Some New Gronwall-Bellmann Type Discrete Fractional Inequalities Arising in the Theory of Discrete Fractional Calculus

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***Abstract*—In this paper, we present some new Gronwall-Bellmann type discrete fractional sum inequalities, and based on them present some Volterra-Fredholm type discrete inequalities. These inequalities are of new forms compared with the existing results in the literature, and can be used in the research of boundedness and continuous dependence on the initial value for solutions of fractional difference equations. As for applications, we apply the presented results to research the initial value problem of a certain fractional difference equation.**

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***Index Terms*—Gronwall-Bellman type inequality, discrete fractional sum inequality, Volterra-Fredholm type discrete inequality, fractional difference equation**

I. Introduction

Fractional differential equations are widely used as models to express many important physical phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on. Recently, there have been much attention paid on the research of properties of fractional differential equations. For example, in [1,2], the authors proposed certain methods for finding analytical solutions of fractional differential equations. In [3-5], qualitative and quantitative properties of solutions of fractional differential equations are investigated.

In the research of qualitative and quantitative properties of solutions of differential equations, difference equations and dynamic equations on time scales, the Gronwall-Bellman inequality [6,7] and its various generalizations play important roles as explicit bounds for unknown functions can be provided. In the last few decades, many Gronwall-Bellman type differential and integral inequalities [8-18], retarded inequalities [19-24], difference inequalities [25-32], and dynamic inequalities on time scales [33-40] have been established, which have proved to be very useful in the research of boundedness, uniqueness, and continuous dependence on initial value and parameter for solutions of differential equations, difference equations and dynamic equations on time scales. Among these inequalities, we notice that most of the inequalities established so far can only be used in the qualitative and quantitative analysis for solutions of differential and difference equations of integer order, while few results are concerned with fractional differential equations [41-44]. Furthermore, there are less inequalities suitable for the qualitative and quantitative analysis for solutions of fractional difference equations.

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Compared to the theory of fractional differential calculus, there has been relatively less development on the theory of fractional difference calculus so far in the literature. In general, there are two types of fractional difference operators: the Δ difference and the ∇ difference. For the Δ difference, we have the following two definitions:

Definition 1. Let $v > 0$, $\sigma(s) = s + 1$, and the function f is defined for $s = a$ mod 1. Then the $v-$th fractional sum of f is defined by

$$\Delta^{-v} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(\alpha-1)} f(s),$$

where $t^{(v)} = \dfrac{\Gamma(t+1)}{\Gamma(t+1-v)}$, $\Delta^{-v} f$ is defined for $s = a + v$ mod 1, and Δ^{-v} maps functions defined on $\mathbf{N}_a$ to functions defined on $\mathbf{N}_{a+v}$.

Definition 2. Let $\mu > 0$, and $m - 1 < \mu < m$, where m is a positive integer. Then the $\mu-$th fractional difference of f is defined by

$$\Delta^{\mu} f(t) = \Delta^{m-(m-\mu)} f(t) = \Delta^{m} \Delta^{-(m-\mu)} f(t).$$

For the Δ difference and fractional sum, we also have the following two theorems.

Theorem A [45, Theorem 1.1] . Let f be a real-valued function defined on $\mathbf{N}_a$, and $\mu,\ v > 0$. Then the following equalities hold:

$$\Delta^{-v}[\Delta^{-\mu} f(t)] = \Delta^{-(v+\mu)} f(t) = \Delta^{-\mu}[\Delta^{-v} f(t)].$$

Theorem B [45, Theorem 2.1]. Let f be a real-valued function defined on $\mathbf{N}_a$, and $v > 0$. Then the following equalities hold:

$$\Delta^{-v} \Delta f(t) = \Delta \Delta^{-v} f(t) - \frac{(t-a)^{(v-1)}}{\Gamma(v)} f(a).$$

For other important properties and conclusions on the discrete fractional calculus, we refer the reader to [45-47].

Motivated by the above analysis, in this paper, we establish some new Gronwall-Bellmann type discrete fractional sum inequalities, and based on them present some Volterra-Fredholm type discrete inequalities. We also apply the presented inequalities to research the initial value problem of a certain fractional difference equation.

The next of this paper is organized as follows. In Section 2, we establish some new Gronwall-Bellmann type discrete fractional sum inequalities as well as some Volterra-Fredholm type discrete inequalities, and deduce explicit bounds for the unknown functions lying in these inequalities. In Section 3, we apply the inequalities established to research boundedness and continuous dependence on the initial value for the solution to a certain initial value problem of a fractional difference equation. In Section 4, some conclusions are presented.

For the sake of convenience, we denote $\mathbf{N}_t = \{t, t+1, t+2, ...\}$, and $\sum_{s=m_0}^{m_1} f(s) = 0$ provided $m_0 > m_1$.

