Existence and Nonexistence of Positive Periodic Solutions in Shifts Delta(+/-) for a Nicholson's Blowflies Model on Time Scales

Meng Hu, Lili Wang

Abstract—**This paper is concerned with a Nicholson's blowflies model with time delays on time scales. Based on the theory of calculus on time scales, sufficient conditions for the existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ are obtained by using Krasnoselskii's fixed point theorem and some mathematical methods. This is the first study to investigate the existence and nonexistence of positive periodic solutions for a Nicholson's blowflies model on time scales, especially for the nonexistence result. Finally, two examples are given to illustrate the feasibility and effectiveness of the results.**

Index Terms—**positive periodic solution; Nicholson's blowflies model; shift operator; time scale.**

I. Introduction

In 1980, Gurney et al. [1] proposed a mathematical model

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)}$$

to describe the dynamics of Nicholson's blowflies, where $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $1/a$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. During the past decades, Nicholson's blowflies model and its analogous equations have received special attention by many researchers; see [2-7].

Notice that, in the nature world, blowflies whose growth processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

A time scale is a nonempty arbitrary closed subset of $\mathbb{R}$. The theory of dynamic equations on time scales was firstly introduced by Hilger in his Ph.D. thesis in 1988 [8]. The study of dynamic equations on time scales can combine the continuous and discrete situations; by choosing the time scale to be $\mathbb{R}$, the general results yields a result for ordinary differential equations; and by choosing the time scale to be $\mathbb{Z}$, the same general results yields a result for difference equations. Since there many other time scales than just the set of real numbers or the set of integers, one has more general results. Nicholson's blowflies model on time scales, one may see [9,10].

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M. Hu and L. Wang are with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000 China e-mail: humeng2001@126.com.

Up to now, there have been many results concerned with periodic solutions of dynamic equations on time scales; see, for example, [11,12]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}}$ and $\sqrt{\mathbb{N}}$ which do not satisfy the condition. Adıvar introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t\pm\omega$ for a fixed $\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [13] and then generalized in [14].

Recently, based on some fixed-point theorems in cones, the existence of positive periodic solutions in shifts $\delta_\pm$ for some nonlinear dynamic equations on time scales have been studied; see [15-17]. However, to the best of our knowledge, there are few papers published on the existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ for a Nicholson's blowflies model with time delays on time scales, especially for the nonexistence results.

Motivated by the above, in the present paper, we consider the following system:

$$\begin{aligned} x^{\Delta}(t) &= -a(t)x(t) \\ &\quad + \sum_{i=1}^{m} b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))}, \quad (1) \end{aligned}$$

where $t \in \mathbb{T}, \mathbb{T} \subset \mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0,\infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $a, b_i \in C(\mathbb{T},(0,\infty))$ $(i = 1,2,\cdots,m)$ are Δ-periodic in shifts $\delta_\pm$ with period ω and $-a \in \mathcal{R}^+$; $c_i \in C(\mathbb{T},(0,\infty))$ are periodic in shifts $\delta_\pm$ with period ω for $i = 1,2,\cdots,m$; $\tau_i(i = 1,2,\cdots,m)$ are fixed if $\mathbb{T} = \mathbb{R}$ and $\tau_i \in [P,\infty)_{\mathbb{T}}$ if $\mathbb{T}$ is periodic in shifts $\delta_\pm$ with period P.

For convenience, we introduce the notation

$$f^* = \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} f(t), \ f_* = \inf_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} f(t),$$

where f is a positive and bounded periodic function.

Take the initial condition

$$x(s) = \phi(s), \phi \in C([\delta_-(\tau^*,0),0]_{\mathbb{T}},(0,\infty)), \phi \not\equiv 0, \quad (2)$$

where $\tau^* = \max_{1\le i\le m} \tau_i$.

It is easy to prove that the initial value problem (1) and (2) has a unique non-negative solution $x(t)$ on $[0,\infty)_{\mathbb{T}}$.

