A Stable Mixed Finite Element Scheme for the Second Order Elliptic Problems

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***Abstract*—A stable mixed finite element method (MFEM) for the second order elliptic problems, in which the scheme just satisfies the discrete $B.B$ condition, is discussed in this paper. The uniqueness and existence of solutions for the corresponding discrete problems are obtained, and the optimal $O(h^2)$ order error estimates are derived.**

***Index Terms*—mixed finite element scheme, error estimates, elliptic problems equations.**

I. Introduction

The mixed finite element methods (MFEM) are widely used in the theoretical analysis and computations of many areas of science and engineering. Because the method could approximate both the scalar variable and its flux variable at the same time, and obtain some convergence order, it becomes more and more popular in the numerical computational areas. A lot of studies on MFEM have been devoted to the second order elliptic problems. For example, [1] proposed a novel discontinuous MFEM, and used the discontinuous piecewise polynomial finite element spaces for the trial and test functions. [2] discussed an MFEM based on the Hellinger-Reissner variational principle, the maximum norm error estimates were obtained. [3] applied the MFEM to the 2D Burgers equation by using the $P_0^2 - P_1$ FE pair and also obtained the corresponding optimal error estimates. [4] considered the a priori and a posteriori error analyses of a mixed finite element method for Darcy's equations with porosity depending exponentially on the pressure by employing the local approximation properties of the Raviart-Thomas and $Clément$ interpolation operators as the main tools for proving the reliability. [5] developed a finite element mathematical model to study the flow of calcium in two dimensions with time, and gave a lot of numerical experiments to illustrate the method. [6] applied the MEFM to numerically study the processes of compaction and fluid flow in a sedimentary basin represented by a two-dimensional mode. [7] developed a nonconforming MFEM to solve the second-order elliptic problems on rectangular meshes, and obtained the optimal order error estimate. Moreover, [14] applied the method of [7] to the elliptic optimal control problem. [9] investigated a mixed variational form for second elliptic problem automatically satisfies strongly elliptic condition and the $B.B$ condition, the optimal order error estimates were obtained.

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Recently, [10] constructed a low-order rectangular MFEM for second order elliptic problems, in which they used (Span$\{1, x, y, xy, x^2\}$,Span$\{1, x, y, xy, y^2\}$) for vector function and Span$\{1, x, y\}$ for scalar function, and the stability and optimal order error estimates were derived. For the Variable-Coefficient elliptic problems, [11] studied the multigrid block krylov subspace spectral method, and some numerical results were shown to demonstrate the effectiveness of this approach. [12] introduced a weak Galerkin method for second order elliptic problems, the error estimates of $H(div)$-norm and L^2-norm for the primary and flux variables are obtained, respectively, but the first mentioned is one order lower that the last one. In this paper, we will propose a nonconforming mixed finite element scheme for the same elliptic problem as above, and obtain the optimal order error estimate. We use (Span$\{1, x, y, xy, x^2\}$,Span$\{1, x, y, xy, y^2\}$) for vector function and Span$\{1, x, y\}$ for scalar function, in which the convergence orders of the primary and flux variables are both $O(h^2)$.

The remainder of this paper is organized as follows. In Section 2, a new MFE scheme for elliptic problems is constructed. In Section 3, the corresponding mixed finite element space is obtained. In Section 4, we study the interpolation property and consistency error of this MFE scheme, which shows that the optimal convergence estimate can be obtained. Throughout this paper, C is a generic positive constant independent of the mesh size h.

II. Analysis of the new mixed variational scheme

Consider the following elliptic problem:

$$\begin{cases} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f, \ in \quad \Omega, \\ u = 0, \ on \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a rectangular domain in R^2, $A = (a_{ij})_{2\times 2}$ is continuous, symmetry and strongly elliptic on $\bar{\Omega}$.

Let $\psi = A\nabla u$, then $div\psi = f$. The following equivalent mixed variational form of (1) is obtained,

$$\begin{cases} a(\psi, \varphi) + b(\varphi, u) = 0, \quad \forall \phi \in H \\ b(\psi, v) = F(v), \quad \forall v \in M, \end{cases} \tag{2}$$

where

$$a(\psi, \varphi) = \int_\Omega A^{-1}\varphi \cdot \psi dx, \ b(\varphi, v) = \int_\Omega v div\varphi dx,$$

$$F(v)=\int_\Omega fvdx,$$

and

$$H=\left\{\phi\Big|\phi\in L^2(\Omega)^2,\ div\phi\in L^2(\Omega)\right\},$$
$$M=\left\{v\Big|v\in L^2(\Omega)\right\}.$$

Define the div-norm as follows:

$$\parallel \phi \parallel_H^2=\parallel \phi \parallel_{0,\Omega}^2+\parallel div\phi \parallel_{0,\Omega}^2. \quad (3)$$

Let $v=div\varphi$ in the second equation of (2), we have

$$\int_\Omega div\psi\cdot div\varphi dx=\int_\Omega f\cdot div\phi dx. \quad (4)$$

