Convergence of Successive Approximations for Fuzzy Differential Equations in The Quotient Space of Fuzzy Numbers

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Abstract—**In this paper, we study the fuzzy differential equations in the quotient space of fuzzy numbers. We deal with the convergence of successive approximations of the initial value problem of the fuzzy differential equations under the general uniqueness assumption of Perron type utilizing the comparison functions that is rather instructive. We also discuss the approximate solutions and the error estimates between the solutions and approximate solutions.**

Index Terms—**fuzzy number, quotient space, fuzzy differential equations, convergence of successive approximations.**

I. Introduction

The fuzzy differential equation is one of the important part of the fuzzy analysis theory. The Hukuhara differentiability (H-derivative) of fuzzy valued mappings were initially studied by Puri and Ralescu [21]. Subsequently, using H-derivatives, the fuzzy differential equation and the initial value problem were regularly treated in [3], [8], [9], [10], [19], [26], [27]. In particular, Wu and Song [29], [30] and Song, Wu and Xue [28] established the relationship between a solution and its approximate solutions to fuzzy differential equations. In [20], Park and Han obtained the global existence and uniqueness of fuzzy solution of fuzzy differential equation using the the properties of Hasegawa's function and successive approximation.

However in many applications the Hukuhara difference appears to have several limitations and to be very restrictive [1], [2], [8], [22]. In [23], Qiu et al. showed that the method of finding the inverse operation of fuzzy numbers in the sense of Mareš [16], [17] is very intuitive. After that, Qiu et al. [25] investigated that the differentiability and integrability properties of such functions and given an existence and uniqueness theorem for a solution to a fuzzy differential equation in the quotient space of fuzzy numbers. In this paper, we will study the convergence of successive approximations for fuzzy differential equations in the quotient space of fuzzy numbers.

II. Preliminaries

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0,1]$. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$, its α-level set. We define the set $[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha\in(0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$ [4]. We denote the class of fuzzy numbers by $\mathscr{F}$.

For any $\widetilde{x}, \widetilde{y} \in \mathscr{F}$ and $a \in \mathbb{R}$, owing to Zadeh's extension principle [32], addition and scalar multiplication are defined for any $x \in \mathbb{R}$ by

$$\mu_{\widetilde{x}+\widetilde{y}}(x) = \sup_{x_1.x_2:x_1+x_2=x} \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{y}}(x_2)\}$$

and

$$\mu_{a\times\widetilde{x}}(x) = \mu_{a\widetilde{x}}(x) = \begin{cases} \mu\left(\dfrac{x}{a}\right), & if\ a \neq 0, \\ \widetilde{0}, & if\ a = 0. \end{cases}$$

For any $\widetilde{x} \in \mathscr{F}$, we define the fuzzy number $-\widetilde{x} \in \mathscr{F}$ by $-\widetilde{x} = (-1) \times \widetilde{x}$, i.e., $\mu_{-\widetilde{x}}(x) = \mu_{\widetilde{x}}(-x)$, for all $x \in \mathbb{R}$. We say that a fuzzy number $\widetilde{s} \in \mathscr{F}$ is symmetric [16], if

$$\mu_{\widetilde{s}}(x) = \mu_{\widetilde{s}}(-x),$$

for all $x \in \mathbb{R}$, i.e., $\widetilde{s} = -\widetilde{s}$. The set of all symmetric fuzzy numbers will be denoted by $\mathscr{S}$.

Definition 2.1: [5] Let $\widetilde{x}, \widetilde{y} \in \mathscr{F}$. We say that $\widetilde{x}$ is equivalent to $\widetilde{y}$ and write $\widetilde{x} \sim \widetilde{y}$ if and only if there exist symmetric fuzzy numbers $\widetilde{s_1}, \widetilde{s_2} \in \mathscr{S}$ such that

$$\widetilde{x} + \widetilde{s_1} = \widetilde{y} + \widetilde{s_2}.$$

The equivalence relation defined above is reflexive, symmetric and transitive [16]. Let $\langle\widetilde{x}\rangle$ denote the equivalence class containing the element $\widetilde{x}$ and denote the set of equivalence classes by $\mathscr{F}/\mathscr{S}$.

Definition 2.2: [12] Let $f : [a,b] \to \mathbb{R}$. f is said be of bounded variation if there exists a $C > 0$ such that

$$\sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leq C$$

for every partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ on $[a,b]$. The set of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

Definition 2.3: [12] Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation. The total variation of f on $[a,b]$ is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|,$$

where p represents all partitions of $[a,b]$.

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Definition 2.4: [23] For a fuzzy number $\widetilde{x}$, we define a function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\widetilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$, i.e.,

$$\widetilde{x}_M(\alpha) = \frac{\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)}{2}.$$

Then the function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\widetilde{x}$.