The main purpose of this paper is to establish sufficient conditions for the existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ of system (1).

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$.

For the basic theory of calculus on time scales, see [18].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \ominus p = -\frac{p}{1 + \mu p}, p \ominus q = p \oplus (\ominus q).$$

Lemma 1. [18] *Suppose that $p : \mathbb{T} \to \mathbb{R}$ is a regressive function, then*

(*i*) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(*iii*) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(*iv*) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(*v*) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$.

The following definitions, lemmas about the shift operators and the new periodicity concept on time scales which can be found in [16,19].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_\mathbb{T}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 1. [19] *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 2. [19] *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 3. [19] *(Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Lemma 2. [19] *$\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ and $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 3. [16] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

(*i*) $e_p(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_p(t, t_0)$ *for* $t, t_0 \in \mathbb{T}^*$;
(*ii*) $e_p(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ *for* $t, s \in \mathbb{T}^*$.

Lemma 4. [19] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^t f(s)\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\Delta s.$$

Lemma 5. [18] *Suppose that r is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\Delta = r(t)y + f(t), \; y(t_0) = y_0$$

is given by

$$y(t) = e_r(t, t_0)y_0 + \int_{t_0}^t e_r(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Set

$$X = \{x : x \in C^1(\mathbb{T}, \mathbb{R}), x(\delta_+^\omega(t)) = x(t)\}$$

with the norm $\|x\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} |x(t)|$, then X is a Banach space.

By Lemma 1 and Lemma 5, we can obtain the following lemma.

Lemma 6. *The function $x \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1) if and only if x is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_t^{\delta_+^\omega(t)} G(t,s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i, s)) \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s, \quad (3)$$

where

$$G(t,s) = \frac{e_{-a}(t,\sigma(s))}{e_{-a}(t_0, \delta_+^\omega(t_0)) - 1}.$$

Proof: If x is an ω-periodic solution in shifts $\delta_\pm$ of system (1). By Lemma 5, for any $s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}$, we have

$$x(s) = e_{-a}(s,t)x(t) + \int_t^s e_{-a}(s,\sigma(\theta)) \times \sum_{i=1}^m b_i(\theta) x(\delta_-(\tau_i,\theta)) e^{-c_i(\theta)x(\delta_-(\tau_i,\theta))} \Delta\theta.$$

Let $s = \delta_+^\omega(t)$ in the above equality, then

$$x(\delta_+^\omega(t)) = e_{-a}(\delta_+^\omega(t), t)x(t) + \int_t^{\delta_+^\omega(t)} e_{-a}(\delta_+^\omega(t), \sigma(\theta)) \times \sum_{i=1}^m b_i(\theta) x(\delta_-(\tau_i,\theta)) e^{-c_i(\theta)x(\delta_-(\tau_i,\theta))} \Delta\theta.$$

Noticing that $e_{-a}(t, \delta_+^\omega(t)) = e_{-a}(t_0, \delta_+^\omega(t_0))$, $x(\delta_+^\omega(t)) = x(t)$, by Lemma 1, then x satisfies (3).

Let x be an ω-periodic solution in shifts $\delta_\pm$ of (3). By (3) and Lemma 2, we have

$$\begin{aligned} x^\Delta(t) &= -a(t)x(t) \\ &\quad + G(\sigma(t), \delta_+^\omega(t)) \sum_{i=1}^m b_i(\delta_+^\omega(t)) \delta_+^{\Delta\omega}(t) \\ &\quad \times x(\delta_-(\tau_i, \delta_+^\omega(t))) e^{-c_i(\delta_+^\omega(t))x(\delta_-(\tau_i,\delta_+^\omega(t)))} \\ &\quad - G(\sigma(t), t) \sum_{i=1}^m b_i(t) x(\delta_-(\tau_i, t)) \\ &\quad \times e^{-c_i(t)x(\delta_-(\tau_i,t))} \\ &= -a(t)x(t) + \sum_{i=1}^m b_i(t) x(\delta_-(\tau_i,t)) \\ &\quad \times e^{-c_i(t)x(\delta_-(\tau_i,t))}. \end{aligned}$$

So, x is an ω-periodic solution in shifts $\delta_\pm$ of system (1). This completes the proof.