Substitution of the above equation into the first equation of (2) yields

$$\int_\Omega A^{-1}\varphi\cdot\psi dx-\int_\Omega u div\varphi dx+\int_\Omega div\psi\cdot div\varphi dx = \int_\Omega f\cdot div\varphi dx \quad (5)$$

Then, a new mixed variational scheme of (1) is obtained, it is to find $(\psi,u)\in H\times M$ such that

$$\begin{cases} a_1(\psi,\varphi)+b(\varphi,u) &= G(\varphi), \quad \forall\varphi\in H,\\ b(\psi,v) &= F(v), \quad \forall v\in M,\end{cases} \quad (6)$$

where

$$a_1(\psi,\varphi)=\int_\Omega A^{-1}\varphi\cdot\psi dx+\int_\Omega div\psi\cdot div\varphi dx$$
$$G(\varphi)=\int_\Omega f\cdot div\varphi dx \quad (7)$$

Lemma 2.1 $a_1(\cdot,\cdot)\in(H\times H)'$ is a positive definite form, i.e., $\forall\psi\in H$, there exists a positive number $\alpha>0$, such that $a_1(\psi,\psi)\geqslant\alpha\parallel\psi\parallel_H^2$

Proof. Firstly, we verify that A^{-1} satisfies the V-ellipticity, where

$$A^{-1}=\frac{1}{|A|}=\begin{bmatrix} a_{22} & -a_{12}\\ -a_{12} & a_{11}\end{bmatrix} \quad (8)$$

By the strong elliptic property of A, we know that there exists a positive number α such that

$$a_{11}(x)>\alpha>0 \quad (9)$$

and

$$a_{11}(x)a_{22}(x)-a_{12}^2(x)>\alpha>0 \quad (10)$$

Obviously, $a_{22}\neq 0$, then there exists another positive number $\beta>0$ such that

$$a_{22}>\beta. \quad (11)$$

On the other hand, considering that $a_{11}a_{22}-(a_{12})^2>0$ and $a_{11}>0$, we have

$$a_{22}>0. \quad (12)$$

Therefore, there exists a positive number α_1, such that

$$a_{22}>\alpha_1>0. \quad (13)$$

Because A is continuous and strong-ellipticity on $\bar{\Omega}$, there exists $M>0$, such that $|A|<M$, which means that $\frac{1}{|A|}>\frac{1}{M}$. Noticing that $a_{11}a_{22}-a_{12}^2>\alpha>0$, all of the order principal minor determinants of A^{-1} are bigger than a positive number.

Thus, A^{-1} satisfies the V-elliptic property, and

$$\begin{aligned} a_1(\psi,\psi) &= \int_\Omega A^{-1}\psi\cdot\psi dx+\int_\Omega div\psi\cdot div\psi dx\\ &\geqslant C\int_\Omega(\psi^2+div\psi^2)dx\\ &= C\parallel\psi\parallel_H^2,\end{aligned} \quad (14)$$

The Lemma 2.1 is proved. □

Lemma 2.2 $b(\cdot,\cdot)\in(H\times M)'$ satisfies the continuous $B.B$ condition, i.e., there exists a positive number $\beta>0$, such that $\forall v\in M$, there holds

$$\sup_{\forall\psi\in H}\frac{b(\psi,v)}{\parallel\psi\parallel_H}\geqslant\beta\parallel v\parallel_M. \quad (15)$$

Proof. Because

$$\begin{aligned}\parallel\psi\parallel_H^2 &=\parallel\psi\parallel_0^2+\parallel div\psi\parallel_0^2\\ &\leqslant\parallel\psi\parallel_0^2+C\parallel\psi\parallel_1^2\\ &\leqslant C\parallel\psi\parallel_1^2,\end{aligned} \quad (16)$$

then, for any given $v\in L^2(\Omega)$, let $w\in H_0^1(\Omega)\cap H^2(\Omega)$ be the solution of the following equation:

$$\begin{cases} -\triangle w &= v \quad in \quad \Omega,\\ w &= 0 \quad on \quad \partial\Omega.\end{cases} \quad (17)$$

We have $\parallel w\parallel_2\leqslant C\parallel v\parallel_0$ by using the regularity theory of partial differential equations, which refers to [13].

If we choose

$$\psi_0=\begin{pmatrix}-w_x\\ -w_y\end{pmatrix}\in H,$$

then,

$$div\psi_0=v\in M.$$

Following the equation

$$\parallel\psi_0\parallel_M\leqslant C\parallel\psi_0\parallel_1\leqslant C\parallel w\parallel_2\leqslant C\parallel v\parallel_0,$$

we have

$$\sup_{\forall\psi\in H}\frac{b(\psi,v)}{\parallel\psi\parallel_H}\geqslant\frac{b(\psi_0,v)}{\parallel\psi_0\parallel_H}\geqslant\frac{\parallel v\parallel_0^2}{C\parallel v\parallel_0}\geqslant\beta\parallel v\parallel_M, \quad (18)$$

The Lemma 2.2 is proved. □

Theorem 2.1 By using the results of Lemmas 2.1-2.2, the problem (6) has a unique solution $(\psi,u)\in H\times M$.