Lemma 2.1: [23] For any $\tilde{x} \in \mathscr{F}$, the midpoint function $\tilde{x}_M$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore it is a function of bounded variation on $[0,1]$.

Definition 2.5: [17] Let $\widetilde{x} \in \mathscr{F}$ and let $\widehat{x}$ be a fuzzy number such that $\widetilde{x} = \widehat{x} + \widetilde{s}$ for some $\widetilde{s} \in \mathscr{S}$, if $\widehat{x} = \widetilde{y} + \widetilde{s}_1$ for some $\widetilde{y} \in \mathscr{F}$ and $\widetilde{s}_1 \in \mathscr{S}$, then $\widetilde{s}_1 = \widetilde{0}$. Then the fuzzy number $\widehat{x}$ will be called the Mareš core of the fuzzy number $\widetilde{x}$.

Definition 2.6: [25] For any $\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S}$, we define a midpoint function $M_{\langle\widetilde{x}\rangle} : [0,1] \to \mathbb{R}$ by

$$M_{\langle\widetilde{x}\rangle}(\alpha) = \widehat{x}_M(\alpha)$$

for all $\alpha \in [0,1]$, where $\widehat{x}$ is Mareš core of $\langle\widetilde{x}\rangle$.

Definition 2.7: [25] For any $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$, we define $\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle$ by

$$\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle = \langle\widetilde{x} + \widetilde{y}\rangle.$$

Remark 2.1: The addition operation defined by Definition 2.7 is a group operation over the set of equivalence classes $\mathscr{F}/\mathscr{S}$ up to the equivalence relation in Definition 2.1. It means that

$\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle = \langle\widetilde{y}\rangle + \langle\widetilde{x}\rangle$,

$(\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle) + \langle\widetilde{z}\rangle = \langle\widetilde{x}\rangle + (\langle\widetilde{y}\rangle + \langle\widetilde{z}\rangle)$,

$\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle = \langle\widetilde{x}\rangle$ if and only if $\langle\widetilde{y}\rangle = \langle\widetilde{0}\rangle$,

$\langle\widetilde{x}\rangle + \langle\widetilde{y}\rangle = \langle\widetilde{0}\rangle$ if and only if $\langle\widetilde{y}\rangle = \langle-\widetilde{x}\rangle = -\langle\widetilde{x}\rangle$,

for any $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle, \langle\widetilde{z}\rangle \in \mathscr{F}/\mathscr{S}$. For the details of the discussion, please see [6], [7].

Definition 2.8: [25] Let $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle, \langle\widetilde{z}\rangle \in \mathscr{F}/\mathscr{S}$. If

$$M_{\langle\widetilde{z}\rangle}(\alpha) = M_{\langle\widetilde{x}\rangle}(\alpha) \cdot M_{\langle\widetilde{y}\rangle}(\alpha)$$

for all $\alpha \in [0,1]$, then we called $\langle\widetilde{z}\rangle$ is the product of $\langle\widetilde{x}\rangle$ and $\langle\widetilde{y}\rangle$, i.e., $\langle\widetilde{z}\rangle = \langle\widetilde{x}\rangle \cdot \langle\widetilde{y}\rangle$.

Definition 2.9: [25] For any $\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S}$ and $\lambda \in \mathbb{R}$, we define $\lambda \cdot \langle\widetilde{x}\rangle = \lambda\langle\widetilde{x}\rangle$ by

$$\lambda\langle\widetilde{x}\rangle = \langle\lambda\widetilde{x}\rangle.$$

It is obvious that $M_{\lambda\langle\widetilde{x}\rangle}(\alpha) = M_{\langle\lambda\widetilde{x}\rangle}(\alpha) = \lambda M_{\langle\widetilde{x}\rangle}(\alpha)$, for all $\alpha \in [0,1]$ and $\lambda\widehat{x}$ is the Mareš core of $\lambda\langle\widetilde{x}\rangle$ if $\widehat{x}$ is the Mareš core of $\langle\widetilde{x}\rangle$.

Definition 2.10: [25] Define $d_{\sup} : \mathscr{F}/\mathscr{S} \times \mathscr{F}/\mathscr{S} \to \mathbb{R}^+ \cup \{0\}$ by

$$d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle) = \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha) - M_{\langle\widetilde{y}\rangle}(\alpha)\right|,$$

for all $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$.

$(\mathscr{F}/\mathscr{S}, d_{\sup})$ is a metric space [23].

III. Convergence of Successive Approximations

Definition 3.1: [25] Define $d^+ : C[J,\mathbb{R}] \to \mathbb{R}$ by

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{1}{h}(m(t+h) - m(t)),$$

for all $m(t) \in C[J,\mathbb{R}]$, where $J = [t_0, t_0 + a]$ and $a > 0$.