II. Main Results

Theorem 1. Assume $0 < \alpha \leq 1$, $u(n),\ a(n),\ b(n),\ c(n)$ are nonnegative function defined on $\mathbf{N}_{\alpha-1}$. If the following inequality satisfies:

$$u(n) \leq a(n)+c(n)\Delta^{-\alpha}[b(n+\alpha-1)u(n+\alpha-1)],\ n \in \mathbf{N}_\alpha, \tag{1}$$

then we have the following estimate for $u(n)$:

$$\begin{aligned} &u(n) \leq a(n) + c(n)\{\theta(n-\alpha, n)a(n-1) \\ &+ \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|a(s+\alpha-1) \\ &+ \sum_{p=\alpha}^{n-1}\{[\theta(p-\alpha,p)a(p-1) \\ &+ \sum_{s=0}^{p-\alpha-1} |\theta(s,p) - \theta(s,p-1)|a(s+\alpha-1)]\times \\ &\prod_{\xi=p+1}^{n} [1 + \theta(\xi-\alpha,\xi)c(\xi-1) \\ &+ \sum_{s=0}^{\xi-\alpha-1} |\theta(s,\xi)-\theta(s,\xi-1)|c(\xi+\alpha-1)]\}\},\ n \in \mathbf{N}_\alpha. \end{aligned} \tag{2}$$

where $\theta(s,n) = \frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$.

Proof. By the definition of the $\alpha-$th fractional sum and (1) one can obtain that

$$\begin{aligned} &u(n) \leq a(n) + \frac{c(n)}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\times \\ &b(s+\alpha-1)u(s+\alpha-1)),\ n \in \mathbf{N}_\alpha. \end{aligned}$$

Denote $\theta(s,n) = \frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$, and $v(n) = \sum_{s=0}^{n-\alpha}\theta(s,n)u(s+\alpha-1)$. Then we have

$$u(n) \leq a(n)+c(n)v(n),\ n \in \mathbf{N}_\alpha. \tag{3}$$

Furthermore, for $n \in \mathbf{N}_\alpha$,

$$\begin{aligned} &v(n) - v(n-1) = \theta(n-\alpha,n)u(n-1) \\ &+ \sum_{s=0}^{n-\alpha-1}[\theta(s,n) - \theta(s,n-1)]u(s+\alpha-1) \\ &\leq \theta(n-\alpha,n)[a(n-1) + c(n-1)v(n-1)] \\ &+ \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|a(s+\alpha-1) \\ &+ \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|c(s+\alpha-1)v(n-1), \end{aligned}$$

which is rewritten by

$$\begin{aligned} &v(n) - \{1 + \theta(n-\alpha,n)c(n-1) \\ &+ \sum_{s=0}^{n-\alpha-1}[|\theta(s,n) - \theta(s,n-1)|c(s+\alpha-1)]\}v(n-1) \\ &\leq \theta(n-\alpha,n)a(n-1)+ \sum_{s=0}^{n-\alpha-1}|\theta(s,n)-\theta(s,n-1)|a(s+\alpha-1). \end{aligned} \tag{4}$$

For $n > \alpha$, substituting n with p in (4), multiplying on both sides by $\prod_{\xi=p+1}^{n}\{1+\theta(\xi-\alpha,\xi)c(\xi-1)+\sum_{s=0}^{\xi-\alpha-1}[|\theta(s,\xi)-\theta(s,\xi-1)|c(\xi+\alpha-1)]\}$, a summation with respect to p from α to $n-1$ together with (4), and using $v(\alpha-1) = 0$, yields that

$$\begin{aligned} &v(n) \leq \theta(n-\alpha,n)a(n-1) \\ &+ \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|a(s+\alpha-1) \\ &+ \sum_{p=\alpha}^{n-1}\{[\theta(p-\alpha,p)a(p-1) \\ &+ \sum_{s=0}^{p-\alpha-1} |\theta(s,p) - \theta(s,p-1)|a(s+\alpha-1)] \times \prod_{\xi=p+1}^{n} \\ &[1+\theta(\xi-\alpha,\xi)c(\xi-1)+ \sum_{s=0}^{\xi-\alpha-1}|\theta(s,\xi)-\theta(s,\xi-1)|c(\xi+\alpha-1)]\}. \end{aligned} \tag{5}$$

Note that (5) also holds for $n = \alpha$. So (5) holds in fact for $n \in \mathbf{N}_\alpha$. Combining (3) and (5) we can deduce the desired result.