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$0 < \frac{1}{\xi - 1} \le G(t,s) \le \frac{\xi}{\xi - 1}, \ \forall s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}, \quad (4)$$

where $\xi = e_{-a}(t_0, \delta_+^\omega(t_0))$. By Lemma 3, we have

$$G(\delta_+^\omega(t), \delta_+^\omega(s)) = G(t,s), \ \forall t \in \mathbb{T}^*, s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}. \quad (5)$$

Define K, a cone in X, by

$$K = \big\{x \in X : x(t) \ge \frac{1}{\xi}\|x\|, \forall t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\big\} \quad (6)$$

and an operator $H : K \to X$ by

$$(Hx)(t) = \int_t^{\delta_+^\omega(t)} G(t,s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s. \quad (7)$$

In the following, we shall give some lemmas concerning K and H defined by (6) and (7), respectively.

Lemma 7. *$H : K \to K$ is well defined.*

Proof: For any $x \in K$, $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$. In view of (7), by Lemma 4 and (5), we have

$$\begin{aligned} &(Hx)(\delta_+^\omega(t)) \\ &= \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t), s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s \\ &= \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t), \delta_+^\omega(s)) \sum_{i=1}^m b_i(\delta_+^\omega(s)) \delta_+^{\Delta\omega}(s) \\ &\quad \times x(\delta_-(\tau_i, \delta_+^\omega(s))) e^{-c_i(\delta_+^\omega(s))x(\delta_-(\tau_i,\delta_+^\omega(s)))} \Delta s \\ &= \int_t^{\delta_+^\omega(t)} G(t,s) \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s \\ &= (Hx)(t), \end{aligned}$$

that is, $Hx \in X$.

Furthermore, for any $x \in K$, $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned} (Hx)(t) &\ge \frac{1}{\xi - 1} \int_t^{\delta_+^\omega(t)} \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s \\ &= \frac{1}{\xi} \cdot \frac{\xi}{\xi - 1} \int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s \\ &\ge \frac{1}{\xi}\|Hx\|, \end{aligned}$$

that is, $Hx \in K$. This completes the proof.

Lemma 8. *$H : K \to K$ is completely continuous.*

Proof: Clearly, H is continuous on $[t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$. For any $x \in K$, $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned} \|Hx\| &= \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} (Hx)(t) \\ &\le \frac{\xi}{\xi - 1} \int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s) x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))} \Delta s \\ &< \frac{\xi}{\xi - 1} \cdot \frac{B}{c_*} := M_1, \end{aligned} \quad (8)$$

where

$$c_* = \min_{1 \le i \le m} c_{i*}, \ B := \int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s) \Delta s.$$

Furthermore, for $t \in \mathbb{T}$, we have

$$(Hx)^\Delta(t) = -a(t)(Hx)(t) + \sum_{i=1}^m b_i(t) x(\delta_-(\tau_i,t)) \times e^{-c_i(t)x(\delta_-(\tau_i,t))},$$

and

$$\begin{aligned}\|(Hx)^{\Delta}(t)\| &= \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} |-a(t)(Hx)(t) \\ &\quad + \sum_{i=1}^{m} b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))}| \\ &\le a^*M_1 + \frac{1}{c_*}\sum_{i=1}^{m} b_i^*.\end{aligned}$$

To sum up, $\{Hx : x \in K\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, the functional H is completely continuous. This completes the proof.

III. Existence result

In this section, we shall state and prove our main result about the existence of at least one positive periodic solution in shifts $\delta_{\pm}$ of system (1).