Definition 3.2: [25] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is differentiable at $t \in J$ if there exists an $F'(t) \in \mathscr{F}/\mathscr{S}$ such that

$$\lim_{h\to 0} d_{\sup}\left(\frac{F(t+h) - F(t)}{h}, F'(t)\right) = 0.$$

If $t = t_0$ (or $t = t_0 + a$), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Definition 3.3: [25] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is measurable if F is measurable with respect to $d_{\sup}$.

A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is called integrably bounded if there exists an integrable function $h : J \to \mathbb{R}^+ \cup \{0\}$ such that $\left|M_{F(t)}(\alpha)\right| \le h(t)$ for all $t \in J$ and $\alpha \in [0,1]$; a mapping $F : J \to \mathscr{F}/\mathscr{S}$ is said to be of uniformly bounded variation with respect to $\alpha \in [0,1]$ (for short, of uniformly bounded variation) if there exists a constant $K > 0$ such that

$$V_0^1\left(M_{F(t)}\right) \le K,$$

for each $t \in J$ [25].

Definition 3.4: [25] Let $F : J \to \mathscr{F}/\mathscr{S}$ be measurable. The integral of F over J, denoted $\int_J F(t)dt$ or $\int_{t_0}^{t_0+a} F(t)dt$, is a mapping $M_{\int_J F(t)dt} : [0,1] \to \mathbb{R}$, which is defined by the equation

$$M_{\int_J F(t)dt}(\alpha) = \int_J M_{F(t)}(\alpha)dt$$

for all $\alpha \in [0,1]$. The mapping $F : J \to \mathscr{F}/\mathscr{S}$ is said to be integrable over J if there exists an $\langle\widetilde{x}_0\rangle \in \mathscr{F}/\mathscr{S}$ such that $M_{\int_J F(t)dt} = M_{\langle\widetilde{x}_0\rangle}$. In this case, we denote the integral by

$$\int_J F(t)dt = \langle\widetilde{x}_0\rangle.$$

Lemma 3.1: [25] Let $F : J \to \mathscr{F}/\mathscr{S}$ be continuous with respect to $d_{\sup}$ and of uniformly bounded variation and $G(t) = \int_{t_0}^t F(s)ds$. Then for $t_0 \le t_1 \le t_2 \le t_0 + a$ we have

$$d_{\sup}(G(t_1), G(t_2)) \le (t_2 - t_1) \sup_{t\in[t_1,t_2]} d_{\sup}\left(F(t), \langle\widetilde{0}\rangle\right).$$

Assume that $f : J \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation. We consider the initial value problem for the fuzzy differential equation

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1}$$

Lemma 3.2: [25] A mapping $x : J \to \mathscr{F}/\mathscr{S}$ is a solution to the initial value problem (1) if and only if it is continuous, of uniformly bounded variation and satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds, \quad t \in J.$$

We shall prove an existence and uniqueness result under an assumption more general than the Lipschitz-type condition considered in Section 6 in [25] by the method of successive approximations.

Theorem 3.1: Assume that

(i) $f : R_0 \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation, where $R_0 = J \times B(x_0, b)(b > 0)$, $B(x_0, b) = \{\langle\widetilde{x}\rangle \in \mathscr{F}/\mathscr{S} : d_{\sup}(\langle\widetilde{x}\rangle, x_0) \le b\}$ and $d_{\sup}(f(t, \langle\widetilde{x}\rangle), \langle\widetilde{0}\rangle) \le M_0$ on R_0;

(ii) $g \in C_1[J \times [0, 2b], \mathbb{R}_+]$, $0 \le g(t, \varphi) \le M_1$ on $J \times C_0[0, 2b]$, $g(t, 0) = 0$, $g(t, \varphi)$ is nondecreasing with respect to φ for each $t \in J$ and $\varphi(t) \equiv 0$ is the unique solution of the scalar differential equation

$$\frac{d\varphi}{dt} = g(t, \varphi), \ \varphi(t_0) = \varphi_0 \ge 0, \tag{2}$$

on J;

(iii) $d_{\sup}(f(t, \langle\widetilde{x}\rangle), f(t, \langle\widetilde{y}\rangle)) \le g(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle))$ on R_0.

Then the successive approximations defined by

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \quad n = 0, 1, 2, \cdots,$$

exist on $[t_0, t_0 + \eta]$, where $\eta = \min\left\{a, \dfrac{b}{M}\right\}$, $M = max\{M_0, M_1\}$ as continuous and of uniformly bounded variation functions and converge uniformly to the unique solution $x(t)$ of (1) on $[t_0, t_0 + \eta]$.