Theorem 2. Assume $0 < \alpha \leq 1$, $u(n),\ a(n),\ b_1(n),\ b_2(n),\ c(n)$ are nonnegative function defined on $\mathbf{N}_{\alpha-1}$. If for $n \in \mathbf{N}_\alpha$, the following inequality satisfies:

$$\begin{aligned} &u(n) \leq a(n) \\ &+\frac{c(n)}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha} u(s+\alpha-1))[(n-s-1)^{(\alpha-1)}b_1(s+\alpha-1) \\ &+\sum_{\xi=0}^{s} b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}], \end{aligned} \tag{6}$$

then for $n \in \mathbf{N}_\alpha$, we have the following estimate for $u(n)$:

$$\begin{aligned} &u(n) \leq a(n) + c(n)\{[\widetilde{\theta}(n-\alpha,n) \\ &+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha} b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]a(n-1) \\ &+ \sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n) - \widetilde{\theta}(s,n-1)| + \frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s} b_2(\xi+\alpha-1)\times \\ &|(n-\xi-1)^{(\alpha-1)} - (n-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1) \\ &+ \sum_{p=\alpha}^{n-1}\{\{[\widetilde{\theta}(p-\alpha,p) + \frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{p-\alpha} b_2(\xi+\alpha-1)\times \end{aligned}$$

$$(p-\xi-1)^{(\alpha-1)}]a(p-1)+\sum_{s=0}^{p-\alpha-1}[|\widetilde{\theta}(s,p)-\widetilde{\theta}(s,p-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(p-\xi-1)^{(\alpha-1)}$$
$$-(p-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1)\}\times$$
$$\prod_{\zeta=p+1}^{n}\{1+[\widetilde{\theta}(\zeta-\alpha,\zeta)+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{\zeta-\alpha}b_2(\xi+\alpha-1)\times$$
$$(\zeta-\xi-1)^{(\alpha-1)}]c(\zeta-1)$$
$$+\sum_{s=0}^{\zeta-\alpha-1}[|\widetilde{\theta}(s,\zeta)-\widetilde{\theta}(s,\zeta-1)|+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)$$
$$|(\zeta-\xi-1)^{(\alpha-1)}-(\zeta-\xi-2)^{(\alpha-1)}|]c(s+\alpha-1)\}\}\}. \quad (7)$$

where $\widetilde{\theta}(s,n)=\frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b_1(s+\alpha-1)$.

Proof. Denote $\widetilde{\theta}(s,n)=\frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b_1(s+\alpha-1)$, and

$$v(n)=\sum_{s=0}^{n-\alpha}\widetilde{\theta}(s,n)u(s+\alpha-1)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}u(s+\alpha-1))\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}.$$

Then we have

$$u(n)\le a(n)+c(n)v(n),\ n\in\mathbf{N}_\alpha. \quad (8)$$

Furthermore, for $n\in\mathbf{N}_\alpha$,

$$v(n)-v(n-1)=[\widetilde{\theta}(n-\alpha,n)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]u(n-1)$$
$$+\sum_{s=0}^{n-\alpha-1}\{\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)[(n-\xi-1)^{(\alpha-1)}$$
$$-(n-\xi-2)^{(\alpha-1)}]\}u(s+\alpha-1)$$
$$\le[\widetilde{\theta}(n-\alpha,n)+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)\times$$
$$(n-\xi-1)^{(\alpha-1)}][a(n-1)+c(n-1)v(n-1)]$$
$$+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}$$
$$-(n-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1)$$
$$+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}$$
$$-(n-\xi-2)^{(\alpha-1)}|](s+\alpha-1)v(n-1),$$

which is rewritten by

$$v(n)-\{1+[\widetilde{\theta}(n-\alpha,n)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]c(n-1)$$
$$+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}$$
$$-(n-\xi-2)^{(\alpha-1)}|]c(s+\alpha-1)\}v(n-1)$$
$$\le[\widetilde{\theta}(n-\alpha,n)+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]$$
$$a(n-1)+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}$$
$$b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}-(n-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1). \quad (9)$$

For $n>\alpha$, substituting n with p in (9), multiplying on both sides by $\prod_{\zeta=p+1}^{n}\{1+[\widetilde{\theta}(\zeta-\alpha,\zeta)+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{\zeta-\alpha}b_2(\xi+\alpha-1)(\zeta-\xi-1)^{(\alpha-1)}]c(\zeta-1)+\sum_{s=0}^{\zeta-\alpha-1}[|\widetilde{\theta}(s,\zeta)-\widetilde{\theta}(s,\zeta-1)|+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(\zeta-\xi-1)^{(\alpha-1)}-(\zeta-\xi-2)^{(\alpha-1)}|]c(s+\alpha-1)\}$, a summation with respect to p from α to $n-1$ together with (9), and using $v(\alpha-1)=0$, yields that

