Oscillation of Even Order Nonlinear Functional Dynamic Equations on Time Scales

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***Abstract*—The paper investigates the oscillation of even order nonlinear functional dynamic equation of the form**

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta}+q(t)\left[x^{\sigma}(\tau(t))\right]^{\beta}=0$$

on $[t_0,\infty)_{\mathbb{T}}$ with $\sup\mathbb{T}=\infty$ and $\lim_{t\to\infty}\tau(t)=\infty$. We establish several new oscillation criteria for the equation by employing Kiguradze's lemma, the Taylor monomials on time scales and a generalized Riccati technique. The obtained results extend and supplement certain known results in the literature. Some interesting examples are shown to illustrate our main results.

***Index Terms*—oscillation, even order, functional dynamic equation, time scale.**

I. Introduction

In this paper, we discuss the oscillation of even order nonlinear functional dynamic equation of the form

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta}+q(t)\left[x^{\sigma}(\tau(t))\right]^{\beta}=0 \qquad (1.1)$$

on $[t_0,\infty)_{\mathbb{T}}$, where $\mathbb{T}$ is an arbitrary time scale, $\sup\mathbb{T}=\infty$, $t_0\in\mathbb{T}$, $[t_0,\infty)_{\mathbb{T}}:=\{t\in\mathbb{T}:t\geq t_0\}$ is a time scale interval in $\mathbb{T}$, $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$ is the forward jump operator on $\mathbb{T}$ and $x^{\sigma}:=x\circ\sigma$. The following conditions will be used:

(H_1) $n\geq 4$ is even, and α and β are quotients of odd positive integers;

(H_2) $\varphi,q\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\int_{t_0}^{\infty}\varphi^{-1/\alpha}(t)\Delta t=\infty$, $\tau\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ and $\lim_{t\to\infty}\tau(t)=\infty$;

(H_3) $\varphi^{\Delta}(t)\geq 0$ for $t\in[t_0,\infty)_{\mathbb{T}}$.

We will consider respectively the following three cases:

$$\int_{t_0}^{\infty}q(u)\Delta u=\infty, \qquad (1.2)$$

$$\left.\begin{array}{l}\int_{t_0}^{\infty}q(u)\Delta u<\infty,\\ \int_{t_0}^{\infty}\left(\varphi^{-1}(s)\int_{s}^{\infty}q(u)\Delta u\right)^{1/\alpha}\Delta s=\infty,\end{array}\right\} \qquad (1.3)$$

$$\left.\begin{array}{l}\int_{t_0}^{\infty}q(u)\Delta u<\infty,\\ \int_{t_0}^{\infty}\left(\varphi^{-1}(s)\int_{s}^{\infty}q(u)\Delta u\right)^{1/\alpha}\Delta s<\infty,\\ \text{and}\\ \int_{t_0}^{\infty}\left[\int_{v}^{\infty}\left(\varphi^{-1}(s)\int_{s}^{\infty}q(u)\Delta u\right)^{1/\alpha}\Delta s\right]\Delta v=\infty.\end{array}\right\} \qquad (1.4)$$

By a solution of (1.1) we mean a nontrivial real function $x\in C_{\mathrm{rd}}^{n}[t_x,\infty)_{\mathbb{T}}$ which has the property that $\varphi(x^{\Delta^{n-1}})^{\alpha}\in C_{\mathrm{rd}}^{1}[t_x,\infty)_{\mathbb{T}}$ for a certain $t_x\geq t_0$ and satisfies (1.1) on $[t_x,\infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.1) which exist on $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|:t>t_*\}>0$ for any $t_*\geq t_x$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$. The study of dynamic equations on time scales is a fairly new topic, and work in this area is rapidly growing. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale. In this way results not only related to the set of real numbers $\mathbb{R}$ or the set of integers $\mathbb{Z}$ but those pertaining to more general time scales are obtained. Therefore, not only can the theory of dynamic equations on time scales unify the theories of differential equations and of difference equations, but it is also able to extend these classical cases to cases "in between," e.g., to the so-called q-difference equations. Dynamic equations on time scales have many applications in biology, engineering, economics, physics, neural networks, social sciences and so on. A book on the subject of time scales, by Bohner and Peterson [38], summarizes and organizes much of time scale calculus, see also the book by Bohner and Peterson [39] for advances in dynamic equations on time scales. For convenience of the readers and completeness of the paper, in the next section we recall some basic concepts and results for the calculus on time scales. More details can be found in [38] and [39].

In the past years, there have been a lot of papers studying the oscillation and nonoscillation of solutions of first to fourth order dynamic equations on time scales, such as the papers [1]–[22]. However, there are few papers dealing with the oscillation of general higher order dynamic equations, and we refer the reader to the papers [23]–[33] and the references cited therein.

When $n=2$, many papers such as the papers [34]–[37] have studied deeply the oscillation of (1.1) or some of its special cases. When $n\geq 4$, to the best of our knowledge, there is very little known about the oscillatory behavior of (1.1).

Recently, the authors in [28] obtained some oscillation criteria for higher order nonlinear dynamic equation on time scales of the form

$$x^{\Delta^{n}}(t)+q(t)\left[x^{\sigma}(\tau(t))\right]^{\lambda}=0 \qquad (1.5)$$

in delay $\tau(t)\leq t$ and non-delay $\tau(t)=t$ cases, where $n\geq 2$ is an integer and λ is a ratio of positive odd integers.

The paper [29] presented several oscillation results for the even order dynamic equation

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta}+q(t)\left[x^{\sigma}(t)\right]^{\beta}=0 \qquad (1.6)$$

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on a time scale $\mathbb{T}$, where $n \geq 2$ is even, and α, β, φ and q are defined as in (H_1)–(H_3).

In [30], Grace *et al.* derived some sufficient conditions for the oscillation of (1.1) when $n \geq 2$ is even, $\alpha, \beta, \varphi, q$ and τ are defined as in (H_1)–(H_3), $\tau(t) \leq t$ and $\tau^{\Delta}(t) > 0$ on $\mathbb{T}$.

It is clear that (1.5) and (1.6) are some special cases of (1.1). The results in [28]–[30] are very significant for the development of the qualitative theory of higher order nonlinear dynamic equations on time scales. However, the papers [28]–[30] did not consider the cases where (1.2), (1.3) and (1.4) hold. Therefore, it is of great interest to establish some oscillation criteria for (1.1) in these cases. In this paper, by using Kiguradze's lemma, the Taylor monomials on time scales and a generalized Riccati substitution, we obtain several oscillation theorems for (1.1) in the cases where (1.2), (1.3) and (1.4) hold. Our results are essentially new, and extend and supplement some of those in [28]–[30]. We also give several examples to illustrate the applications of our main results.

In what follows, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

The rest of the paper is organized as follows. In the next section, we present some basic definitions and results concerning the calculus on time scales. In Section III, we give several useful lemmas. In Section IV, we state and prove our main results. In the last section, some examples are considered to illustrate our main results.

II. Preliminaries on time scales

In this section, we recall the following concepts and results related to the notion of time scales, which are contained in [38] and [39].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Some examples of time scales are as follows: the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$, $h\mathbb{Z} := \{hk : k \in \mathbb{Z}, h > 0\}$ and $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$. But the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$ and the open interval $(0,1)$ are not time scales. Many other interesting time scales exist, and they give rise to plenty of applications (see [38]).

For $t \in \mathbb{T}$, the forward jump operator and the backward jump operator are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

and

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively, where $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), here $\emptyset$ denotes the empty set.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \tag{2.1}$$

We also need below the set $\mathbb{T}^{\kappa}$: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$, then we define the function $f^{\sigma} : \mathbb{T}^{\kappa} \to \mathbb{R}$ by

$$f^{\sigma}(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}^{\kappa},$$

i.e., $f^{\sigma} := f \circ \sigma$.

For $a, b \in \mathbb{T}$ with $a < b$, we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals, etc. are defined accordingly.

Fix $t \in \mathbb{T}^{\kappa}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. In this case, we say that $f^{\Delta}(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t.