Proof. We shall define the successive approximations of the scalar differential equation (2) as follows

$$\varphi_0(t) = M(t - t_0), \ \cdots, \ \varphi_{n+1}(t) = \int_{t_0}^{t} g(s, \varphi_n(s))ds, \ \cdots,$$

for each $t \in [t_0, t_0 + \eta]$. Since $0 \le g(t, \varphi) \le M_1$ on $J \times C_0[0, 2b]$, we obtain

$$|\varphi_n(t_2) - \varphi_n(t_1)| \le \int_{t_1}^{t_2} |g(s, \varphi_{n-1}(s))| \, ds \le M_1(t_2 - t_1),$$

$n = 1, 2, \cdots$. Thus, for any $\varepsilon > 0$, there exists a $\delta = \dfrac{\varepsilon}{M_1} > 0$ such that if $|t_1 - t_2| < \delta$, then

$$|\varphi_n(t_2) - \varphi_n(t_1)| \le M_1 |t_1 - t_2| = \varepsilon, \quad n = 1, 2, \cdots.$$

Hence, $\varphi_n(t)$ is continuous with respect to $t \in [t_0, t_0 + \eta]$, $n = 1, 2, \cdots$, which implies that the family of functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ is equicontinuous on $[t_0, t_0+\eta]$. Since $[t_0, t_0+\eta]$ is a closed and bounded interval and by the Arzelà-Ascoli theorem, we get that the family of functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ has a uniformly convergent subsequence $\{\varphi_{n_k}(t)\}_{k=1}^{\infty}$. We see that

$$\varphi_1(t) = \int_{t_0}^{t} g(s, \varphi_0(s))ds \le M_1(t - t_0) \le M(t - t_0) = \varphi_0(t).$$

Suppose that $\varphi_n(t) \le \varphi_{n-1}(t)$, for each $t \in [t_0, t_0 + \eta]$ and some given n. Using the monotonicity of $g(t, \varphi)$ with respect to φ, we get

$$\varphi_{n+1}(t) = \int_{t_0}^{t} g(s, \varphi_n(s))ds \le \int_{t_0}^{t} g(s, \varphi_{n-1}(s))ds = \varphi_n(t),$$

for each $t \in [t_0, t_0+\eta]$. Thus, by the mathematical induction, we get

$$\varphi_0(t) \ge \varphi_2(t) \ge \cdots \ge \varphi_n(t) \ge \cdots,$$

for each $t \in [t_0, t_0 + \eta]$. Hence, the family of functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ that converges uniformly, that is, there exists $\varphi(t)$ such that $\lim\limits_{n\to\infty} \varphi_n(t) = \varphi(t)$ uniformly on $[t_0, t_0+\eta]$. It is obvious that $\varphi(t)$ satisfies the scalar differential equation (2). Thus, by condition (2), we have $\varphi(t) \equiv 0$, for each $t \in [t_0, t_0 + \eta]$.

Since the successive approximations defined by

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \ n = 0, 1, 2, \cdots,$$

for each $t \in [t_0, t_0 + \eta]$, we have

$$\begin{aligned} d_{\sup}(x_1(t), x_0) &= d_{\sup}\left(\int_{t_0}^{t} f(s, x_0)ds, \langle\widetilde{0}\rangle\right) \\ &\le \int_{t_0}^{t} d_{\sup}(f(s, x_0), \langle\widetilde{0}\rangle)ds \le M_0(t - t_0) \\ &\le M(t - t_0) = \varphi_0(t) \\ &\le M\eta \le b. \end{aligned}$$

Suppose that

$$d_{\sup}(x_n(t), x_{n-1}(t)) \le \varphi_{n-1}(t) \ \ and \ \ d_{\sup}(x_n(t), x_0) \le b$$

for each $t \in [t_0, t_0+\eta]$ and some given n. Using the condition (3) and the monotonicity of $g(t, \varphi)$ with respect to φ for each $t \in [t_0, t_0 + \eta]$, we get

$$\begin{aligned} & d_{\sup}(x_{n+1}(t), x_n(t)) \\ =\ & d_{\sup}\left(\int_{t_0}^{t} f(s, x_n(s))ds, \int_{t_0}^{t} f(s, x_{n-1}(s))ds\right) \\ \le\ & \int_{t_0}^{t} d_{\sup}(f(s, x_n(s)), f(s, x_{n-1}(s)))ds \\ \le\ & \int_{t_0}^{t} g(s, d_{\sup}(x_n(s), x_{n-1}(s)))ds \\ \le\ & \int_{t_0}^{t} g(s, \varphi_{n-1}(s))ds = \varphi_n(t) \end{aligned}$$

and

$$\begin{aligned} & d_{\sup}(x_{n+1}(t), x_0) \\ =\ & d_{\sup}\left(x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, x_0\right) \\ =\ & d_{\sup}\left(\int_{t_0}^{t} f(s, x_n(s))ds, \langle\widetilde{0}\rangle\right) \\ \le\ & \int_{t_0}^{t} d_{\sup}(f(s, x_n(s)), \langle\widetilde{0}\rangle)ds \\ \le\ & M_0(t - t_0) \le M_0\eta \le b, \end{aligned}$$

for each $t \in [t_0, t_0+\eta]$. Thus, by the mathematical induction, we get

$$d_{\sup}(x_{n+1}(t), x_n(t)) \le \varphi_n(t)$$

and

$$d_{\sup}(x_{n+1}(t), x_0) \le b, \quad n = 0, 1, 2, \cdots,$$

for each $t \in [t_0, t_0 + \eta]$, Hence, we obtain $\{x_n(t)\}_{n=1}^{\infty} \subseteq B(x_0, b)$ for each $t \in [t_0, t_0 + \eta]$.