$$v(n)\le[\widetilde{\theta}(n-\alpha,n)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]a(n-1)$$
$$+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}$$
$$-(n-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1)+\sum_{p=\alpha}^{n-1}\{\{[\widetilde{\theta}(p-\alpha,p)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{p-\alpha}b_2(\xi+\alpha-1)(p-\xi-1)^{(\alpha-1)}]a(p-1)$$
$$+\sum_{s=0}^{p-\alpha-1}[|\widetilde{\theta}(s,p)-\widetilde{\theta}(s,p-1)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(p-\xi-1)^{(\alpha-1)}$$
$$-(p-\xi-2)^{(\alpha-1)}|]a(s+\alpha-1)\}\prod_{\zeta=p+1}^{n}\{1+[\widetilde{\theta}(\zeta-\alpha,\zeta)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{\zeta-\alpha}b_2(\xi+\alpha-1)(\zeta-\xi-1)^{(\alpha-1)}]c(\zeta-1)$$
$$+\sum_{s=0}^{\zeta-\alpha-1}[|\widetilde{\theta}(s,\zeta)-\widetilde{\theta}(s,\zeta-1)|+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)$$
$$|(\zeta-\xi-1)^{(\alpha-1)}-(\zeta-\xi-2)^{(\alpha-1)}|]c(s+\alpha-1)\}\}. \quad (10)$$

Note that (10) also holds for $n=\alpha$. So (10) holds in fact for $n\in\mathbf{N}_\alpha$. Combining (8) and (10) we can get the desired result.

In the following two theorems we establish some Volterra-Fredholm type discrete inequalities based on the results of Theorems 1-2.

Theorem 3. Assume $0<\alpha\le1$, $u(n)$, $a(n)$, $b(n)$ are nonnegative functions defined on $\mathbf{N}_{\alpha-1}$, $C>0$ is a constant, and $T \in \mathbf{N}_\alpha$ is a constant. If the following inequality satisfies:

$$u(n) \le C + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} b(s+\alpha-1)$$
$$u(s+\alpha-1)) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} \times$$
$$b(s+\alpha-1)u(s+\alpha-1)),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha, \tag{11}$$

then we have

$$u(n) \le \frac{C}{2-\mu(\alpha,T)} \mu(\alpha,n),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha, \tag{12}$$

provided that $\mu(\alpha, T) < 2$, where $\theta(s,n)$ is defined as in Theorem 1, and

$$\mu(\alpha,n) = 1 + \theta(n-\alpha,n) + \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|$$
$$+ \sum_{p=\alpha}^{n-1} \{[\theta(p-\alpha,p) + \sum_{s=0}^{p-\alpha-1} |\theta(s,p) - \theta(s,p-1)|] \times$$
$$\prod_{\xi=p+1}^{n} [1+\theta(\xi-\alpha,\xi) + \sum_{s=0}^{\xi-\alpha-1} |\theta(s,\xi) - \theta(s,\xi-1)|]\}. \tag{13}$$

Proof. Denote the right-hand side of (11) by $v(n)$. Then we have

$$u(n) \le v(n),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha. \tag{14}$$

Using $v(\alpha-1) = C + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} b(s+\alpha-1)u(s+\alpha-1))$, one can obtain that

$$v(n) = v(\alpha-1) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} \times$$
$$b(s+\alpha-1)u(s+\alpha-1))$$
$$\le v(\alpha-1) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} b(s+\alpha-1) \times$$
$$v(s+\alpha-1),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha. \tag{15}$$

Applying Theorem 1 to (15) yields that

$$v(n) \le v(\alpha-1)\{1 + \theta(n-\alpha,n)$$
$$+ \sum_{s=0}^{n-\alpha-1} |\theta(s,n) - \theta(s,n-1)|$$
$$+ \sum_{p=\alpha}^{n-1} \{[\theta(p-\alpha,p) + \sum_{s=0}^{p-\alpha-1} |\theta(s,p) - \theta(s,p-1)|] \times$$
$$\prod_{\xi=p+1}^{n} [1+\theta(\xi-\alpha,\xi) + \sum_{s=0}^{\xi-\alpha-1} |\theta(s,\xi) - \theta(s,\xi-1)|]\}\}$$
$$= v(\alpha-1)\mu(\alpha,n),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha, \tag{16}$$

where $\theta(s,n)$ is defined as in Theorem 1, and $\mu(\alpha,n)$ is defined in (13).

Setting $n = T$ in (16) one can obtain that

$$v(T) \le v(\alpha-1)\mu(\alpha,T).$$

So

$$2v(\alpha-1) - C = v(T) \le v(\alpha-1)\mu(\alpha,T),$$

which is followed by

$$v(\alpha-1) \le \frac{C}{2-\mu(\alpha,T)}. \tag{17}$$

Combining (14), (16) and (17) we can deduce the desired inequality (12).