Lemma 9. *(Guo-Krasnoselskii [20]) Let X be a Banach space and $K \subset X$ be a cone in X. Suppose that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $H : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that, either*

(1) $\|Hx\| \le \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Hx\| \ge \|x\|, x \in K \cap \partial\Omega_2$; *or*

(2) $\|Hx\| \ge \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Hx\| \le \|x\|, x \in K \cap \partial\Omega_2$.

Then H has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 10. *Let*

$$\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}. \tag{9}$$

Then there are positive constants M_1 and M_2 such that for $x \in K$,

$$M_2 \le \|Hx\| \le M_1. \tag{10}$$

Proof: From (8), for any $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$\|Hx\| \le M_1. \tag{11}$$

From (9), there is $q > 1$ such that

$$\sum_{i=1}^{m} b_i(t) > qa(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}. \tag{12}$$

For any $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$\begin{aligned}(Hx)(t) &= \int_t^{\delta_+^{\omega}(t)} G(t,s)\sum_{i=1}^{m} b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\Delta s \\ &> q\int_{t_0}^{\delta_+^{\omega}(t_0)} \frac{a(s)e_{-a}(t_0,\sigma(s))}{e_{-a}(t_0,\delta_+^{\omega}(t_0))-1} \\ &\quad \times \min_{1\le i\le m} \inf_{s\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \{x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\}\Delta s \\ &= q\int_{t_0}^{\delta_+^{\omega}(t_0)} \frac{1}{e_{-a}(t_0,\delta_+^{\omega}(t_0))-1} \\ &\quad \times \min_{1\le i\le m} \inf_{s\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \{x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\}\Delta[e_{-a}(t_0,s)] \\ &\ge q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\},\end{aligned} \tag{13}$$

where $c^* = \max_{1\le i\le m} c_i^*$.

Comparing (3) with (7), we also have for $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$x(t) > q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\},$$

which implies that

$$x_* > q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\}. \tag{14}$$

In the same way as (8), $x(t) \le M_1$, which implies that

$$x^* \le M_1. \tag{15}$$

If $\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\} = x^*e^{-c^*x^*}$, then

$$(Hx)(t) > qM_1e^{-c^*M_1} := M_{21} > 0. \tag{16}$$

If $\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\} = x_*e^{-c^*x_*}$, from (14), $x_* > qx_*e^{-c^*x_*}$, which implies that

$$x_* > \frac{\ln q}{c^*}.$$

From (13), we obtain

$$(Hx)(t) > q\frac{\ln q}{c^*}e^{-c^*\cdot\frac{\ln q}{c^*}} = \frac{\ln q}{c^*} := M_{22} > 0. \tag{17}$$

Let $M_2 = \min\{M_{21}, M_{22}\}$, then for $x \in K$,

$$\|Hx\| \ge M_2. \tag{18}$$

This completes the proof.

Theorem 1. *Suppose that*

$$\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}.$$

Then system (1) has at least one positive ω-periodic solution in shifts $\delta_{\pm}$.

Proof: Let

$$\Omega_1 = \{x \in X : \|x\| \le M_2\},$$

and

$$\Omega_2 = \{x \in X : \|x\| \le M_1\}.$$

Clearly, Ω_1 and Ω_2 are open bounded subsets in X, and $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. From Lemma 8, $H : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous.

If $x \in K \cap \partial\Omega_2$, which implies that $\|x\| = M_1$, from Lemma 10, $\|Hx\| \le M_1$. Hence $\|Hx\| \le \|x\|$ for $x \in K \cap \partial\Omega_2$.

If $x \in K \cap \partial\Omega_1$, which implies that $\|x\| = M_2$, from Lemma 10, $\|Hx\| \le M_2$. Hence $\|Hx\| \ge \|x\|$ for $x \in K \cap \partial\Omega_1$.