For any positive integer n, let $v_n(t) = d_{\sup}(x_{n+1}(t), x_n(t))$ for each $t \in [t_0, t_0 + \eta]$. Then $v_n(t_0) = d_{\sup}(x_{n+1}(t_0), x_n(t_0)) \le \varphi_n(t_0) = 0$, and for any fixed $t \in [t_0, t_0 + \eta]$ and $h \ne 0$ with $t + h \in [t_0, t_0 + \eta]$, we have

$$\begin{aligned} & v_n(t + h) - v_n(t) \\ =\ & d_{\sup}(x_{n+1}(t + h), x_n(t + h)) - d_{\sup}(x_{n+1}(t), x_n(t)). \end{aligned}$$

Since

$$\begin{aligned} & d_{\sup}(x_{n+1}(t+h), x_n(t+h)) \\ \leq\ & d_{\sup}(x_{n+1}(t+h), x_{n+1}(t) + hf(t, x_n(t))) \\ +\ & d_{\sup}(x_{n+1}(t) + hf(t, x_n(t)), x_n(t+h)), \end{aligned}$$

and

$$\begin{aligned} & d_{\sup}(x_{n+1}(t) + hf(t, x_n(t)), x_n(t+h)) \\ \leq\ & d_{\sup}(x_n(t+h), x_n(t) + hf(t, x_{n-1}(t))) \\ +\ & d_{\sup}(x_{n+1}(t) + hf(t, x_n(t)), x_n(t) + hf(t, x_{n-1}(t)) \end{aligned}$$

we get

$$\begin{aligned} & \frac{v_n(t+h) - v_n(t)}{h} \\ \leq\ & \frac{1}{h} d_{\sup}(x_{n+1}(t+h), x_{n+1}(t) + hf(t, x_n(t))) \\ +\ & \frac{1}{h} d_{\sup}(x_n(t+h), x_n(t) + hf(t, x_n(t))) \\ +\ & \frac{|h|}{h} d_{\sup}(f(t, x_n(t)), f(t, x_{n-1}(t))), \end{aligned}$$

for each $t \in [t_0, t_0 + \eta]$. Thus, by the condition (3) and Definition 3.1, we have

$$\begin{aligned} d^+ v_n(t) &= \overline{\lim_{h \to 0^+}} \frac{1}{h}(v_n(t+h) - v_n(t)) \\ &\leq g(t, d_{\sup}(x_n(t), x_{n-1}(t))). \end{aligned}$$

By the monotonicity of $g(t, \varphi)$ with respect to φ for each $t \in [t_0, t_0 + \eta]$, we get

$$d^+ v_n(t) \leq g(t, d_{\sup}(x_n(t), x_{n-1}(t))) \leq g(t, \varphi_{n-1}(t)).$$

Let $n \leq m$ and $w_n(t) = d_{\sup}(x_n(t), x_m(t))$ for each $t \in [t_0, t_0 + \eta]$. Then $w_n(t_0) = d_{\sup}(x_n(t_0), x_m(t_0)) \leq \sum_{i=n}^{m-1} \varphi_i(t_0) = 0 = \varphi_n(t_0)$ and by the similar proof, we obtain

$$d^+ w_n(t) \leq d_{\sup}(f(t, x_{n-1}(t)), f(t, x_{m-1}(t))),\ t \in [t_0, t_0+\eta].$$

By the monotonicity of the sequence $\{\varphi_n(t)\}_{n=1}^{\infty}$ and $g(t, \varphi)$ with respect to φ for each $t \in [t_0, t_0 + \eta]$, we have

$$\begin{aligned} & d_{\sup}(f(t, x_{n-1}(t)), f(t, x_{m-1}(t))) \\ \leq\ & g(t, d_{\sup}(x_{n-1}(t), x_n(t))) + g(t, d_{\sup}(x_n(t), x_m(t))) \\ +\ & g(t, d_{\sup}(x_m(t), x_{m-1}(t))) \\ \leq\ & 2g(t, \varphi_{n-1}(t)) + g(t, d_{\sup}(x_n(t), x_m(t))). \end{aligned}$$