Theorem 4. Assume $0 < \alpha \le 1$, $u(n)$, $a(n)$, $b_1(n)$, $b_2(n)$, $c(n)$ are nonnegative functions defined on $\mathbf{N}_{\alpha-1}$, $C > 0$ is a constant, and $T \in \mathbf{N}_\alpha$ is a constant. If the following inequality satisfies:

$$u(n) \le C + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} u(s+\alpha-1))[(n-s-1)^{(\alpha-1)} \times$$
$$b_1(s+\alpha-1) + \sum_{\xi=0}^{s} b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} u(s+\alpha-1))[(T-s-1)^{(\alpha-1)} b_1(s+\alpha-1)$$
$$+ \sum_{\xi=0}^{s} b_2(\xi+\alpha-1)(T-\xi-1)^{(\alpha-1)}], \tag{18}$$

then we have

$$u(n) \le \frac{C}{2-\widetilde{\mu}(\alpha,T)} \widetilde{\mu}(\alpha,n),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha, \tag{19}$$

provided that $\widetilde{\mu}(\alpha, T) < 2$, where $\widetilde{\theta}(s,n)$ is defined as in Theorem 2, and

$$\widetilde{\mu}(\alpha,n) = 1 + \widetilde{\theta}(n-\alpha,n)z(n-1)$$
$$+ \sum_{s=0}^{n-\alpha-1} |\widetilde{\theta}(s,n) - \widetilde{\theta}(s,n-1)|$$
$$+ \sum_{p=\alpha}^{n-1} \{[\widetilde{\theta}(p-\alpha,p) + \sum_{s=0}^{p-\alpha-1} |\widetilde{\theta}(s,p) - \widetilde{\theta}(s,p-1)|] \times$$
$$\prod_{\xi=p+1}^{n} [1+\widetilde{\theta}(\xi-\alpha,\xi) + \sum_{s=0}^{\xi-\alpha-1} |\widetilde{\theta}(s,\xi) - \widetilde{\theta}(s,\xi-1)|]\}. \tag{20}$$

Proof. Denote the right-hand side of (18) by $v(n)$. Then we have

$$u(n) \le v(n),\ n \in [\alpha, T] \bigcap \mathbf{N}_\alpha. \tag{21}$$

Using $v(\alpha-1) = C + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} u(s+\alpha-1))[(T-s-1)^{(\alpha-1)} b_1(s+\alpha-1) + \sum_{\xi=0}^{s} b_2(\xi+\alpha-1)(T-\xi-1)^{(\alpha-1)}]$, one can obtain that

$$v(n) = v(\alpha-1) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} u(s+\alpha-1))[(n-s-1)^{(\alpha-1)}$$
$$b_1(s+\alpha-1) + \sum_{\xi=0}^{s} b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]$$

$$\leq v(\alpha-1)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}v(s+\alpha-1)[(n-s-1)^{(\alpha-1)}\times$$

$$b_1(s+\alpha-1)+\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}],\ n\in[\alpha,T]\bigcap\mathbf{N}_\alpha. \tag{22}$$

Applying Theorem 2 to (22) yields that

$$\begin{aligned}
&v(n)\leq v(\alpha-1)\{1+[\widetilde{\theta}(n-\alpha,n)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{n-\alpha}b_2(\xi+\alpha-1)(n-\xi-1)^{(\alpha-1)}]\\
&+\sum_{s=0}^{n-\alpha-1}[|\widetilde{\theta}(s,n)-\widetilde{\theta}(s,n-1)|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(n-\xi-1)^{(\alpha-1)}-(n-\xi-2)^{(\alpha-1)}|]\\
&+\sum_{p=\alpha}^{n-1}\{\{[\widetilde{\theta}(p-\alpha,p)+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{p-\alpha}b_2(\xi+\alpha-1)\\
&(p-\xi-1)^{(\alpha-1)}]+\sum_{s=0}^{p-\alpha-1}[|\widetilde{\theta}(s,p)-\widetilde{\theta}(s,p-1)|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(p-\xi-1)^{(\alpha-1)}\\
&-(p-\xi-2)^{(\alpha-1)}|]\}\times\prod_{\zeta=p+1}^{n}\{1+[\widetilde{\theta}(\zeta-\alpha,\zeta)\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{\zeta-\alpha}b_2(\xi+\alpha-1)(\zeta-\xi-1)^{(\alpha-1)}]\\
&+\sum_{s=0}^{\zeta-\alpha-1}[|\widetilde{\theta}(s,\zeta)-\widetilde{\theta}(s,\zeta-1)|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s}b_2(\xi+\alpha-1)|(\zeta-\xi-1)^{(\alpha-1)}\\
&-(\zeta-\xi-2)^{(\alpha-1)}|]\}\}\}\\
&=v(\alpha-1)\widetilde{\mu}(\alpha,n),\ n\in[\alpha,T]\bigcap\mathbf{N}_\alpha,
\end{aligned}\tag{23}$$

where $\widetilde{\theta}(s,n)$ is defined as in Theorem 2, and $\widetilde{\mu}(\alpha,n)$ is defined in (20).

Setting $n=T$ in (23) one can obtain that

$$v(T)\leq v(\alpha-1)\widetilde{\mu}(\alpha,T).$$

So

$$2v(\alpha-1)-C=v(T)\leq v(\alpha-1)\widetilde{\mu}(\alpha,T),$$

which is followed by

$$v(\alpha-1)\leq\frac{C}{2-\widetilde{\mu}(\alpha,T)}. \tag{24}$$

Combining (21), (23) and (24) we can deduce the desired inequality (19).

