cdots x_m}\frac{dx_1dx_2\cdots dx_m}{1-zx_1x_2\cdots x_m} = H(z,\vec{s},\vec{u})\prod_{k=1}^{m}\Gamma(s_k). \quad (38)$$

Proof. Using power series expansions, it directly follows from the left side of (37) that

$$\begin{aligned}
&\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right)\frac{dx_1\cdots dx_m}{1-zx_1\cdots x_m}\\
=&\sum_{k=0}^{\infty} z^k\times \int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1+k}\ln^{s_j-1}\frac{1}{x_j}\right)dx_1\cdots dx_m\\
=&\sum_{k=0}^{\infty} z^k\times\prod_{j=1}^{m}\int_0^1 x_j^{u_j-1+k}\ln^{s_j-1}\frac{1}{x_j}dx_j.
\end{aligned} \quad (39)$$

With the help of variable substitution and (4), (39) gives

$$\begin{aligned}
&\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right)\frac{dx_1\cdots dx_m}{1-zx_1\cdots x_m}\\
=&\sum_{k=0}^{\infty} z^k\times\prod_{j=1}^{m}\int_0^{\infty} e^{-t_j(u_j+k)}t_j^{s_j-1}dt_j\\
=&\sum_{k=0}^{\infty} z^k\times\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}}\int_0^{\infty} e^{-x_j}x_j^{s_j-1}dx_j\\
=&\Phi(z,\vec{s},\vec{u})\prod_{k=1}^{m}\Gamma(s_k),
\end{aligned} \quad (40)$$

where $\Re u_j > 0, \Re s_j > 0 (j=1,2,\ldots,m)$ and either $|z|<1$ or $|z|=1, \Re(\sum\limits_{j=1}^{m} s_j) > 1$. Therefore, (37) holds.

With the aid of (37), we have

$$\begin{aligned}
&\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j}\ln^{s_j-1}\frac{1}{x_j}\right) \times\frac{dx_1dx_2\cdots dx_m}{(1-Zx_1x_2\cdots x_m)(1-zx_1x_2\cdots x_m)}\\
=&\frac{\Phi(z,\vec{s},\vec{u})-\Phi(Z,\vec{s},\vec{u})}{z-Z}\prod_{k=1}^{m}\Gamma(s_k)
\end{aligned} \quad (41)$$

for $z\neq Z$. Integrating from 0 to z on Z for (41) and using (9), (10), we yield

$$\begin{aligned}
&\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-1}\ln^{s_j-1}\frac{1}{x_j}\right) \times\ln\frac{1}{1-zx_1x_2\cdots x_m}\frac{dx_1dx_2\cdots dx_m}{1-zx_1x_2\cdots x_m}\\
=&\int_0^z\sum_{k=0}^{\infty}\frac{z^k-Z^k}{z-Z}\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}}dZ \times\prod_{k=1}^{m}\Gamma(s_k)\\
=&\sum_{k=0}^{\infty}\int_0^z\sum_{l=1}^{k} z^{k-l}Z^{l-1}dZ\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}} \times\prod_{k=1}^{m}\Gamma(s_k)\\
=&\sum_{k=0}^{\infty}\sum_{l=1}^{k}\frac{1}{l}z^k\prod_{j=1}^{m}\frac{1}{(u_j+k)^{s_j}}\prod_{k=1}^{m}\Gamma(s_k)\\
=&H(z,\vec{s},\vec{u})\prod_{k=1}^{m}\Gamma(s_k),
\end{aligned} \quad (42)$$

which means that (38) holds. ■

Calculating n-order partial derivatives on z for (37), we can obtain the following relation.