Then we obtain

$$d^+ w_n(t) \leq 2g(t, \varphi_{n-1}(t)) + g(t, d_{\sup}(x_n(t), x_m(t))).$$

By Theorem 1.4.1 in [13], we have

$$w_n(t) \leq r_n(t), \quad t \in [t_0, t_0 + \eta],$$

where $r_n(t)$ is the maximal solution of

$$\frac{dr_n(t)}{dt} = g(t, r_n(t)) + 2g(t, \varphi_{n-1}(t)), \quad r_n(t_0) = 0.$$

Since as $n \to \infty$, $2g(t, \varphi_{n-1}(t)) \to 0$ uniformly on $[t_0, t_0 + \eta]$ and by Lemma 1.3.1 in [13], we have $r_n \to 0$ uniformly on $[t_0, t_0 + \eta]$. By the definition of $w_n(t)$, there exists $x(t)$ such that $x_n(t)$ converges uniformly to $x(t)$, that is,

$$\lim_{n \to +\infty} \sup_{t \in [t_0, t_0+\eta]} d_{\sup}(x_n(t), x(t)) = 0.$$

We know that

$$d_{\sup}(x(t), x_0) \leq d_{\sup}(x_n(t), x(t)) + b.$$

If as $n \to \infty$, then $d_{\sup}(x(t), x_0) \leq b$, which implies that $x(t) \in B(x_0, b)$ for each $t \in [t_0, t_0 + \eta]$. Since $d_{\sup}(f(t, x_n(t)), \langle \widetilde{0} \rangle) \leq M_0$ and

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s)) ds,$$

on $[t_0, t_0 + \eta]$, we get $d_{\sup}(f(t, x(t)), \langle \widetilde{0} \rangle) \leq M_0$ and

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds,$$

on $[t_0, t_0 + \eta]$. Hence, for any $\alpha \in [0, 1]$, we have

$$M_{x(t)}(\alpha) = M_{x_0}(\alpha) + \int_{t_0}^{t} M_{f(s, x(s))}(\alpha) ds,$$

for each $t \in [t_0, t_0 + \eta]$. We get that

$$V_0^1 \left(M_{x(t)} \right) \leq V_0^1 \left(M_{x_0} \right) + \int_{t_0}^{t} V_0^1 \left(M_{f(s, x(s))} \right) ds,$$

for each $t \in [t_0, t_0 + \eta]$. Since f is of uniformly bounded variation, there exists a constant $K_1 > 0$ such that

$$V_0^1 \left(M_{f(t, \langle \widetilde{x} \rangle)} \right) \leq K_1$$

for each $t \in J$ and $\langle \widetilde{x} \rangle \in B(x_0, b)$. Thus we have that there exists K_2 such that $V_0^1 \left(M_{x_0} \right) \leq K_2$. Let $K = \eta K_1 + K_2$. Then

$$V_0^1 \left(M_{x(t)} \right) \leq (t - t_0) K_1 + K_2 \leq \eta K_1 + K_2 = K,$$

for each $t \in [t_0, t_0 + \eta]$. Thus, $x(t)$ is of uniformly bounded variation. For any $t_1, t_2 \in [t_0, t_0 + \eta], t_1 \leq t_2$, by Lemma 3.1, we have

$$\begin{aligned} d_{\sup}(x(t_1), x(t_2)) &\leq (t_2 - t_1) \sup_{t \in [t_1, t_2]} \sup_{\alpha \in [0,1]} \left| M_{f(t, x(t))}(\alpha) \right| \\ &= (t_2 - t_1) \sup_{t \in [t_1, t_2]} d_{\sup}(f(t, x(t)), \langle \widetilde{0} \rangle) \\ &\leq M_0 (t_2 - t_1). \end{aligned}$$

Thus, $x(t)$ is continuous with respect to $t \in [t_0, t_0 + \eta]$. By Lemma 3.2, we know that $x(t)$ is a solution of (1) on $[t_0, t_0 + \eta]$.

Next we shall prove uniqueness, let $y(t)$ be another solution of (1) and $m(t) = d_{\sup}(x(t), y(t))$. Then $m(t_0) = d_{\sup}(x_0, x_0) = 0 \leq \varphi_0$ and

$$d^+ m(t) \leq g(t, d_{\sup}(x(t), y(t))) = g(t, m(t)),$$

for each $t \in [t_0, t_0 + \eta]$. By Theorem 4.2 in [25], we get

$$m(t) \leq r(t, t_0, \varphi_0), \quad t \in [t_0, t_0 + \eta],$$

where $r(t, t_0, \varphi_0)$ is the maximal solution of (2). By the assumptions $r(t, t_0, \varphi_0) \equiv 0$, we obtain $y(t) = x(t)$ for each $t \in [t_0, t_0 + \eta]$. □

IV. Approximate Solutions

In this section, we shall obtain an error estimate between the solutions and approximate solutions of the initial value problem (1). Denote by $CBV(J, \mathscr{F}/\mathscr{S})$ the set of all continuous and of uniformly bounded variation mappings from J to $\mathscr{F}/\mathscr{S}$.