From the cone fixed point theorem (Lemma 9), the operator H has at least one fixed point lying in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e., system (1) has at least one positive ω-periodic solution in shifts $\delta_{\pm}$. This completes the proof.

IV. Nonexistence result

In this section, we shall state and prove our main result about the nonexistence of positive periodic solution in shifts $\delta_\pm$ of system (1).

Lemma 11. *Suppose that*

$$\sum_{i=1}^{m} b_i(t) \leq \frac{1}{2}a(t), t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}. \tag{19}$$

Then every positive solution of system (1) tends to zero as $t \to \infty$.

Proof: Let $x(t)$ be any positive solution of system (1). By Lemma 5, integrating system (1) from t_0 to $t(> t_0)$, we have

$$\begin{aligned} x(t) &= e_{-a}(t,t_0)x(t_0) + \int_{t_0}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} b_i(s) \\ &\quad \times x(\delta_-(\tau_i,s))e^{-c_i(s)x(\delta_-(\tau_i,s))}\Delta s. \end{aligned} \tag{20}$$

From (19),

$$\begin{aligned} x(t) &\leq e_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}\int_{t_0}^{t} a(s)e_{-a}(t,\sigma(s))\Delta s \\ &= e_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}\int_{t_0}^{t} \Delta[e_{-a}(t,s)] \\ &= e_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}[1 - e_{-a}(t,t_0)]. \end{aligned}$$

Let $\beta = \limsup_{t\to\infty} x(t)$, then $0 \leq \beta < \infty$.

Next, we shall prove $\beta = 0$. We have some possible cases to consider.

Case 1. $x^\Delta(t) > 0$ eventually. Choose $t_0 > 0$ such that $x^\Delta(t) > 0$ for $t \geq t_0$. Let $\eta > 0$ be a sufficient large number with $\delta_-(\tau_i,t) > t_0, i = 1,2,\cdots,m$ for $t > t_0 + \eta$. Then $0 < x(\delta_-(\tau_i,t)) < x(t)$ for $t \geq t_0 + \eta$ and $i = 1,2,\cdots,m$. From (1), for $t \geq t_0 + \eta$,

$$\begin{aligned} 0 &< -a(t)x(t) + \sum_{i=1}^{m} b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))} \\ &< \left[\sum_{i=1}^{m} b_i(t) - a(t)\right]x(t) < 0. \end{aligned}$$

This contradiction shows that Case 1 is impossible.

Case 2. $x^\Delta(t) < 0$ eventually. Choose $t_0 > 0$ such that $x^\Delta(t) < 0$ for $t \geq t_0$. Then $\beta < x(\delta_-(\tau_i,t)) < x(\delta_-(\tau_i,t_0))$ for $t \geq t_0 + \eta$ and $i = 1,2,\cdots,m$. From (19) and (20), we have

$$\begin{aligned} x(t) &\leq e_{-a}(t,t_0)x(t_0) + \frac{1}{2}\max_{1\leq i\leq m} x(\delta_-(\tau_i,t_0))e^{-c_*\beta} \\ &\quad \times[1 - e_{-a}(t,t_0)]. \end{aligned} \tag{21}$$

Let $t \to \infty$ in (21), we obtain

$$\beta \leq \frac{1}{2}\max_{1\leq i\leq m} x(\delta_-(\tau_i,t_0))e^{-c_*\beta}. \tag{22}$$

Again let $t_0 \to \infty$ in (22), we have that $\beta \leq \beta(\frac{1}{2}e^{-c_*\beta})$, which implies that $\beta = 0$.