For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative f^{Δ^2} provided f^{Δ} is (delta) differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative

$$f^{\Delta^2} = (f^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}.$$

Similarly, we define higher order derivatives

$$f^{\Delta^n} = (f^{\Delta^{n-1}})^{\Delta} : \mathbb{T}^{\kappa^n} \to \mathbb{R} \quad \text{for } n = 3, 4, \cdots,$$

where $\mathbb{T}^{\kappa^n} := \left(\mathbb{T}^{\kappa^{n-1}}\right)^{\kappa}$. For convenience we also put

$$f^{\Delta^0} = f, f^{\Delta^1} = f^{\Delta}, f^{\Delta\Delta} = f^{\Delta^2}, f^{\Delta\Delta\Delta} = f^{\Delta^3},$$

$\mathbb{T}^{\kappa^0} = \mathbb{T}$ and $\mathbb{T}^{\kappa^1} = \mathbb{T}^{\kappa}$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^{\kappa}$. If f is (delta) differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by

$$C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are (delta) differentiable and whose (delta) derivative is rd-continuous is denoted by

$$C^1_{\mathrm{rd}}(\mathbb{T}) = C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

We will make use of the following product and quotient rules for the (delta) derivatives of the product fg and the quotient f/g of two (delta) differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma} \tag{2.2}$$

and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}, \tag{2.3}$$

where $g^{\sigma} = g \circ \sigma$ and $g g^{\sigma} \neq 0$.

For $a, b \in \mathbb{T}$ and a (delta) differentiable function f, the Cauchy (delta) integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t) \Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t \tag{2.4}$$

or

$$\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t.$$

The infinite integral is defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

III. Lemmas

In this section, we present some lemmas which are used in the following sections.

Lemma 3.1. ([40, Theorem 5]) *Let* $\sup\mathbb{T} = \infty$ *and* $x \in C^m_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$. *If* $x^{\Delta^m}(t)$ *is of constant sign on* $[t_0,\infty)_{\mathbb{T}}$ *and not identically zero on* $[t_1,\infty)_{\mathbb{T}}$ *for any* $t_1 \ge t_0$, *then there exist a* $t_x \ge t_0$ *and an integer* l, $0 \le l \le m$, *with* $m+l$ *even for* $x^{\Delta^m}(t) \ge 0$, *or* $m+l$ *odd for* $x^{\Delta^m}(t) \le 0$, *such that*

$$l > 0 \text{ implies } x^{\Delta^i}(t) > 0 \quad \text{for } t \ge t_x, i = 0,1,\cdots,l, \tag{3.1}$$

and

$$l \le m-1 \text{ implies } (-1)^{l+i}x^{\Delta^i}(t) > 0 \tag{3.2}$$

for $t \ge t_x, i = l, l+1,\cdots,m-1$.

The next lemma needs the Taylor monomials $\{h_i\}_{i=0}^{\infty}$ (see [38, Section 1.6]), which are defined recursively by

$$h_0(t,s) = 1 \quad \text{and} \quad h_i(t,s) = \int_s^t h_{i-1}(\zeta,s)\Delta\zeta \tag{3.3}$$

for $t,s \in [t_0,\infty)_{\mathbb{T}}$ and $i \in \mathbb{N}$.

Lemma 3.2. *Let* x *be an eventually positive solution of (1.1) and assume that* (H_1)*–*(H_3) *and (1.4) hold, then there exists* $t_x \in [t_0,\infty)_{\mathbb{T}}$ *such that for* $t \in [t_x,\infty)_{\mathbb{T}}$ *and* $i = 0,1,\cdots,n-1$,

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta} < 0, \quad x^{\Delta^i}(t) > 0, \tag{3.4}$$

$$\left[\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta} < 0, \tag{3.5}$$

and

$$x^{\Delta}(t) \ge x^{\Delta^{n-1}}(t)h_{n-2}(t,t_x), \tag{3.6}$$

where h_{n-2} *is defined as in (3.3).*

Proof. Since x is an eventually positive solution of (1.1), there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$x(t) > 0 \quad \text{and} \quad x^{\sigma}(\tau(t)) > 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.7}$$

Therefore, from (1.1) we have

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta} = -q(t)\left[x^{\sigma}(\tau(t))\right]^{\beta} < 0 \tag{3.8}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$, which implies that $\varphi(t)\big(x^{\Delta^{n-1}}(t)\big)^{\alpha}$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$ and eventually of one sign. Hence, $x^{\Delta^{n-1}}(t)$ is eventually of one sign, i.e., $x^{\Delta^{n-1}}(t)$ is either eventually positive or eventually negative. We now claim

$$x^{\Delta^{n-1}}(t) > 0 \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.9}$$

Assume that (3.9) doesn't hold, then there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $x^{\Delta^{n-1}}(t) < 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Thus, we conclude that $-\varphi(t)\big(x^{\Delta^{n-1}}(t)\big)^{\alpha} = \varphi(t)\big(-x^{\Delta^{n-1}}(t)\big)^{\alpha}$ is positive and strictly increasing on $[t_2,\infty)_{\mathbb{T}}$ and that $\varphi^{1/\alpha}(t)\big(-x^{\Delta^{n-1}}(t)\big)$ is also positive and strictly increasing on $[t_2,\infty)_{\mathbb{T}}$. Therefore, we get

$$\begin{aligned} &x^{\Delta^{n-2}}(t) \\ &= x^{\Delta^{n-2}}(t_2) + \int_{t_2}^t x^{\Delta^{n-1}}(s)\Delta s \\ &= x^{\Delta^{n-2}}(t_2) - \int_{t_2}^t \left[\varphi^{1/\alpha}(s)\big(-x^{\Delta^{n-1}}(s)\big)\right]\varphi^{-1/\alpha}(s)\Delta s \\ &\le x^{\Delta^{n-2}}(t_2) - c_1\int_{t_2}^t \varphi^{-1/\alpha}(s)\Delta s \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}, \end{aligned}$$

where $c_1 := \varphi^{1/\alpha}(t_2)\big(-x^{\Delta^{n-1}}(t_2)\big) > 0$. Letting $t \to \infty$ and using the condition $\int_{t_0}^{\infty}\varphi^{-1/\alpha}(t)\Delta t = \infty$, we obtain $\lim_{t\to\infty} x^{\Delta^{n-2}}(t) = -\infty$. Analogously, we have

$$\begin{aligned} \lim_{t\to\infty} x^{\Delta^{n-3}}(t) &= \lim_{t\to\infty} x^{\Delta^{n-4}}(t) = \cdots \\ &= \lim_{t\to\infty} x^{\Delta}(t) = \lim_{t\to\infty} x(t) = -\infty, \end{aligned}$$

which contradicts the fact that $x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Hence, (3.9) holds. Take $m = n-1$. From Lemma 3.1, we get that there exist $t_x \in [t_2,\infty)_{\mathbb{T}}$ and an odd integer $l, 1 \le l \le m = n-1$, such that (3.1) and (3.2) hold. Thus, we have

$$x^{\Delta}(t) > 0 \quad \text{for } t \in [t_x,\infty)_{\mathbb{T}}. \tag{3.10}$$

Take $t_3 \in [t_x,\infty)_{\mathbb{T}}$ such that $\tau(t) \ge t_x$ for $t \in [t_3,\infty)_{\mathbb{T}}$, then we have $\sigma(\tau(t)) \ge \tau(t) \ge t_x$ for $t \in [t_3,\infty)_{\mathbb{T}}$. From (3.10) we conclude $x^{\sigma}(\tau(t)) \ge x(t_x) > 0$ for $t \in [t_3,\infty)_{\mathbb{T}}$. Hence, from (3.8) we get

$$\left[\varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right]^{\Delta} \le -q(t)x^{\beta}(t_x) := -c_2q(t) \tag{3.11}$$

for $t \in [t_3,\infty)_{\mathbb{T}}$, where $c_2 := x^{\beta}(t_x) > 0$. Integrating (3.11) from t to v, we obtain

$$\begin{aligned} \varphi(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha} &\ge \varphi(v)\left(x^{\Delta^{n-1}}(v)\right)^{\alpha} + c_2\int_t^v q(u)\Delta u \\ &> c_2\int_t^v q(u)\Delta u \end{aligned} \tag{3.12}$$

for $v \ge t \ge t_3$. Letting $v \to \infty$, we have

$$x^{\Delta^{n-1}}(t) \ge c_2^{1/\alpha}\left(\varphi^{-1}(t)\int_t^{\infty} q(u)\Delta u\right)^{1/\alpha} \tag{3.13}$$

for $t \in [t_3,\infty)_{\mathbb{T}}$. Now, we declare $l = m = n-1$. Otherwise, we obtain $l \le m-2 = n-3$. Therefore, it follows from (3.2) that $x^{\Delta^{m-1}}(t) = x^{\Delta^{n-2}}(t) < 0$ and