Corollary 2.5 Let n, m, s_i be positive integers and $\Re u_i > 0 (i=1,2,\ldots,m)$. Then

$$\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j+n-1}\ln^{s_j-1}\frac{1}{x_j}\right) \times\frac{dx_1\cdots dx_m}{(1-zx_1\cdots x_m)^{n+1}} = \frac{1}{n!}\Phi^{(n)}(z,\vec{s},\vec{u})\prod_{k=1}^{m}\Gamma(s_k) \quad (43)$$

for $|z|<1$ or $|z|=1, \sum\limits_{j=1}^{m} s_j > n+1$, where $\Phi^{(n)}(z,\vec{s},\vec{u})$ are given by (21) or (25).

Integrating from 0 to z on z for (38), the following relation holds.

Corollary 2.6 If $\Re u_j>0, \Re s_j>0 (j=1,2,\ldots,m)$ and either $|z|<1$ or $|z|=1, \Re(\sum\limits_{j=1}^{m} s_j)>1$, then

$$\begin{aligned}&\int_{[0,1]^m}\prod_{j=1}^{m}\left(x_j^{u_j-2}\ln^{s_j-1}\frac{1}{x_j}\right)\\&\times\ln^2\left(1-zx_1\cdots x_m\right)dx_1\cdots dx_m\\=\ &2zH(z,\vec{s}_{m+1},\vec{u}_{m+1})\prod_{k=1}^{m}\Gamma(s_k),\end{aligned}\tag{44}$$

where $\vec{s}_{m+1}=(\vec{s},1)$ and $\vec{u}_{m+1}=(\vec{u},1)$.

Remark 2.7 If $u_i=u_j, i<j (i,j=1,2,\ldots,m)$ in the above-mentioned multiple integrals, we just need replace $(u_i+k)^{-s_i}(u_j+k)^{-s_j}$, m with $(u_i+k)^{-s_i-s_j}$, $m-1$ in $\Phi(z,\vec{s},\vec{u})$ or $H(z,\vec{s},\vec{u})$, respectively.

III. The expressions of the closed forms for multiple integrals

If $u_j, s_j (j=1,2,\ldots,m)$ are integers and $z=1,-1$ or $\frac{1}{2}$, we can obtain the closed forms of multiple integrals (37) and (38), which can be expressed by the special constants and Riemann zeta functions in the following examples by using the Mathematica software.

Example 3.1 Setting $z=1, \vec{u}=(4,3,2,5), \vec{s}=(2,4,5,3)$ and $m=4$ in (37), we have

$$\begin{aligned}&\int_{[0,1]^4}\frac{x_1^3x_2^2x_3x_4^4}{1-x_1x_2x_3x_4}\\&\times\ln\frac{1}{x_1}\ln^3\frac{1}{x_2}\ln^4\frac{1}{x_3}\ln^2\frac{1}{x_4}dx_1\cdots dx_4\\=\ &\frac{4607653}{7776}-\frac{8855\pi^2}{162}-\frac{26\pi^4}{45}\\&+\frac{10}{27}\zeta(3)+\frac{8}{3}\zeta(5).\end{aligned}\tag{45}$$

Example 3.2 Setting $z=1, \vec{u}=(4,4,5,5), \vec{s}=(2,4,5,3)$ and $m=4$ in (37), we have

$$\begin{aligned}&\int_{[0,1]^4}\frac{x_1^3x_2^3x_3^4x_4^4}{1-x_1x_2x_3x_4}\\&\times\ln\frac{1}{x_1}\ln^3\frac{1}{x_2}\ln^4\frac{1}{x_3}\ln^2\frac{1}{x_4}dx_1\cdots dx_4\\=\ &\frac{16\pi^2}{525}\left(1247400+17010\pi^2+220\pi^4+\pi^6\right)\\&+1728\left[22\zeta(3)+8\zeta(5)+\zeta(7)\right]\\&-\frac{737850270985}{1492992}.\end{aligned}\tag{46}$$