Case 3. $x^\Delta(t)$ is oscillatory. By the definition of oscillatory, then

(i) there is $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$x^\Delta(t_n) = 0 \text{ for } n = 1,2,\cdots, \text{ and}$$

$$\lim_{n\to\infty} x(t_n) = \beta;$$

or

(ii) there is $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$x^\Delta(t_n)x^\Delta(\rho(t_n)) < 0 \text{ for } n = 1,2,\cdots, \text{ and}$$
$$\lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} x(\rho(t_n)) = \beta.$$

In case (i), from (1),

$$\begin{aligned} a(t_n)x(t_n) &= \sum_{i=1}^{m} b_i(t_n)x(\delta_-(\tau_i,t_n))e^{-c_i(t_n)x(\delta_-(\tau_i,t_n))} \\ &\leq x(\delta_-(\tau_l,t_n))e^{-c_*x(\delta_-(\tau_l,t_n))} \\ &\quad \times\sum_{i=1}^{m} b_i(t_n), \end{aligned} \tag{23}$$

where $l = l(n) \in \{1,2,\cdots,m\}$ such that

$$\begin{aligned} &x(\delta_-(\tau_l,t_n))e^{-c_*x(\delta_-(\tau_l,t_n))} \\ &= \max_{1\leq i\leq m} x(\delta_-(\tau_i,t_n))e^{-c_i(t_n)x(\delta_-(\tau_i,t_n))}. \end{aligned}$$

From (19) and (23), we have

$$2x(t_n)e^{c_*x(\delta_-(\tau_l,t_n))} \leq x(\delta_-(\tau_l,t_n)). \tag{24}$$

Set $\alpha = \limsup_{n\to\infty} x(\delta_-(\tau_l,t_n))$, then $\alpha \leq \beta$. Finding the superior limit of both sides of (24), we obtain

$$\beta(2e^{c_*\alpha}) \leq \alpha,$$

then,

$$\beta(2e^{c_*\alpha}) \leq \alpha \leq \beta,$$

which implies that $\beta = \alpha = 0$.

In case (ii), from (1),

$$\begin{aligned} &a(t_n)a(\rho(t_n))x(t_n)x(\rho(t_n)) \\ &+\sum_{i=1}^{m} b_i(t_n)x(\delta_-(\tau_i,t_n))e^{-c_i(t_n)x(\delta_-(\tau_i,t_n))} \\ &\times\sum_{i=1}^{m} b_i(\rho(t_n))x(\delta_-(\tau_i,\rho(t_n))) \\ &\times e^{-c_i(\rho(t_n))x(\delta_-(\tau_i,\rho(t_n)))} \\ <\ & a(t_n)x(t_n)\sum_{i=1}^{m} b_i(\rho(t_n))x(\delta_-(\tau_i,\rho(t_n))) \\ &\times e^{-c_i(\rho(t_n))x(\delta_-(\tau_i,\rho(t_n)))} \\ &+a(\rho(t_n))x(\rho(t_n))\sum_{i=1}^{m} b_i(t_n)x(\delta_-(\tau_i,t_n)) \\ &\times e^{-c_i(t_n)x(\delta_-(\tau_i,t_n))} \\ \leq\ & [a(t_n)x(t_n)\sum_{i=1}^{m} b_i(\rho(t_n)) + a(\rho(t_n))x(\rho(t_n)) \\ &\times\sum_{i=1}^{m} b_i(t_n)]x(\delta_-(\tau_l,\hat{t}_n))e^{-c_*x(\delta_-(\tau_l,\hat{t}_n))}, \end{aligned} \tag{25}$$

where $l = l(n) \in \{1,2,\cdots,m\}$, $\hat{t}_n = \{t_n, \rho(t_n)\}$, such that

$$\begin{aligned} &x(\delta_-(\tau_l,\hat{t}_n))e^{-c_*x(\delta_-(\tau_l,\hat{t}_n))} \\ &= \max_{1\leq i\leq m}\{x(\delta_-(\tau_i,t_n))e^{-c_i(t_n)x(\delta_-(\tau_i,t_n))}, \\ &\quad x(\delta_-(\tau_i,\rho(t_n)))e^{-c_i(\rho(t_n))x(\delta_-(\tau_i,\rho(t_n)))}\}. \end{aligned}$$