$x^{\Delta^{m-2}}(t) = x^{\Delta^{n-3}}(t) > 0$ for $t \in [t_x, \infty)_{\mathbb{T}}$. Integrating both sides of (3.13) from t to v, we get

$$\begin{aligned} -x^{\Delta^{n-2}}(t) &> x^{\Delta^{n-2}}(v) - x^{\Delta^{n-2}}(t) \\ &\geq c_2^{1/\alpha} \int_t^v \left(\varphi^{-1}(s) \int_s^\infty q(u)\Delta u\right)^{1/\alpha} \Delta s \end{aligned}$$

for $v \geq t \geq t_3$. Letting $v \to \infty$, we obtain

$$-x^{\Delta^{n-2}}(t) \geq c_2^{1/\alpha} \int_t^\infty \left(\varphi^{-1}(s) \int_s^\infty q(u)\Delta u\right)^{1/\alpha} \Delta s$$

for $t \in [t_3, \infty)_{\mathbb{T}}$. Integrating both sides of the last inequality from t_3 to t, we have for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$\begin{aligned} &x^{\Delta^{n-3}}(t_3) \\ &> -x^{\Delta^{n-3}}(t) + x^{\Delta^{n-3}}(t_3) \\ &\geq c_2^{1/\alpha} \int_{t_3}^t \left[\int_v^\infty \left(\varphi^{-1}(s) \int_s^\infty q(u)\Delta u\right)^{1/\alpha} \Delta s\right]\Delta v. \end{aligned}$$

Letting $t \to \infty$, we find

$$\begin{aligned} &\int_{t_3}^\infty \left[\int_v^\infty \left(\varphi^{-1}(s) \int_s^\infty q(u)\Delta u\right)^{1/\alpha} \Delta s\right]\Delta v \\ &\leq c_2^{-1/\alpha} x^{\Delta^{n-3}}(t_3) \\ &< \infty, \end{aligned}$$

which contradicts the third formula in (1.4). Thus, we have $l = m = n-1$ and then from (3.1) and (3.8) we get that (3.4) holds. Since $\varphi^\Delta(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and

$$\begin{aligned} &\left[\varphi(t)\Big(x^{\Delta^{n-1}}(t)\Big)^\alpha\right]^\Delta \\ &= \varphi^\Delta(t)\Big(x^{\Delta^{n-1}}(t)\Big)^\alpha + \varphi^\sigma(t)\left[\Big(x^{\Delta^{n-1}}(t)\Big)^\alpha\right]^\Delta \\ &< 0 \quad \text{for } t \in [t_x, \infty)_{\mathbb{T}}, \end{aligned}$$

we obtain $\left[\Big(x^{\Delta^{n-1}}(t)\Big)^\alpha\right]^\Delta < 0$ for $t \in [t_x, \infty)_{\mathbb{T}}$, which is namely (3.5). Therefore, we see that $x^{\Delta^{n-1}}(t)$ is strictly decreasing on $[t_x, \infty)_{\mathbb{T}}$. Hence, from (3.4) we obtain

$$\begin{aligned} x^{\Delta^{n-2}}(t) &= x^{\Delta^{n-2}}(t_x) + \int_{t_x}^t x^{\Delta^{n-1}}(\zeta)\Delta\zeta \\ &\geq x^{\Delta^{n-1}}(t) \int_{t_x}^t \Delta\zeta \\ &:= x^{\Delta^{n-1}}(t) h_1(t, t_x) \quad \text{for } t \in [t_x, \infty)_{\mathbb{T}}. \end{aligned}$$

Integrating the last inequality from t_x to t, we find

$$\begin{aligned} x^{\Delta^{n-3}}(t) &\geq x^{\Delta^{n-3}}(t_x) + \int_{t_x}^t x^{\Delta^{n-1}}(\zeta) h_1(\zeta, t_x)\Delta\zeta \\ &\geq x^{\Delta^{n-1}}(t) \int_{t_x}^t h_1(\zeta, t_x)\Delta\zeta \\ &:= x^{\Delta^{n-1}}(t) h_2(t, t_x) \quad \text{for } t \in [t_x, \infty)_{\mathbb{T}}. \end{aligned}$$

Repeating the above argument, we have

$$x^\Delta(t) \geq x^{\Delta^{n-1}}(t) h_{n-2}(t, t_x) \quad \text{for } t \in [t_x, \infty)_{\mathbb{T}},$$

which is namely (3.6). The proof is complete.

Lemma 3.3. ([41, Lemma 2.3]) *Suppose that following condition holds:*

(H_4) *$\tau^\Delta(t) > 0$ is rd-continuous on $\mathbb{T}$, $\tilde{\mathbb{T}} := \tau(\mathbb{T}) = \{\tau(t) : t \in \mathbb{T}\} \subset \mathbb{T}$ is a time scale, and $(\tau^\sigma)(t) = (\sigma \circ \tau)(t)$ for all $t \in \mathbb{T}$, where $(\tau^\sigma)(t) := (\tau \circ \sigma)(t)$.*

Let $y : \mathbb{T} \to \mathbb{R}$. If $y^\Delta(t)$ exists for all sufficiently large $t \in \mathbb{T}$, then $(y \circ \tau)^\Delta(t) = (y^\Delta \circ \tau)(t)\tau^\Delta(t)$ for all sufficiently large $t \in \mathbb{T}$.

Lemma 3.4. ([41, Lemma 2.4]) *Let $F : \mathbb{T} \to \mathbb{R}$ and $\gamma > 0$ be a constant. Furthermore, assume $F^\Delta(t) > 0$ and $F(t) > 0$ for all sufficiently large $t \in \mathbb{T}$. Then we have the following:*

(i) If $0 < \gamma < 1$, then $(F^\gamma)^\Delta(t) \geq \gamma(F^\sigma)^{\gamma-1}(t)F^\Delta(t)$ for all sufficiently large $t \in \mathbb{T}$, where $F^\sigma := F \circ \sigma$;

(ii) If $\gamma \geq 1$, then $(F^\gamma)^\Delta(t) \geq \gamma F^{\gamma-1}(t)F^\Delta(t)$ for all sufficiently large $t \in \mathbb{T}$.

Lemma 3.5. ([42]) *If X and Y are nonnegative, then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1)Y^\lambda \quad for\ \lambda > 1,$$

where the equality holds if and only if X = Y.

Lemma 3.6. ([39, Theorem 5.68]) *Suppose that $\mathbb{T}$ is unbounded above. Let $t_0 \in \mathbb{T}, t_0 > 0$ and $0 \leq \lambda \leq 1$. Then we have $\int_{t_0}^\infty \frac{1}{t^\lambda}\Delta t = \infty$.*

Lemma 3.7. ([38, Parts (ii) and (iii) of Theorem 1.79]) *Let $a, b \in \mathbb{T}, a < b$ and $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.*

(i) If $[a, b]$ consists of finitely many isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t),$$

where μ is defined as in (2.1).

(ii) If $\mathbb{T} = c\mathbb{Z} := \{ck : k \in \mathbb{Z}\}$, where $c > 0$ is a constant, then

$$\int_a^b f(t)\Delta t = \sum_{k=a/c}^{k=b/c-1} cf(kc).$$

Lemma 3.8. ([38, Example 1.104]) *Let $q_0 > 1$ be a constant and $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} := q_0^{\mathbb{Z}} \cup \{0\} := \{q_0^k : k \in \mathbb{Z}\} \cup \{0\}$, then*

$$h_k(t, s) = \prod_{j=0}^{k-1} \frac{t - q_0^j s}{\sum_{i=0}^j q_0^i} \quad for\ all\ t, s \in \mathbb{T}\ and\ all\ k \in \mathbb{N}_0,$$

where $h_k(t, s) (k \in \mathbb{N}_0)$ are defined as in (3.3).

IV. Main results

In this section, we state and prove our main results. We establish several new oscillation criteria for (1.1).

Theorem 4.1. *Suppose that (H_1), (H_2) and (1.2) hold, then (1.1) is oscillatory.*

Proof. Assume that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Proceeding as in the proof of Lemma 3.2, we find that there exists $t_3 \in [t_0, \infty)_{\mathbb{T}}$ such that (3.12) holds. Take $t = t_3$, then from (3.12) we conclude

$$\int_{t_3}^v q(u)\Delta u < c_2^{-1}\varphi(t_3)\big(x^{\Delta^{n-1}}(t_3)\big)^\alpha \quad \text{for } v \in [t_3, \infty)_{\mathbb{T}}.$$

Letting $v \to \infty$, we get

$$\int_{t_3}^{\infty} q(u)\Delta u \le c_2^{-1}\varphi(t_3)\big(x^{\Delta^{n-1}}(t_3)\big)^{\alpha} < \infty,$$

which contradicts (1.2). The proof is complete.