Definition 4.1: A function $y(t, t_0, y_0, \varepsilon)$, $\varepsilon > 0$, is said to be an ε-approximate solution of (1), if $y \in CBV(J, \mathscr{F}/\mathscr{S})$, $y(t_0, t_0, y_0, \varepsilon) = y_0$ and

$$d_{\sup}(y'(t), f(t, y(t))) \leq \varepsilon, \quad t \in J.$$

In case $\varepsilon = 0$, $y(t)$ is a solution of (1).

Theorem 4.1: Assume that $f : J \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation and for $t \in J$, $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$,

$$d_{\sup}\left(f\left(t, \langle\widetilde{x}\rangle\right), f\left(t, \langle\widetilde{y}\rangle\right)\right) \leq g\left(t, d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right)\right),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$. Suppose that $r(t) = r(t, t_0, \varphi_0, \varepsilon)$ is the maximal solution of

$$\frac{d\varphi(t)}{dt} = g(t, \varphi(t)) + \varepsilon, \quad \varphi(t_0) = \varphi_0 \geq 0,$$

existing for $t \in J$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) and $y(t) = y(t, t_0, y_0, \varepsilon)$ be an ε-approximate solution of (1) existing for $t \in J$. If $d_{\sup}(x_0, y_0) \leq \varphi_0$, then

$$d_{\sup}(x(t), y(t)) \leq r(t, t_0, \varphi_0, \varepsilon), \quad t \in J.$$

Proof. Let $m(t) = d_{\sup}(x(t), y(t))$ for each $t \in J$. Then $m(t_0) = d_{\sup}(x_0, y_0) \leq \varphi_0$ and

$$\begin{aligned} & d^+ m(t) \\ = \;& \overline{\lim_{h \to 0^+}} \frac{1}{h}(m(t+h) - m(t)) \\ \leq \;& \overline{\lim_{h \to 0^+}} d_{\sup}\left(\frac{x(t+h) - x(t)}{h}, f(t, x(t))\right) \\ + \;& \overline{\lim_{h \to 0^+}} d_{\sup}\left(f(t, y(t)), \frac{y(t+h) - y(t)}{h}\right) \\ + \;& d_{\sup}(f(t, x(t)), f(t, y(t))), \end{aligned}$$

for each $t \in J$. In fact, we can show this assertion by a similar method of Theorem 3.1. By Definition 4.1, we have

$$d^+ m(t) \leq g(t, d_{\sup}(x(t), y(t))) + \varepsilon = g(t, m(t)) + \varepsilon,$$

for each $t \in J$. Hence, by Theorem 4.2 in [25], we obtain $d_{\sup}(x(t), y(t)) \leq r(t, t_0, \varphi_0, \varepsilon)$, for each $t \in J$. □

Corollary 4.1: Assume that $f : J \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation and for $t \in J$, $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$,

$$d_{\sup}\left(f\left(t, \langle\widetilde{x}\rangle\right), f\left(t, \langle\widetilde{y}\rangle\right)\right) \leq g\left(t, d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right)\right),$$

where the function $g(t, \varphi) = L\varphi$, $L > 0$, is admissible in Theorem 4.1 and $\varphi(t_0) = d_{\sup}(x_0, y_0)$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) and $y(t) = y(t, t_0, y_0, \varepsilon)$ is an ε-approximate solution of (1) existing for $t \in J$. Then

$$d_{\sup}(x(t), y(t)) \leq d_{\sup}(x_0, y_0) e^{L(t - t_0)} + \frac{\varepsilon}{L}(e^{L(t - t_0)} - 1).$$

Proof. We see that the scalar differential equation

$$\frac{d\varphi(t)}{dt} = g(t, \varphi(t)) + \varepsilon = L\varphi(t) + \varepsilon, \quad \varphi(t_0) = d_{\sup}(x_0, y_0)$$

has a unique solution

$$\varphi(t) = d_{\sup}(x_0, y_0) e^{L(t - t_0)} + \frac{\varepsilon}{L}(e^{L(t - t_0)} - 1),$$

on J. By Theorem 4.1, we obtain

$$d_{\sup}(x(t), y(t)) \leq d_{\sup}(x_0, y_0) e^{L(t - t_0)} + \frac{\varepsilon}{L}(e^{L(t - t_0)} - 1),$$

for each $t \in J$. □

Example 4.1: Define two fuzzy mapping $F, G : J \to \mathscr{F}/\mathscr{S}$ by the level sets

$$\left[\widehat{F(t)}\right]^\alpha = \left[0, 3e^{\frac{a^{-(1+\alpha)}(t - t_0)^{1+\alpha}}{1+\alpha}}\right]$$

and

$$\left[\widehat{G(t)}\right]^\alpha = \left[-2a^{-(1+\alpha)}(t - t_0)^{1+\alpha}, 0\right],$$