From (19) and (25), we have

$$\begin{aligned}&2x(t_n)x(\rho(t_n))e^{c_*x(\delta_-(\tau_l,\hat{t}_n))}\\ \le\ &[x(t_n)+x(\rho(t_n))]x(\delta_-(\tau_l,\hat{t}_n)).\end{aligned} \tag{26}$$

Set $\alpha = \limsup_{n\to\infty} x(\delta_-(\tau_l, t_n))$, then $\alpha \le \beta$. Finding the superior limit of both sides of (26), we obtain

$$\beta e^{c_*\alpha} \le \alpha,$$

then,

$$\beta e^{c_*\alpha} \le \alpha \le \beta,$$

which implies that $\beta = \alpha = 0$. This completes the proof.

From Lemma 11, we can get the following Theorem.

Theorem 2. *Suppose that the condition (19) holds. Then system (1) has no positive ω-periodic solution in shifts $\delta_\pm$.*

V. Numerical examples

Example 1. Consider system (1) with

$$\begin{aligned}&a(t) = a_0 + 0.05|\sin 2t + \cos 3t|,\\ &b_1(t) = e^{-1}(0.7 + 0.02|\sin t|),\\ &b_2(t) = e^{-1}(0.4 + 0.03|\cos t|),\\ &c_1(t) = c_2(t) = 0.25 + 0.025|\sin 3t + \cos 2t|.\end{aligned}$$

Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then $\omega = \pi$ and $\delta_+^\omega(t) = t + \pi$. It is easy to verify $a(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ satisfy

$$\begin{aligned}&a(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = a(t), b_i(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b_i(t),\\ &c_i(\delta_+^\omega(t)) = c_i(t),\ \forall t \in \mathbb{T}^*,\ i = 1, 2,\end{aligned}$$

and $-a \in \mathcal{R}^+$.

Case I. If $a_0 = 0.2$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) \ge 1.1e^{-1} = 0.4047 > a(t), t \in \mathbb{R}.$$

According to Theorem 1, when $\mathbb{T} = \mathbb{R}$, system (1) exists at least one positive π-periodic solution in shifts $\delta_\pm$.

Case II. If $a_0 = 0.85$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) \le 1.15e^{-1} = 0.4231 < \frac{1}{2}a(t), t \in \mathbb{R}.$$

According to Theorem 2, when $\mathbb{T} = \mathbb{R}$, system (1) has no positive periodic solution in shifts $\delta_\pm$.

Example 2. Consider system (1) with

$$\begin{aligned}&a(t) = \frac{1}{a_0 t}, b_1(t) = \frac{1}{2t},\ b_2(t) = \frac{1}{3t},\\ &c_1(t) = c_2(t) = 0.25.\end{aligned}$$

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, then $\omega = 4$ and $\delta_+^\omega(t) = 4t$. It is easy to verify $a(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ satisfy

$$\begin{aligned}&a(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = a(t),\ b_i(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b_i(t),\\ &c_i(\delta_+^\omega(t)) = c_i(t),\ \forall t \in \mathbb{T}^*,\ i = 1, 2,\end{aligned}$$

and $-a \in \mathcal{R}^+$.

Case I. If $a_0 = 6$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} > a(t), t \in 2^{\mathbb{N}_0}.$$

According to Theorem 1, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (1) exists at least one positive 4-periodic solution in shifts $\delta_\pm$.

Case II. If $a_0 = \frac{1}{2}$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} < \frac{1}{2}a(t), t \in 2^{\mathbb{N}_0}.$$

According to Theorem 2, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (1) has no positive periodic solution in shifts $\delta_\pm$.

VI. Conclusion

Two problems for a Nicholson's blowflies model with time delays on time scales have been studied, namely, existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ on time scales. It is important to notice that the methods used in this paper can be extended to other types of biological models [21-23]. Future work will include biological dynamic systems modeling and analysis on time scales.

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