Theorem 4.2. *Suppose that (H_1), (H_2) and (1.3) hold, then every bounded solution of (1.1) is oscillatory.*

Proof. Assume on the contrary that there is a nonoscillatory bounded solution x of (1.1). Without loss of generality, we may assume that x is a bounded eventually positive solution of (1.1). Proceeding as in the proof of Lemma 3.2, we see that there exists $t_3 \in [t_0,\infty)_{\mathbb{T}}$ such that (3.13) holds. Integrating both sides of (3.13) from t_3 to t, we get

$$\begin{aligned} & x^{\Delta^{n-2}}(t) \\ & \ge x^{\Delta^{n-2}}(t_3) + c_2^{1/\alpha}\int_{t_3}^{t}\left(\varphi^{-1}(s)\int_s^{\infty} q(u)\Delta u\right)^{1/\alpha}\Delta s \end{aligned}$$

for $t \in [t_3,\infty)_{\mathbb{T}}$. Letting $t \to \infty$ and using the second formula in (1.3), we get $\lim_{t\to\infty} x^{\Delta^{n-2}}(t) = \infty$. Take a constant $c_3 > 0$, then there exists $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $x^{\Delta^{n-2}}(t) \ge c_3$ for $t \in [t_4,\infty)_{\mathbb{T}}$. Integrating from t_4 to t, we obtain

$$\begin{aligned} x^{\Delta^{n-3}}(t) &\ge x^{\Delta^{n-3}}(t_4) + \int_{t_4}^{t} c_3\Delta s \\ &= x^{\Delta^{n-3}}(t_4) + c_3(t-t_4) \quad \text{for } t \in [t_4,\infty)_{\mathbb{T}}. \end{aligned}$$

Letting $t \to \infty$, we conclude $\lim_{t\to\infty} x^{\Delta^{n-3}}(t) = \infty$. Similarly, we find

$$\lim_{t\to\infty} x^{\Delta^{n-4}}(t) = \cdots = \lim_{t\to\infty} x^{\Delta}(t) = \lim_{t\to\infty} x(t) = \infty,$$

which contradicts the boundedness of x. The proof is complete.

Theorem 4.3. *Suppose that (H_1)–(H_4), (1.4) and the following conditions hold:*

(H_5) *$\tau(t) \le t$ for $t \in [t_0,\infty)_{\mathbb{T}}$;*

(H_6) *There exists a positive function $\rho \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$\begin{aligned} &\limsup_{t\to\infty}\int_{T_2}^{t}\Bigg[\rho(s)q(s) - \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s,T_1)(\alpha+1)^{\alpha+1}} \times \\ &\qquad \frac{(\rho_+^{\Delta}(s))^{\alpha+1}}{[\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s),T_1)]^{\alpha}}\Bigg]\Delta s \\ &= \infty \end{aligned} \tag{4.1}$$

for all sufficiently large $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$, where

$$\Phi(s,T_1) := \begin{cases} \varepsilon, if\ \beta > \alpha, \\ 1, if\ \beta = \alpha, \\ \left[\varepsilon_0 + \varepsilon_1 h_1(\tau^{\sigma}(s),T_1) + \varepsilon_2 h_2(\tau^{\sigma}(s),T_1) + \cdots \right. \\ \left. +\varepsilon_{n-1}h_{n-1}(\tau^{\sigma}(s),T_1)\right]^{\frac{\beta}{\alpha}-1}, if\ \beta < \alpha, \end{cases}$$

$\varepsilon, \varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1}$ are arbitrary positive constants, T_2 satisfies $\tau(s) > T_1$ for $s \in [T_2,\infty)_{\mathbb{T}}$, $\rho_+^{\Delta}(s) := \max\{\rho^{\Delta}(s), 0\}$ and h_i ($i = 1, 2, \cdots, n-1$) are defined as in (3.3).

Then (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Then there exists $t_x \in [t_0,\infty)_{\mathbb{T}}$ such that (3.4)–(3.6) hold. Take $t_5 \in [t_x,\infty)_{\mathbb{T}}$ such that $\tau(t) > t_x$ for $t \in [t_5,\infty)_{\mathbb{T}}$, then we obtain

$$\sigma(\tau(t)) \ge \tau(t) > t_x \quad \text{for } t \in [t_5,\infty)_{\mathbb{T}}. \tag{4.2}$$

Hence, from (3.4) we get

$$x^{\sigma}(\tau(t)) \ge x(\tau(t)) > x(t_x) > 0 \quad \text{for } t \in [t_5,\infty)_{\mathbb{T}}. \tag{4.3}$$

Define the function w by the generalized Riccati substitution

$$w(t) = \rho(t)\frac{\varphi(t)\big(x^{\Delta^{n-1}}(t)\big)^{\alpha}}{[x(\tau(t))]^{\beta}} \quad \text{for } t \in [t_5,\infty)_{\mathbb{T}}. \tag{4.4}$$

We have $w(t) > 0$ for $t \in [t_5,\infty)_{\mathbb{T}}$. By the product and quotient rules (2.2) and (2.3) for taking delta derivatives, from (4.4) and then from (1.1) and (4.3) we obtain

$$\begin{aligned} w^{\Delta} &= \Big[\varphi\big(x^{\Delta^{n-1}}\big)^{\alpha}\Big]^{\Delta}\frac{\rho}{(x\circ\tau)^{\beta}} \\ &\quad + \Big(\varphi\big(x^{\Delta^{n-1}}\big)^{\alpha}\Big)^{\sigma}\left[\frac{\rho}{(x\circ\tau)^{\beta}}\right]^{\Delta} \\ &= \Big[\varphi\big(x^{\Delta^{n-1}}\big)^{\alpha}\Big]^{\Delta}\frac{\rho}{(x\circ\tau)^{\beta}} + \Big(\varphi\big(x^{\Delta^{n-1}}\big)^{\alpha}\Big)^{\sigma}\times \\ &\quad \left[\frac{\rho^{\Delta}}{((x\circ\tau)^{\beta})^{\sigma}} - \frac{\rho[(x\circ\tau)^{\beta}]^{\Delta}}{(x\circ\tau)^{\beta}((x\circ\tau)^{\beta})^{\sigma}}\right] \\ &= \Big[\varphi\big(x^{\Delta^{n-1}}\big)^{\alpha}\Big]^{\Delta}\frac{\rho}{(x^{\sigma}\circ\tau)^{\beta}}\frac{(x^{\sigma}\circ\tau)^{\beta}}{(x\circ\tau)^{\beta}} + \frac{\rho^{\Delta}}{\rho^{\sigma}}w^{\sigma} \\ &\quad - \rho\frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}[(x\circ\tau)^{\beta}]^{\Delta}}{(x\circ\tau)^{\beta}(x\circ\tau^{\sigma})^{\beta}} \\ &\le -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}}w^{\sigma} - \rho\frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}[(x\circ\tau)^{\beta}]^{\Delta}}{(x\circ\tau)^{\beta}(x\circ\tau^{\sigma})^{\beta}}, \end{aligned} \tag{4.5}$$

where ρ_+^{Δ} is defined as in Theorem 4.3. By Lemma 3.3 we get

$$(x\circ\tau)^{\Delta} = (x^{\Delta}\circ\tau)\tau^{\Delta} > 0. \tag{4.6}$$

If $0 < \beta < 1$, then by taking $F = x\circ\tau$ and by Lemma 3.4 (i) and (4.6) we get

$$\begin{aligned} [(x\circ\tau)^{\beta}]^{\Delta} &\ge \beta(x\circ\tau^{\sigma})^{\beta-1}(x\circ\tau)^{\Delta} \\ &= \beta(x\circ\tau^{\sigma})^{\beta-1}(x^{\Delta}\circ\tau)\tau^{\Delta}. \end{aligned} \tag{4.7}$$

From (4.5) and (4.7), it follows that

$$\begin{aligned} w^{\Delta} &\le -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}}w^{\sigma} \\ &\quad - \rho\frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}\cdot\beta(x\circ\tau^{\sigma})^{\beta-1}(x^{\Delta}\circ\tau)\tau^{\Delta}}{(x\circ\tau)^{\beta}(x\circ\tau^{\sigma})^{\beta}} \\ &= -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}}w^{\sigma} \\ &\quad - \beta\tau^{\Delta}\rho\frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}}{(x\circ\tau^{\sigma})^{\beta+1}}\cdot\frac{(x\circ\tau^{\sigma})^{\beta}}{(x\circ\tau)^{\beta}}(x^{\Delta}\circ\tau). \end{aligned} \tag{4.8}$$

If $\beta \ge 1$, then by taking $F = x\circ\tau$ and by Lemma 3.4 (ii) and (4.6) we have

$$\begin{aligned} [(x\circ\tau)^{\beta}]^{\Delta} &\ge \beta(x\circ\tau)^{\beta-1}(x\circ\tau)^{\Delta} \\ &= \beta(x\circ\tau)^{\beta-1}(x^{\Delta}\circ\tau)\tau^{\Delta}. \end{aligned} \tag{4.9}$$