Remark. We note that the inequalities as well as the bounds established in Theorems 1-4 are new results in the literature.

III. Applications

In this section, we apply the inequalities established above to research boundedness and continuous dependence on the initial value for the solution to a fractional difference equation.

Consider the IVP of the following fractional difference equation:

$$\begin{cases}\Delta^\alpha u(k)=f(k+\alpha-1,u(k+\alpha-1)),\ k=0,1,2,...,\\ \Delta^{\alpha-1}u(k)|_{k=0}=C,\end{cases} \tag{25}$$

where $0<\alpha<1$, $u(n)$ is an unknown function defined on $\mathbf{N}_{\alpha-1}$, $f:\ \mathbf{N}_{\alpha-1}\times\mathbf{R}\to\mathbf{R}$.

Theorem 5. For the IVP (25), if $|f(k+\alpha-1,u(k+\alpha-1))|\leq b(k+\alpha-1)|u(k+\alpha-1)|$, where b is a nonnegative function defined on $\mathbf{N}_{\alpha-1}$, then we have the following estimate for $u(n)$:

$$\begin{aligned}
&|u(n)|\leq\frac{|C|}{\Gamma(\alpha)}\{n^{(\alpha-1)}+\theta(n-\alpha,n)(n-1)^{(\alpha-1)}\\
&+\sum_{s=0}^{n-\alpha-1}|\theta(s,n)-\theta(s,n-1)|(s+\alpha-1)^{(\alpha-1)}\\
&+\sum_{p=\alpha}^{n-1}\{[\theta(p-\alpha,p)(p-1)^{(\alpha-1)}\\
&+\sum_{s=0}^{p-\alpha-1}|\theta(s,p)-\theta(s,p-1)|(s+\alpha-1)^{(\alpha-1)}]\times\\
&\prod_{\xi=p+1}^{n}[1+\theta(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\theta(s,\xi)-\theta(s,\xi-1)|]\}\},\ n\in\mathbf{N}_\alpha.
\end{aligned}\tag{26}$$

where $\theta(s,n)=\frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$.

Proof. By [45, Eq. (4)], the equivalent discrete fractional sum equation of the IVP (25) can be denoted as follows:

$$\begin{aligned}
&u(n)=\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}C\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s+\alpha-1,u(s+\alpha-1)).
\end{aligned}$$

So

$$\begin{aligned}
&|u(n)|\leq\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|C|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}|f(s+\alpha-1,u(s+\alpha-1))|\\
&\leq\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|C|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)|u(s+\alpha-1)|\\
&=\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|C|+\Delta^{-\alpha}[b(n+\alpha-1)|u(n+\alpha-1)|].
\end{aligned}\tag{27}$$

Then a suitable application of Theorem 1 (with $c(n)\equiv 1$) to (27) yields the desired result.

Now we research the continuous dependence on the initial value for the solution of the IVP (25).

Theorem 6. For the IVP (25), if $|f(k+\alpha-1,u)-f(k+\alpha-1,v)|\leq b(k+\alpha-1)|u-v|$, where b is a nonnegative function defined on $\mathbf{N}_{\alpha-1}$, then the solution of the IVP (25) depends continuously on the initial value C.

Proof. Let $\widetilde{u}(n)$ be the solution of the following IVP:

$$\begin{cases} \Delta^{\alpha}\widetilde{u}(k) = f(k+\alpha-1, \widetilde{u}(k+\alpha-1)),\ k = 0,1,2...,\\ \Delta^{\alpha-1}\widetilde{u}(k)|_{k=0} = \widetilde{C}, \end{cases} \tag{28}$$

Similar to Theorem 5, the equivalent discrete fractional sum equation of the IVP (28) can be denoted as follows:

$$\widetilde{u}(n) = \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}\widetilde{C}$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s+\alpha-1, \widetilde{u}(s+\alpha-1)). \tag{29}$$

So we have

$$u(n) - \widetilde{u}(n) = \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}(C-\widetilde{C})$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\times$$
$$[f(s+\alpha-1, u(s+\alpha-1)) - f(s+\alpha-1, \widetilde{u}(s+\alpha-1))]. \tag{30}$$

Furthermore,

$$|u(n) - \widetilde{u}(n)| \le \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|C-\widetilde{C}|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$$
$$|u_1(s+\alpha-1) - u_2(s+\alpha-1)|. \tag{31}$$

Applying Theorem 1 to (31), after some basic computation we can deduce that

$$|u(n)-\widetilde{u}(n)| \le \frac{|C-\widetilde{C}|}{\Gamma(\alpha)}\{n^{(\alpha-1)}+\theta(n-\alpha,n)(n-1)^{(\alpha-1)}$$
$$+\sum_{s=0}^{n-\alpha-1}|\theta(s,n)-\theta(s,n-1)|(s+\alpha-1)^{(\alpha-1)}$$
$$+\sum_{p=\alpha}^{n-1}\{[\theta(p-\alpha,p)(p-1)^{(\alpha-1)}$$
$$+\sum_{s=0}^{p-\alpha-1}|\theta(s,p)-\theta(s,p-1)|(s+\alpha-1)^{(\alpha-1)}]\times$$
$$\prod_{\xi=p+1}^{n}[1+\theta(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\theta(s,\xi)-\theta(s,\xi-1)|]\}\},\ n\in\mathbf{N}_{\alpha}. \tag{32}$$

where $\theta(s,n) = \frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$.