Example 3.3 Setting $z=1, \vec{u}=(4,3,2,5,6), \vec{s}=(2,4,5,3,6)$ and $m=5$ in (37), we have

$$\begin{aligned}&\int_{[0,1]^5}\frac{x_1^3x_2^2x_3x_4^4x_5^5}{1-x_1x_2x_3x_4}\ln\frac{1}{x_1}\ln^3\frac{1}{x_2}\ln^4\frac{1}{x_3}\\&\times\ln^2\frac{1}{x_4}\ln^5\frac{1}{x_5}dx_1\cdots dx_5\\=\ &\frac{3085376801}{57600000}-\frac{16251235\pi^2}{2985984}-\frac{5575\pi^4}{124416}\\&+\frac{\pi^6}{9072}+\frac{124775}{41472}\zeta(3)+\frac{55}{72}\zeta(5).\end{aligned}\tag{47}$$

Example 3.4 Setting $z=1, \vec{u}=(4,3,3,6,6), \vec{s}=(2,4,5,3,6)$ and $m=5$ in (37), we have

$$\begin{aligned}&\int_{[0,1]^5}\frac{x_1^3x_2^2x_3^2x_4^5x_5^5}{1-x_1x_2x_3x_4}\ln\frac{1}{x_1}\ln^3\frac{1}{x_2}\\&\times\ln^4\frac{1}{x_3}\ln^2\frac{1}{x_4}\ln^5\frac{1}{x_5}dx_1\cdots dx_5\\=\ &\frac{99329287241977737267}{22958251200000000}\\&-\frac{\pi^2}{74401740}\Big[4002823825+48\pi^2\Big(2472295\\&+93820\pi^2+1728\pi^4\Big)\Big]+\frac{10644565}{59049}\zeta(3)\\&+\frac{528620}{6561}\zeta(5)+\frac{560}{27}\zeta(7)+\frac{320}{243}\zeta(9).\end{aligned}\tag{48}$$

Example 3.5 Setting $z=\frac{1}{2}, \vec{u}=(5,4,2,3), \vec{s}=(2,2,2,2)$ and $m=4$ in (38), we have

$$\begin{aligned}&2\int_{[0,1]^4}\frac{x_1^4x_2^3x_3x_4^2}{2-x_1x_2x_3x_4}\prod_{j=1}^{4}\ln x_j\\&\times\ln\frac{2}{2-x_1x_2x_3x_4}dx_1\cdots dx_4\\=\ &-\frac{275593}{15552}+\frac{83\pi^2}{81}+\frac{7\zeta(3)}{8}\\&+\frac{\ln 2}{864}\Big(10885-48\ln 2(67+21\ln 2)\Big).\end{aligned}\tag{49}$$

Example 3.6 Setting $z=-1, \vec{u}=(5,4,2,3,6), \vec{s}=(2,2,2,2,2)$ and $m=5$ in (38), we have

$$\begin{aligned}&\int_{[0,1]^5}\frac{x_1^4x_2^3x_3x_4^2x_5^5}{1+x_1x_2x_3x_4x_5}\prod_{j=1}^{5}\ln x_j\\&\times\ln\frac{1}{1+x_1x_2x_3x_4x_5}dx_1\cdots dx_5\\=\ &\frac{1}{20736000}\Big[406541-35850\pi^2\\&-122880\ln 2+27000\zeta(3)\Big].\end{aligned}\tag{50}$$

IV. Conclusion

In this paper, we focus on some properties of the generalized Lerch transcendent $\Phi(z, \vec{s}, \vec{u})$ and the generalized Euler sum with parameters $H(z, \vec{s}, \vec{u})$ and their connections with multiple integrals (7) and (8). Moreover, Lemma 2.2 and Theorem 2.3 show that $\Phi^{(n)}(z, \vec{s}, \vec{u})$ and $H^{(n)}(z, \vec{s}, \vec{u})(n \in \mathbb{N}_0)$ can be expressed by $\Phi(z, s, u)$ and $H(z, s, u)$, respectively. It means that multiple integrals in (37), (38), (43) and (44) can be expressed as the linear combination of $\Phi(z, s, u)$ or $H(z, s, u)$.

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