It follows from (4.5) and (4.9) that

$$\begin{aligned} w^{\Delta} &\le -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}} w^{\sigma} \\ &\quad - \rho \frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma} \cdot \beta (x\circ\tau)^{\beta-1}(x^{\Delta}\circ\tau)\tau^{\Delta}}{(x\circ\tau)^{\beta}(x\circ\tau^{\sigma})^{\beta}} \\ &= -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}} w^{\sigma} \\ &\quad - \beta\tau^{\Delta}\rho \frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}}{(x\circ\tau^{\sigma})^{\beta+1}} \cdot \frac{(x\circ\tau^{\sigma})}{(x\circ\tau)}(x^{\Delta}\circ\tau). \quad (4.10) \end{aligned}$$

From (H_4) we see that $\tau(t)$ is increasing on $\mathbb{T}$. Since $t \le \sigma(t)$, we have $\tau(t) \le \tau^{\sigma}(t)$. Since $x^{\Delta}(t) > 0$, we obtain $(x\circ\tau)(t) \le (x\circ\tau^{\sigma})(t)$. Thus, for all $\beta > 0$, from (4.8) and (4.10) we get

$$w^{\Delta} \le -\rho q + \frac{\rho_+^{\Delta}}{\rho^{\sigma}} w^{\sigma} - \beta\tau^{\Delta}\rho \frac{\big(\varphi(x^{\Delta^{n-1}})^{\alpha}\big)^{\sigma}}{(x\circ\tau^{\sigma})^{\beta+1}}(x^{\Delta}\circ\tau). \quad (4.11)$$

Since $\varphi(t)\big(x^{\Delta^{n-1}}(t)\big)^{\alpha}$ is decreasing on $[t_x, \infty)$ and $\tau(t) \le t \le \sigma(t)$, we have

$$((\varphi\circ\tau)(x^{\Delta^{n-1}}\circ\tau)^{\alpha})(t) \ge (\varphi(x^{\Delta^{n-1}})^{\alpha})^{\sigma}(t)$$

and

$$(x^{\Delta^{n-1}}\circ\tau)(t) \ge [(\varphi(x^{\Delta^{n-1}})^{\alpha})^{\sigma}(t)]^{1/\alpha}/[(\varphi\circ\tau)^{1/\alpha}(t)].$$

Thus, from (3.6) we have

$$\begin{aligned} &(x^{\Delta}\circ\tau)(t) \\ &\ge (x^{\Delta^{n-1}}\circ\tau)(t)h_{n-2}(\tau(t), t_x) \\ &\ge \frac{[(\varphi(x^{\Delta^{n-1}})^{\alpha})^{\sigma}(t)]^{1/\alpha} h_{n-2}(\tau(t), t_x)}{(\varphi\circ\tau)^{1/\alpha}(t)}. \quad (4.12) \end{aligned}$$

From (4.11) and (4.12), we conclude

$$\begin{aligned} w^{\Delta}(t) &\le -\rho(t)q(t) + \frac{\rho_+^{\Delta}(t)}{\rho^{\sigma}(t)} w^{\sigma}(t) - \beta\tau^{\Delta}(t)\rho(t)\times \\ &\quad \frac{[(\varphi(x^{\Delta^{n-1}})^{\alpha})^{\sigma}(t)]^{1+1/\alpha}}{(x\circ\tau^{\sigma})^{\beta+1}(t)} \frac{h_{n-2}(\tau(t), t_x)}{(\varphi\circ\tau)^{1/\alpha}(t)}. \quad (4.13) \end{aligned}$$

From (4.4) and (4.13) there exists $t_6 \in [t_5, \infty)$ such that

$$\begin{aligned} w^{\Delta}(t) &\le -\rho(t)q(t) + \frac{\rho_+^{\Delta}(t)}{\rho^{\sigma}(t)} w^{\sigma}(t) - \beta\tau^{\Delta}(t)\rho(t)\times \\ &\quad \frac{h_{n-2}(\tau(t), t_x)(x\circ\tau^{\sigma})^{\beta/\alpha-1}(t)}{(\varphi\circ\tau)^{1/\alpha}(t)} \left(\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}\right)^{1+1/\alpha} \end{aligned} \quad (4.14)$$

for $t \in [t_6, \infty)_{\mathbb{T}}$. Next, we consider respectively the three cases: $\beta > \alpha$, $\beta = \alpha$ and $\beta < \alpha$.

Case (i). Let $\beta > \alpha$. Since $\tau(t)$ is increasing on $\mathbb{T}$ and $\sigma(t) \ge t \ge t_6$ on $[t_6, \infty)_{\mathbb{T}}$, we get $\tau^{\sigma}(t) \ge \tau(t_6)$ and $(x\circ\tau^{\sigma})(t) \ge (x\circ\tau)(t_6) := c_4$ for $t \in [t_6, \infty)_{\mathbb{T}}$. Therefore, for $t \in [t_6, \infty)_{\mathbb{T}}$ we conclude

$$(x\circ\tau^{\sigma})^{\frac{\beta}{\alpha}-1}(t) \ge c_4^{\beta/\alpha-1} := \varepsilon > 0. \quad (4.15)$$

Case (ii). Let $\beta = \alpha$. Then we have

$$(x\circ\tau^{\sigma})^{\frac{\beta}{\alpha}-1}(t) = 1 \quad \text{for } t \in [t_6, \infty)_{\mathbb{T}}. \quad (4.16)$$

Case (iii). Let $\beta < \alpha$. From (3.5), we see that $x^{\Delta^{n-1}}(t)$ is strictly decreasing on $[t_6, \infty)_{\mathbb{T}}$. Take $t_7 \in [t_6, \infty)_{\mathbb{T}}$ such that $\tau(t) > t_6$ for $t \in [t_7, \infty)_{\mathbb{T}}$. Since $\tau(t)$ is increasing and $\sigma(t) \ge t$ on $\mathbb{T}$, we obtain

$$\tau^{\sigma}(t) \ge \tau(t) > t_6 \quad \text{for } t \in [t_7, \infty)_{\mathbb{T}} \quad (4.17)$$

and

$$x^{\Delta^{n-1}}(t) \le x^{\Delta^{n-1}}(t_6) \quad \text{for } t \in [t_6, \infty)_{\mathbb{T}}. \quad (4.18)$$

Integrating both sides of (4.18) from t_6 to t, we get

$$\begin{aligned} x^{\Delta^{n-2}}(t) &\le x^{\Delta^{n-2}}(t_6) + x^{\Delta^{n-1}}(t_6)\int_{t_6}^{t} \Delta s \\ &= x^{\Delta^{n-2}}(t_6) + x^{\Delta^{n-1}}(t_6)h_1(t, t_6) \end{aligned}$$

for $t \in [t_6, \infty)_{\mathbb{T}}$, where h_1 is defined as in (3.3). Similarly, Integrating both sides of (4.18) twice from t_6 to t, we conclude

$$\begin{aligned} x^{\Delta^{n-3}}(t) &\le x^{\Delta^{n-3}}(t_6) + x^{\Delta^{n-2}}(t_6)h_1(t, t_6) \\ &\quad + x^{\Delta^{n-1}}(t_6)h_2(t, t_6) \end{aligned}$$

for $t \in [t_6, \infty)_{\mathbb{T}}$. Integrating both sides of (4.18) (n-1)-times from t_6 to t, we find

$$\begin{aligned} x(t) &\le x(t_6) + x^{\Delta}(t_6)h_1(t, t_6) + x^{\Delta^2}(t_6)h_2(t, t_6) + \cdots \\ &\quad + x^{\Delta^{n-1}}(t_6)h_{n-1}(t, t_6) \\ &:= \varepsilon_0 + \varepsilon_1 h_1(t, t_6) + \varepsilon_2 h_2(t, t_6) + \cdots \\ &\quad + \varepsilon_{n-1}h_{n-1}(t, t_6) \quad \text{for } t \in [t_6, \infty)_{\mathbb{T}}, \quad (4.19) \end{aligned}$$

where $\varepsilon_i := x^{\Delta^i}(t_6) (i = 0, 1, 2, \cdots, n-1)$. From (4.17) and (4.19), it follows that for $t \in [t_7, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x(\tau^{\sigma}(t)) &\le \varepsilon_0 + \varepsilon_1 h_1(\tau^{\sigma}(t), t_6) + \varepsilon_2 h_2(\tau^{\sigma}(t), t_6) + \cdots \\ &\quad + \varepsilon_{n-1}h_{n-1}(\tau^{\sigma}(t), t_6). \end{aligned}$$

Therefore, we derive for $t \in [t_7, \infty)_{\mathbb{T}}$,

$$\begin{aligned} &\Big(x(\tau^{\sigma}(t))\Big)^{\beta/\alpha-1} \\ &\ge \Big[\varepsilon_0 + \varepsilon_1 h_1(\tau^{\sigma}(t), t_6) + \varepsilon_2 h_2(\tau^{\sigma}(t), t_6) + \cdots \\ &\quad + \varepsilon_{n-1}h_{n-1}(\tau^{\sigma}(t), t_6)\Big]^{\beta/\alpha-1}. \quad (4.20) \end{aligned}$$