Seen from (32) one can see that small change in the initial value C leads to small change in the solution $u(n)$ of the IVP (25). So the proof is complete.

IV. Further results on Gronwall-Bellmann type discrete fractional inequalities

In Theorems 1-4 established in Section 2, the unknown function $u(n)$ are all linear function terms, which can be extended to nonlinear power function terms with arbitrary power.

In this section, based on the main results presented in Section 2, we investigate some further results on Gronwall-Bellmann type inequalities , and establish some discrete fractional sum inequalities involving nonlinear power function terms with arbitrary power for unknown functions.

Lemma 7 [**12**]: Assume that $a \ge 0, p \ge q \ge 0$, and $p \ne 0$, then for any $K > 0$,

$$a^{\frac{q}{p}} \le \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 8. Assume $0 < \alpha \le 1$, $u(n)$, $a(n)$, $b(n)$ are nonnegative functions defined on $\mathbf{N}_{\alpha-1}$, n_1, n_2 are constants with $n_1 \ge n_2 > 0$. If for $n \in \mathbf{N}_{\alpha}$, the following inequality holds:

$$u^{n_1}(n) \le a(n)$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)u^{n_2}(s+\alpha-1), \tag{33}$$

then for $n \in \mathbf{N}_{\alpha}$, one has

$$u(n) \le \{a(n) + \widehat{e}(n) + \widehat{\theta}(n-\alpha,n)\widehat{e}(n-1)$$
$$+\sum_{s=0}^{n-\alpha-1}|\widehat{\theta}(s,n)-\widehat{\theta}(s,n-1)|\widehat{e}(s+\alpha-1)$$
$$+\sum_{p=\alpha}^{n-1}\{[\widehat{\theta}(p-\alpha,p)\widehat{e}(p-1)$$
$$+\sum_{s=0}^{p-\alpha-1}|\widehat{\theta}(s,p)-\widehat{\theta}(s,p-1)|\widehat{e}(s+\alpha-1)]\times$$
$$\prod_{\xi=p+1}^{n}[1+\widehat{\theta}(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\widehat{\theta}(s,\xi)-\widehat{\theta}(s,\xi-1)|]\}\}^{\frac{1}{n_1}}, \tag{34}$$

where

$$\begin{cases} \widehat{b}(n+\alpha-1) = \frac{n_2}{n_1}K^{\frac{n_2-n_1}{n_1}}b(n+\alpha-1),\\ \widehat{e}(n) = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)\\ \qquad [\frac{n_2}{n_1}K^{\frac{n_2-n_1}{n_1}}a(s+\alpha-1) + \frac{n_1-n_2}{n_1}K^{\frac{n_2}{n_1}}],\\ \widehat{\theta}(s,n) = \frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}\widehat{b}(s+\alpha-1), \end{cases} \tag{35}$$

and $K > 0$ is an arbitrary constant.

Proof. Denote

$$v(n) = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)u^{n_2}(s+\alpha-1).$$

Then

$$u^{n_1}(n) \le a(n) + v(n),\ n \in \mathbf{N}_{\alpha}. \tag{36}$$

and it holds that

$$v(n) \le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$$
$$[a(s+\alpha-1)+v(s+\alpha-1)]^{\frac{n_2}{n_1}}$$
$$\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s+\alpha-1)$$

$$[\frac{n_2}{n_1}K^{\frac{n_2-n_1}{n_1}}(a(s+\alpha-1)+v(s+\alpha-1))+\frac{n_1-n_2}{n_1}K^{\frac{n_2}{n_1}}]$$
$$=\widehat{e}(n)+\Delta^{-\alpha}[\widehat{b}(n+\alpha-1)v(n+\alpha-1)],\ n\in\mathbf{N}_\alpha,\quad(37)$$

where $\widehat{e},\ \widehat{b}$ are defined as in (35), and $K>0$ is an arbitrary constant.