for each $\alpha \in [0, 1]$, where $\widehat{F(t)}$ and $\widehat{G(t)}$ are the Mareš core of $F(t)$ and $G(t)$, respectively, for each $t \in J$. Thus, we have

$$M_{F(t)}(\alpha) = \frac{3}{2} e^{\frac{a^{-(1+\alpha)}(t - t_0)^{1+\alpha}}{1+\alpha}}$$

and

$$M_{G(t)}(\alpha) = -a^{-(1+\alpha)}(t - t_0)^{1+\alpha},$$

for each $\alpha \in [0, 1]$ and $t \in J$. It is obvious that $M_{F(t)}(\alpha)$ and $M_{G(t)}(\alpha)$ are continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Since $M_{F(t)}(\alpha)$ and $M_{G(t)}(\alpha)$ is decreasing with respect to α, we get

$$V_0^1(M_{F(t)}) \leq \frac{3}{2} e^{a^{-1}(t - t_0)} \leq \frac{3}{2} e$$

and

$$V_0^1(M_{G(t)}) = -\left(\frac{t - t_0}{a} - \frac{1}{2}\right)^2 + \frac{1}{4} \leq \frac{1}{4},$$

for each $t \in J$. Thus, we get that $F(t)$ and $G(t)$ are of uniformly bounded variation. Since $M_{F(t)}(\alpha)$ and $M_{G(t)}(\alpha)$ are uniformly continuous with respect to $t \in J$, we get that $F(t)$ and $G(t)$ are continuous with respect to $d_{\sup}$. Define $f : J \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ by

$$f\left(t, \langle\widetilde{x}\rangle\right) = F(t)\langle\widetilde{x}\rangle + G(t).$$

It is obvious that f is continuous with respect to $d_{\sup}$ and of uniformly bounded variation. We get

$$\begin{aligned} & d_{\sup}\left(f\left(t, \langle\widetilde{x}\rangle\right), f\left(t, \langle\widetilde{y}\rangle\right)\right) \\ = \;& d_{\sup}\left(F(t)\langle\widetilde{x}\rangle + G(t), F(t)\langle\widetilde{y}\rangle + G(t)\right) \\ = \;& \sup_{\alpha \in [0,1]} \left|M_{F(t)}(\alpha) M_{\langle\widetilde{x}\rangle}(\alpha) - M_{F(t)}(\alpha) M_{\langle\widetilde{y}\rangle}(\alpha)\right| \\ \leq \;& \sup_{\alpha \in [0,1]} \left|M_{F(t)}(\alpha)\right| d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right) \\ \leq \;& \frac{3}{2} e d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right) \end{aligned}$$

for each $t \in J$ and $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$. Define the scalar differential equation

$$\frac{d\varphi(t)}{dt} = g(t, \varphi(t)) + \varepsilon, \quad \varphi(t_0) = d_{\sup}(x_0, y_0),$$

where $\varepsilon > 0$ and the function $g(t, \varphi) = \frac{3}{2} e\varphi$ for each $t \in J$. It is obvious that $g \in C_1[J \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, \varphi)$ is nondecreasing with respect to φ for each $t \in J$. Then

$$g\left(t, d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right)\right) = \frac{3}{2} e d_{\sup}\left(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle\right)$$

for each $t \in J$ and $\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle \in \mathscr{F}/\mathscr{S}$. Hence, we obtain

$$d_{\sup}\left(f\left(t, \langle \widetilde{x} \rangle\right), f\left(t, \langle \widetilde{y} \rangle\right)\right) \leq g\left(t, d_{\sup}\left(\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle\right)\right)$$

for each $t \in J$ and $\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle \in \mathscr{F}/\mathscr{S}$. By Corollary 4.1, we conclude that

$$d_{\sup}(x(t), y(t)) \leq d_{\sup}(x_0, y_0) e^{\frac{3}{2}e(t-t_0)} + \frac{2\varepsilon}{3e}(e^{\frac{3}{2}e(t-t_0)} - 1),$$

where $x(t) = x(t, t_0, x_0)$ is a solution and $y(t) = y(t, t_0, y_0, \varepsilon)$ is an ε-approximate solution of the fuzzy differential equation

$$x'(t) = f(t, x(t))$$

through (t_0, x_0) and (t_0, y_0) on J, respectively.

V. Conclusions

In this paper, we have researched the convergence of successive approximations for fuzzy differential equations in the quotient space of fuzzy numbers. We have solved the convergence of successive approximations of the initial value problem for the fuzzy differential equations, provided that f is a continuous with respect to $d_{\sup}$, of uniformly bounded variation on T and bounded function, under the general uniqueness assumption of Perron type utilizing the comparison functions. And then we have discuss the approximate solutions and the error estimates between the solutions and approximate solutions. We also hope that our results in this paper may lead to significant, new and innovative results in other related fields [11], [14], [15], [18], [24], [31].

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