Combining (4.15), (4.16) and (4.20), we conclude

$$\Big(x(\tau^{\sigma}(t))\Big)^{\beta/\alpha-1} \ge \Phi(t, t_6) > 0 \quad (4.21)$$

for $t \in [t_7, \infty)_{\mathbb{T}}$ and for all α, β, where Φ is defined as in Theorem 4.3. Hence, from (4.14) and (4.21) we find for $t \in [t_7, \infty)_{\mathbb{T}}$,

$$\begin{aligned} w^{\Delta}(t) &\le -\rho(t)q(t) + \frac{\rho_+^{\Delta}(t)}{\rho^{\sigma}(t)} w^{\sigma}(t) - \beta\tau^{\Delta}(t)\rho(t)\times \\ &\quad \frac{h_{n-2}(\tau(t), t_x)\Phi(t, t_6)}{(\varphi\circ\tau)^{1/\alpha}(t)} \left(\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}\right)^{1+1/\alpha}. \quad (4.22) \end{aligned}$$

Since $t_x \le t_6$ and $h_i(t, s)$ are decreasing in s provided that $t \ge s$ and $i \in \mathbb{N}_0$ (see [33, Corollary 2.3] and [43, Property 2.1]), we get

$$h_{n-2}(\tau(t), t_x) \ge h_{n-2}(\tau(t), t_6) > 0 \quad \text{for } t \in [t_7, \infty)_{\mathbb{T}}.$$

Therefore, it follows from (4.22) that for $t \in [t_7, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(t) \leq -\rho(t)q(t) + \frac{\rho_+^{\Delta}(t)}{\rho^{\sigma}(t)} w^{\sigma}(t) - \beta\tau^{\Delta}(t)\rho(t) \times \frac{h_{n-2}(\tau(t), t_6)\Phi(t, t_6)}{(\varphi \circ \tau)^{1/\alpha}(t)} \left(\frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}\right)^{1+1/\alpha}. \quad (4.23)$$

Taking $\lambda = 1 + 1/\alpha$,

$$X = \Big[\frac{\beta\tau^{\Delta}(t)\rho(t)h_{n-2}(\tau(t), t_6)\Phi(t, t_6)}{(\varphi \circ \tau)^{1/\alpha}(t)}\Big]^{1/\lambda} \frac{w^{\sigma}(t)}{\rho^{\sigma}(t)}$$

and

$$Y = \frac{[\rho_+^{\Delta}(t)]^{\alpha}}{\lambda^{\alpha}} \left[\frac{\beta\tau^{\Delta}(t)\rho(t)h_{n-2}(\tau(t), t_6)\Phi(t, t_6)}{(\varphi \circ \tau)^{1/\alpha}(t)}\right]^{-\alpha/\lambda}$$

for $t \in [t_7, \infty)_{\mathbb{T}}$, by Lemma 3.5 and (4.23) we have

$$w^{\Delta}(t) \leq -\rho(t)q(t) + \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(t))}{\Phi(t, t_6)(\alpha+1)^{\alpha+1}} \times \frac{(\rho_+^{\Delta}(t))^{\alpha+1}}{[\tau^{\Delta}(t)\rho(t)h_{n-2}(\tau(t), t_6)]^{\alpha}}$$

for $t \in [t_7, \infty)_{\mathbb{T}}$. Integrating both sides of the last inequality from t_7 to t ($t \geq t_7$), we obtain

$$w(t) - w(t_7) \leq -\int_{t_7}^{t} \left[\rho(s)q(s) - \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s, t_6)(\alpha+1)^{\alpha+1}} \times \frac{(\rho_+^{\Delta}(s))^{\alpha+1}}{[\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s), t_6)]^{\alpha}}\right] \Delta s.$$

Since $w(t) > 0$ for $t \in [t_5, \infty)_{\mathbb{T}}$, we have

$$\int_{t_7}^{t} \left[\rho(s)q(s) - \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s, t_6)(\alpha+1)^{\alpha+1}} \times \frac{(\rho_+^{\Delta}(s))^{\alpha+1}}{[\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s), t_6)]^{\alpha}}\right] \Delta s \leq w(t_7) - w(t) < w(t_7)$$

for $t \in [t_7, \infty)_{\mathbb{T}}$. Therefore, we get

$$\limsup_{t\to\infty} \int_{t_7}^{t} \left[\rho(s)q(s) - \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s, t_6)(\alpha+1)^{\alpha+1}} \times \frac{(\rho_+^{\Delta}(s))^{\alpha+1}}{[\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s), t_6)]^{\alpha}}\right] \Delta s \leq w(t_7) < \infty,$$

which yields a contradiction to (4.1). The proof is complete.

Theorem 4.4. *Assume that (H_1)–(H_5) and (1.4) hold and that there exist a positive function $\rho \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and a function $G \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$, such that*

$$G(t, t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \quad \textit{and} \quad G(t, s) > 0$$

for $(t, s) \in \mathbb{D}_0$, where $\mathbb{D}_0 := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$. Furthermore, suppose that G has a rd-continuous delta partial derivative $G^{\Delta_s}(t, s)$ on $\mathbb{D}$ with respect to the second variable and that there exists a function $g \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$ such that

$$G^{\Delta_s}(t, s) + G(t, s)\frac{\rho_+^{\Delta}(s)}{\rho^{\sigma}(s)} = \frac{g(t, s)}{\rho^{\sigma}(s)} G^{\frac{\alpha}{\alpha+1}}(t, s) \quad (4.24)$$

for $(t.s) \in \mathbb{D}$ and

$$\limsup_{t\to\infty} \frac{1}{G(t, T_2)} \int_{T_2}^{t} \left[G(t, s)\rho(s)q(s) - \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s, T_1)} \times \frac{(g_+(t, s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s), T_1)]^{\alpha}}\right] \Delta s = \infty \quad (4.25)$$

for all sufficiently large $T_1, T_2 \in [t_0, \infty)_{\mathbb{T}}$, where T_2 satisfies $\tau(s) > T_1$ for $s \in [T_2, \infty)_{\mathbb{T}}$, $\rho_+^{\Delta}(s) := \max\{\rho^{\Delta}(s), 0\}$, $g_+(t, s) := \max\{g(t, s), 0\}$, Φ is defined as in Theorem 4.3 and h_{n-2} is defined as in (3.3). Then (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Then there exists $t_x \in [t_0, \infty)_{\mathbb{T}}$ such that (3.4)–(3.6) hold. Proceeding as in the proof of Theorem 4.3, we get (4.2)–(4.23). Multiplying (4.23) by $G(t, s)$ and then integrating from t_7 to t, we conclude

$$\begin{aligned}
&\int_{t_7}^{t} G(t, s)\rho(s)q(s)\Delta s \\
&\leq -\int_{t_7}^{t} G(t, s)w^{\Delta}(s)\Delta s + \int_{t_7}^{t} G(t, s)\frac{\rho_+^{\Delta}(s)}{\rho^{\sigma}(s)} w^{\sigma}(s)\Delta s \\
&\quad - \int_{t_7}^{t} G(t, s)\xi(s, t_6)\left(\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\right)^{1+1/\alpha} \Delta s \qquad (4.26)
\end{aligned}$$

for $t \in [t_7, \infty)$, where

$$\xi(s, t_6) := \frac{\beta\tau^{\Delta}(s)\rho(s)h_{n-2}(\tau(s), t_6)\Phi(s, t_6)}{(\varphi \circ \tau)^{1/\alpha}(s)}.$$

By the integration by parts formula (2.4), we obtain

$$\begin{aligned}
&-\int_{t_7}^{t} G(t, s)w^{\Delta}(s)\Delta s \\
&= \Big[-G(t, s)w(s)\Big]_{s=t_7}^{s=t} + \int_{t_7}^{t} G^{\Delta_s}(t, s)w^{\sigma}(s)\Delta s \\
&= G(t, t_7)w(t_7) + \int_{t_7}^{t} G^{\Delta_s}(t, s)w^{\sigma}(s)\Delta s. \qquad (4.27)
\end{aligned}$$

It follows from (4.26) and (4.27) that for $t \in [t_7, \infty)$,

$$\begin{aligned}
&\int_{t_7}^{t} G(t, s)\rho(s)q(s)\Delta s \\
&\leq G(t, t_7)w(t_7) + \int_{t_7}^{t} \left\{\left[G^{\Delta_s}(t, s) + G(t, s)\frac{\rho_+^{\Delta}(s)}{\rho^{\sigma}(s)}\right] w^{\sigma}(s) \right. \\
&\quad \left. - G(t, s)\xi(s, t_6)\left(\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\right)^{1+1/\alpha}\right\}\Delta s.
\end{aligned}$$