Applying Theorem 1 to (37) one can deduce that

$$\begin{aligned}v(n)\le{}&\widehat{e}(n)+\widehat{\theta}(n-\alpha,n)\widehat{e}(n-1)\\&+\sum_{s=0}^{n-\alpha-1}|\widehat{\theta}(s,n)-\widehat{\theta}(s,n-1)|\widehat{e}(s+\alpha-1)\\&+\sum_{p=\alpha}^{n-1}\{[\widehat{\theta}(p-\alpha,p)\widehat{e}(p-1)\\&+\sum_{s=0}^{p-\alpha-1}|\widehat{\theta}(s,p)-\widehat{\theta}(s,p-1)|\widehat{e}(s+\alpha-1)]\times\\&\prod_{\xi=p+1}^{n}[1+\widehat{\theta}(\xi-\alpha,\xi)\\&+\sum_{s=0}^{\xi-\alpha-1}|\widehat{\theta}(s,\xi)-\widehat{\theta}(s,\xi-1)|]\},\ n\in\mathbf{N}_\alpha,\end{aligned}\quad(38)$$

where $\widehat{\theta}$ is defined as in (35).

Combining (36) and (38) one can obtain the desired inequality (34).

Now we present one application for Theorem 8.

Consider the following IVP of fractional difference equation:

$$\begin{cases}\Delta^{\frac{1}{2}}u^3(k)=g(k-\frac{1}{2})u(k-\frac{1}{2}),\ k\in\mathbf{N}_0,\\ \Delta^{-\frac{1}{2}}u^3(k)|_{k=0}=u_0,\end{cases}\quad(39)$$

where $u(n)$ is an unknown function defined on $\mathbf{N}_{-\frac{1}{2}}$.

In the following theorem, explicit bound for the unknown function u is obtained.

Theorem 9. For the IVP (39), if g is a nonnegative function defined on $\mathbf{N}_{\alpha-1}$, then we have the following estimate for $u(n)$:

$$\begin{aligned}|u(n)|\le{}&\{\frac{n^{(-\frac{1}{2})}}{\Gamma(\frac{1}{2})}|u_0|+\widehat{e}(n)+\widehat{\theta}(n-\tfrac{1}{2},n)\widehat{e}(n-1)\\&+\sum_{s=0}^{n-\frac{3}{2}}|\widehat{\theta}(s,n)-\widehat{\theta}(s,n-1)|\widehat{e}(s-\tfrac{1}{2})\\&+\sum_{p=\frac{1}{2}}^{n-1}\{[\widehat{\theta}(p-\tfrac{1}{2},p)\widehat{e}(p-1)\\&+\sum_{s=0}^{p-\frac{3}{2}}|\widehat{\theta}(s,p)-\widehat{\theta}(s,p-1)|\widehat{e}(s-\tfrac{1}{2})]\times\\&\prod_{\xi=p+1}^{n}[1+\widehat{\theta}(\xi-\tfrac{1}{2},\xi)\\&+\sum_{s=0}^{\xi-\frac{3}{2}}|\widehat{\theta}(s,\xi)-\widehat{\theta}(s,\xi-1)|]\}\}^{\frac{1}{3}},\ n\in\mathbf{N}_{\frac{1}{2}},\end{aligned}\quad(40)$$

where

$$\begin{cases}\widehat{\theta}(s,n)=\frac{1}{\Gamma(\frac{1}{2})}(n-s-1)^{(-\frac{1}{2})}\frac{1}{3}K^{-\frac{2}{3}}g(s-\frac{1}{2}),\\ \widehat{e}(n)=\frac{1}{\Gamma(\frac{1}{2})}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}g(s-\frac{1}{2})\\ \qquad[\frac{1}{3}K^{-\frac{2}{3}}\frac{(s-\frac{1}{2})^{(-\frac{1}{2})}}{\Gamma(\frac{1}{2})}|u_0|+\frac{2}{3}K^{\frac{1}{3}}],\end{cases}\quad(41)$$

and $K>0$ is an arbitrary constant.

Proof. Similar to Theorem 5, the equivalent discrete fractional sum equation of the IVP (39) can be denoted as follows:

$$u^3(n)=\frac{n^{(-\frac{1}{2})}}{\Gamma(\frac{1}{2})}u_0+\frac{1}{\Gamma(\frac{1}{2})}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}g(s-\tfrac{1}{2})u(s-\tfrac{1}{2}).$$

So

$$|u(n)|^3\le\frac{n^{(-\frac{1}{2})}}{\Gamma(\frac{1}{2})}|u_0|+\frac{1}{\Gamma(\frac{1}{2})}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}g(s-\tfrac{1}{2})|u(s-\tfrac{1}{2})|.\quad(42)$$

Applying Theorem 8 (with $n_1=3,\ n_2=1$) to (42) yields the desired result.

V. Conclusions

In this paper, by use of the properties of discrete fractional calculus, we have presented some new Gronwall-Bellmann type discrete fractional sum inequalities as well as some Volterra-Fredholm type discrete inequalities. Based on these inequalities, explicit estimates for the unknown functions concerned have been established. As for applications, we apply the results to research boundedness and continuous dependence on the initial value for the solution of the IVP of one certain fractional difference equation. It is worthy to note that the main inequalities in this paper can be extended to other discrete inequalities of more general forms, especially to 2D case, which are supposed to further research.

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