In view of (4.24), we have

$$\begin{aligned}
&\int_{t_7}^{t} G(t, s)\rho(s)q(s)\Delta s \\
&\leq G(t, t_7)w(t_7) + \int_{t_7}^{t} \left[\frac{g(t, s)}{\rho^{\sigma}(s)} G^{\frac{\alpha}{\alpha+1}}(t, s)w^{\sigma}(s) \right. \\
&\quad \left. - G(t, s)\xi(s, t_6)\left(\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\right)^{1+1/\alpha}\right]\Delta s \\
&\leq G(t, t_7)w(t_7) + \int_{t_7}^{t} \left[\frac{g_+(t, s)}{\rho^{\sigma}(s)} G^{\frac{\alpha}{\alpha+1}}(t, s)w^{\sigma}(s) \right. \\
&\quad \left. - G(t, s)\xi(s, t_6)\left(\frac{w^{\sigma}(s)}{\rho^{\sigma}(s)}\right)^{1+1/\alpha}\right]\Delta s \qquad (4.28)
\end{aligned}$$

for $t \in [t_7, \infty)$, where g_+ is defined as in Theorem 4.4. Taking $\lambda = 1 + \frac{1}{\alpha}, X = [G(t,s)\xi(s,t_6)]^{1/\lambda} \frac{w^\sigma(s)}{\rho^\sigma(s)}$ and $Y = \left[g_+(t,s)G^{\frac{\alpha}{\alpha+1}}(t,s)\right]^\alpha / \{\lambda^\alpha [G(t,s)\xi(s,t_6)]^{\alpha/\lambda}\}$ for $t \in [t_7, \infty)_{\mathbb{T}}$, by Lemma 3.5 and (4.28) we get

$$\begin{aligned}&\int_{t_7}^t G(t,s)\rho(s)q(s)\Delta s\\ &\leq G(t,t_7)w(t_7)\\ &+ \int_{t_7}^t \frac{(\alpha/\beta)^\alpha \varphi(\tau(s))(g_+(t,s))^{\alpha+1}}{\Phi(s,t_6)(\alpha+1)^{\alpha+1}[\tau^\Delta(s)\rho(s)h_{n-2}(\tau(s),t_6)]^\alpha}\Delta s\end{aligned}$$

for $t \in [t_7, \infty)_{\mathbb{T}}$. Therefore, we obtain

$$\begin{aligned}&\frac{1}{G(t,t_7)}\int_{t_7}^t \Big[G(t,s)\rho(s)q(s)\\ &-\frac{(\alpha/\beta)^\alpha \varphi(\tau(s))(g_+(t,s))^{\alpha+1}}{\Phi(s,t_6)(\alpha+1)^{\alpha+1}[\tau^\Delta(s)\rho(s)h_{n-2}(\tau(s),t_6)]^\alpha}\Big]\Delta s\\ &\leq w(t_7) \quad \text{for } t \in (t_7, \infty)_{\mathbb{T}}\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{G(t,t_7)}\int_{t_7}^t \Big[G(t,s)\rho(s)q(s)\\ &-\frac{(\alpha/\beta)^\alpha \varphi(\tau(s))(g_+(t,s))^{\alpha+1}}{\Phi(s,t_6)(\alpha+1)^{\alpha+1}[\tau^\Delta(s)\rho(s)h_{n-2}(\tau(s),t_6)]^\alpha}\Big]\Delta s\\ &\leq w(t_7) < \infty,\end{aligned}$$

which contradicts (4.25). The proof is complete.

Remark 4.1. From Theorems 4.3 and 4.4, we can get a lot of different sufficient conditions for the oscillation of (1.1) with different choices of the functions ρ and G. For example, let $\rho(s) = s$, then Theorem 4.3 yields the following result.

Corollary 4.1. *Assume that (H$_1$)–(H$_5$), (1.4) and the following condition hold:*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{T_2}^t \Big[sq(s)\\ &-\frac{(\alpha/\beta)^\alpha \varphi(\tau(s))}{\Phi(s,T_1)(\alpha+1)^{\alpha+1}[s\tau^\Delta(s)h_{n-2}(\tau(s),T_1)]^\alpha}\Big]\Delta s\\ &= \infty \end{aligned} \tag{4.29}$$

for all sufficiently large $T_1, T_2 \in [t_0, \infty)_{\mathbb{T}}$, where T_2 satisfies $\tau(s) > T_1$ for $s \in [T_2, \infty)_{\mathbb{T}}$, Φ is defined as in Theorem 4.3 and h_{n-2} is defined as in (3.3). Then (1.1) is oscillatory.

V. Examples

In this section, we give some examples to illustrate our main results.

Example 5.1. Consider the nonlinear functional dynamic equation

$$\left[\sqrt{t}\left(x^{\Delta^{n-1}}(t)\right)^{5/9}\right]^\Delta + \frac{1}{t}\left[x^\sigma(t+a+b)\right]^{3/13} = 0 \tag{5.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $n \geq 4$ is even, a and b are arbitrary positive real numbers, $\mathbb{T} = \mathbb{R}_{a,b} := \bigcup_{k\in\mathbb{Z}} [k(a+b), k(a+b)+a]$ and $t_0 = a+b$.

In (5.1), $\alpha = \frac{5}{9}, \beta = \frac{3}{13}, \varphi(t) = \sqrt{t}, q(t) = \frac{1}{t}, \tau(t) = t+a+b$. By Lemma 3.6, we have

$$\int_{t_0}^\infty \varphi^{-1/\alpha}(t)\Delta t = \int_{a+b}^\infty \frac{1}{t^{9/10}}\Delta t = \infty$$

and $\int_{t_0}^\infty q(u)\Delta u = \int_{a+b}^\infty \frac{1}{u}\Delta u = \infty$. Therefore, the conditions (H$_1$), (H$_2$) and (1.2) are satisfied. By Theorem 4.1, every solution of (5.1) is oscillatory.

Example 5.2. Consider the nonlinear functional dynamic equation

$$\left[t^2\left(x^{\Delta^{n-1}}(t)\right)^{11/3}\right]^\Delta + \frac{1}{t^{3/2}}\left[x^\sigma(t-c)\right]^{7/9} = 0 \tag{5.2}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $n \geq 4$ is even, $c > 0$ is a constant, $\mathbb{T} = c\mathbb{Z} := \{ck : k \in \mathbb{Z}\}$, $t_0 \in \mathbb{T}$ and $t_0 \geq 1$.

In (5.2), $\alpha = \frac{11}{3}, \beta = \frac{7}{9}, \varphi(t) = t^2, q(t) = \frac{1}{t^{3/2}}, \tau(t) = t - c$. From Lemma 3.6, we see $\int_{t_0}^\infty \varphi^{-1/\alpha}(t)\Delta t = \int_{t_0}^\infty \frac{1}{t^{6/11}}\Delta t = \infty$. Therefore, the conditions (H$_1$) and (H$_2$) are satisfied. To apply Theorem 4.2, it remains to satisfy the condition (1.3). From Lemma 3.7 (ii) and the definition of improper integrals, we conclude

$$\begin{aligned}\int_{t_0}^\infty q(u)\Delta u &= \int_{t_0}^\infty \frac{1}{u^{3/2}}\Delta u = \sum_{k=t_0/c}^\infty \frac{c}{(kc)^{3/2}}\\ &= c^{-1/2}\sum_{k=t_0/c}^\infty \frac{1}{k^{3/2}} < \infty.\end{aligned} \tag{5.3}$$

Furthermore, we find

$$\begin{aligned}\int_s^\infty q(u)\Delta u = \int_s^\infty \frac{1}{u^{3/2}}\Delta u &\geq \int_s^\infty \frac{1}{u(u+c)}\Delta u\\ &= \int_s^\infty \left(-\frac{1}{u}\right)^\Delta \Delta u\\ &= -\frac{1}{u}\Big|_s^\infty = \frac{1}{s} \quad \text{for } s \geq t_0.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\int_{t_0}^\infty \left(\varphi^{-1}(s)\int_s^\infty q(u)\Delta u\right)^{1/\alpha}\Delta s\\ &\geq \int_{t_0}^\infty \left(\frac{1}{s^2 s}\right)^{3/11}\Delta s = \int_{t_0}^\infty \frac{1}{s^{9/11}}\Delta s = \infty.\end{aligned} \tag{5.4}$$

In view of (5.3) and (5.4), we see that the condition (1.3) is satisfied. Therefore, by Theorem 4.2, every bounded solution of (5.2) is oscillatory.

Example 5.3. Consider the nonlinear functional dynamic equation

$$\left[t^{9/5}\left(x^{\Delta^3}(t)\right)^{7/3}\right]^\Delta + \frac{1}{t\sigma(t)}\left[x^\sigma(q_0^{-1}t)\right]^{13/5} = 0 \tag{5.5}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $q_0 > 1$ is a constant, $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} := q_0^{\mathbb{Z}} \cup \{0\} := \{q_0^k : k \in \mathbb{Z}\} \cup \{0\}$ and $t_0 = q_0$.

In (5.5), $n = 4, \alpha = \frac{7}{3}, \beta = \frac{13}{5}, \varphi(t) = t^{9/5}, q(t) = \frac{1}{t\sigma(t)}, \tau(t) = q_0^{-1}t$. It follows from Lemma 3.6 that $\int_{t_0}^\infty \varphi^{-1/\alpha}(t)\Delta t = \int_{q_0}^\infty \frac{1}{t^{27/35}}\Delta t = \infty$. It is easy to see that the forward jump operator $\sigma(t) = q_0 t$ on $\mathbb{T}$. Hence, the conditions (H$_1$)–(H$_5$) are satisfied. We will apply Corollary

4.1. It remains to satisfy the conditions (1.4) and (4.29). We see

$$\int_{t_0}^{\infty} q(u)\Delta u = \int_{q_0}^{\infty} \frac{1}{u\sigma(u)}\Delta u = \int_{q_0}^{\infty} \Big(-\frac{1}{u}\Big)^{\Delta}\Delta u = -\frac{1}{u}\Big|_{q_0}^{\infty} = \frac{1}{q_0} < \infty.$$

Since $\mu(t) = \sigma(t) - t = (q_0 - 1)t$ on $\mathbb{T}$, by Lemma 3.7 (i) and the definition of improper integrals, we conclude

$$\begin{aligned}&\int_{t_0}^{\infty} \Big(\varphi^{-1}(s)\int_s^{\infty} q(u)\Delta u\Big)^{1/\alpha}\Delta s\\ &= \int_{q_0}^{\infty} \Big(\frac{1}{s^{9/5}s}\Big)^{3/7}\Delta s = \int_{q_0}^{\infty} \frac{1}{s^{6/5}}\Delta s\\ &= \sum_{k=1}^{\infty}(q_0-1)q_0^k\frac{1}{(q_0^k)^{6/5}} = \sum_{k=1}^{\infty}(q_0-1)\Big(q_0^{-1/5}\Big)^k\\ &= \frac{q_0-1}{q_0^{1/5}-1} < \infty.\end{aligned}$$

We also get

$$\begin{aligned}&\int_{t_0}^{\infty}\Big[\int_v^{\infty}\Big(\varphi^{-1}(s)\int_s^{\infty} q(u)\Delta u\Big)^{1/\alpha}\Delta s\Big]\Delta v\\ &= \int_{t_0}^{\infty}\Big[\int_v^{\infty}\frac{1}{s^{6/5}}\Delta s\Big]\Delta v\\ &\geq \int_{t_0}^{\infty}\Big[\int_v^{\infty}\frac{1}{s^2}\Delta s\Big]\Delta v\\ &= \int_{t_0}^{\infty}\Big[\int_v^{\infty}\frac{q_0}{sq_0s}\Delta s\Big]\Delta v = \int_{t_0}^{\infty} q_0\Big[\int_v^{\infty}\frac{1}{s\sigma(s)}\Delta s\Big]\Delta v\\ &= \int_{t_0}^{\infty} q_0\Big[\int_v^{\infty}\Big(-\frac{1}{s}\Big)^{\Delta}\Delta s\Big]\Delta v = \int_{t_0}^{\infty} q_0\Big[-\frac{1}{s}\Big|_v^{\infty}\Big]\Delta v\\ &= \int_{t_0}^{\infty}\frac{q_0}{v}\Delta v = \infty.\end{aligned}$$

Thus, the condition (1.4) is satisfied. Next, we will verify that the condition (4.29) is satisfied. For all sufficiently large $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$, where T_2 satisfies $\tau(s) > T_1$ for $s \in [T_2,\infty)_{\mathbb{T}}$, by Lemma 3.6 we have

$$\begin{aligned}\lim_{t\to\infty}\int_{T_2}^t sq(s)\Delta s &= \lim_{t\to\infty}\int_{T_2}^t s\frac{1}{s\sigma(s)}\Delta s = \int_{T_2}^{\infty}\frac{1}{\sigma(s)}\Delta s\\ &= \int_{T_2}^{\infty}\frac{1}{q_0 s}\Delta s = \infty. \end{aligned}\tag{5.6}$$

By Lemma 3.8, we obtain $h_{n-2}(t,s) = h_2(t,s) = (1+q_0)^{-1}(t-s)(t-q_0s)$ for all $t,s \in \mathbb{T}$. From the definition of Φ in Theorem 4.3, we have $\Phi(s,T_1) = \varepsilon$, where $\varepsilon > 0$ is a given arbitrary constant. Consequently, we conclude

$$\begin{aligned}&\lim_{t\to\infty}\int_{T_2}^t \frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s,T_1)(\alpha+1)^{\alpha+1}[s\tau^{\Delta}(s)h_{n-2}(\tau(s),T_1)]^{\alpha}}\Delta s\\ &= \lim_{t\to\infty}\int_{T_2}^t \frac{(35/39)^{7/3}(q_0^{-1}s)^{9/5}}{\varepsilon(10/3)^{10/3}\big[sq_0^{-1}(1+q_0)^{-1}(q_0^{-1}s-T_1)\big]^{7/3}}\times\\ &\qquad \frac{1}{(q_0^{-1}s-q_0T_1)^{7/3}}\Delta s\\ &= \int_{T_2}^{\infty}\frac{\varepsilon_* s^{9/5}}{\big[s(q_0^{-1}s-T_1)(q_0^{-1}s-q_0T_1)\big]^{7/3}}\Delta s\\ &= \sum_{k=T_2/q_0}^{\infty}(q_0-1)q_0^k\frac{\varepsilon_*(q_0^k)^{9/5}}{\big[q_0^k(q_0^{-1}q_0^k-T_1)(q_0^{-1}q_0^k-q_0T_1)\big]^{7/3}}\\ &= \sum_{k=T_2/q_0}^{\infty}\frac{(q_0-1)\varepsilon_* q_0^{7k/15}}{\big[(q_0^{-1}q_0^k-T_1)(q_0^{-1}q_0^k-q_0T_1)\big]^{7/3}}, \end{aligned}\tag{5.7}$$

where $\varepsilon_* := (35/39)^{7/3}q_0^{-9/5}/\big\{\varepsilon(10/3)^{10/3}[q_0^{-1}(1+q_0)^{-1}]^{7/3}\big\} > 0$. Since

$$\begin{aligned}&\lim_{k\to\infty}\left\{\frac{(q_0-1)\varepsilon_* q_0^{7k/15}}{\big[(q_0^{-1}q_0^k-T_1)(q_0^{-1}q_0^k-q_0T_1)\big]^{7/3}}\Big/\Big(q_0^{-21/5}\Big)^k\right\}\\ &= \frac{(q_0-1)\varepsilon_*}{q_0^{-14/3}} > 0\end{aligned}$$

and $\sum_{k=T_2/q_0}^{\infty}\Big(q_0^{-21/5}\Big)^k < \infty$, it follows that

$$\sum_{k=T_2/q_0}^{\infty}\frac{(q_0-1)\varepsilon_* q_0^{7k/15}}{\big[(q_0^{-1}q_0^k-T_1)(q_0^{-1}q_0^k-q_0T_1)\big]^{7/3}} < \infty. \tag{5.8}$$

From (5.6)–(5.8), we get

$$\begin{aligned}&\limsup_{t\to\infty}\int_{T_2}^t\Big[sq(s)\\ &\qquad -\frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s,T_1)(\alpha+1)^{\alpha+1}[s\tau^{\Delta}(s)h_{n-2}(\tau(s),T_1)]^{\alpha}}\Big]\Delta s\\ &= \lim_{t\to\infty}\int_{T_2}^t\Big[sq(s)\\ &\qquad -\frac{(\alpha/\beta)^{\alpha}\varphi(\tau(s))}{\Phi(s,T_1)(\alpha+1)^{\alpha+1}[s\tau^{\Delta}(s)h_{n-2}(\tau(s),T_1)]^{\alpha}}\Big]\Delta s\\ &= \infty,\end{aligned}\tag{5.9}$$

which implies that the condition (4.29) is satisfied. Therefore, all the conditions of Corollary 4.1 are satisfied. By Corollary 4.1, (5.5) is oscillatory.

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