III. Construction of the Mixed Finite Element Space

Let J_h be a rectangular subdivision of Ω, and K be a general element with four vertices (x_e-h_1,y_e-h_2), (x_e+h_1,y_e-h_2), (x_e+h_1,y_e+h_2) and (x_e-h_1,y_e+h_2), F_K be an affine mapping from $\widehat{K}$ to K,

$$F_K:\begin{cases} x_1 &= x_e+h_1\widehat{x_1},\\ x_2 &= y_e+h_2\widehat{x_2},\end{cases} \quad (19)$$

where $\widehat{K}$ is the rectangular reference element with four vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$ in $\widehat{x_1}-\widehat{x_2}$ plane.

Define the following functions on the reference element $\widehat{K}$:

$$\begin{aligned} \widehat{\phi}_1(\widehat{x}) = 1+\widehat{x}_1, \qquad \widehat{\phi}_2(\widehat{x}) = 1+\widehat{x}_2, \\ \widehat{\phi}_3(\widehat{x}) = 1-\widehat{x}_1, \qquad \widehat{\phi}_4(\widehat{x}) = 1-\widehat{x}_2. \end{aligned} \tag{20}$$

We denote the four edges of $\widehat{K}$ by $\widehat{l}_i$, on which $\widehat{\phi}_i = 0$ $(i = 1,2,3,4)$. $\widehat{B}_i$ denotes the midpoint of $\widehat{l}_i$, $B_{K,i} = F_K(\widehat{B}_i)$, $l_{K,i} = F_K(\widehat{l}_i)$, $n_{K,i}(x)$ denotes the unit outside normal vector. $\psi_{K,i} = n_{K,i}(B_{K,i}) \prod_{\substack{j=1 \\ i\neq j}}^{4} (\widehat{\varphi}_j) \in Q_2(K)$, which is zero on $l_{K,j} (j \neq i)$.

Then, $Q_2^-(K)$ can be defined as:

$$\begin{aligned} Q_2^-(K) &= Q_1(K) \oplus span\{\psi_{K,1}, \cdots, \psi_{K,4}\} \\ &= Q_1(K) \oplus \left\{ \begin{array}{c} (1+x_1)(1-x_1)(1+x_2), \\ (1+x_1)(1-x_1)(1-x_2), \\ (1+x_1)(1+x_2)(1-x_2), \\ (1-x_1)(1+x_2)(1-x_2) \end{array} \right\} \\ &= Q_1(K) \oplus \{x_1^2 x_2, x_1 x_2^2, x_1^2, x_2^2\}, \end{aligned} \tag{21}$$

Define

$$\begin{aligned} X_h &= \left\{ \phi_h \middle| \phi_h|_K \circ F_K^{-1} \in Q_2^-(\widehat{K}) \right\}, \\ M_h &= \left\{ v_h \middle| \; v_h|_K \in P_1^{disc}(K) \right\} \end{aligned} \tag{22}$$

be two conforming finite element spaces, where $P_1^{disc}(K)$ is a nonuniform continuous piecewise one order polynomial space.

The discrete form of (6) is to find $(\psi_h, u_h) \in (X_h \times M_h)$, such that

$$\left\{ \begin{array}{rcll} a_1(\psi_h, \varphi_h) + b(\varphi_h, u_h) &=& G(\psi_h), & \forall \varphi_h \in X_h, \\ b(\psi_h, v_h) &=& F(v_h), & \forall v_h \in M_h. \end{array} \right. \tag{23}$$

For the convergence analysis in next section, we will give the definition of several subspaces. $\bar{X}_h$, a subspace of X_h, can be defined as

$$\begin{aligned} &\bar{X}_h \\ &= \left\{ v \in C^0(\Omega)^2 \middle| v|_K \in Q_2^-(K), \forall K \in J_h, \; v|_{\partial\Omega} = 0 \right\} \\ &\subset X_h. \end{aligned} \tag{24}$$

We divide Ω to R non-overlapping domains $\Omega_r (r = 1, \cdots, R)$, each one is $Lipschitz$ continues open domain:

$$\bar{\Omega} = \bigcup_{r=1}^{R} \bar{\Omega}_r, \Omega_r \cap \Omega_s = \phi \; (r \neq s). \tag{25}$$

Then, $\widehat{P}_1$ is denoted by the first order polynomial spaces, and define

$$P_1(K) = \left\{ v = \widehat{v} \circ F_K^{-1} \middle| \quad \widehat{v} \in \widehat{P}_1 \right\}, \tag{26}$$

$$\begin{aligned} Q_h &= P_1(J_h) \\ &= \left\{ v \in L^2(\Omega) \middle| v|_K \in P_1(K), \forall K \in J_h \right\}. \end{aligned} \tag{27}$$

Then, we have the following subspaces

$$X_h(\Omega_r) : = \left\{ v|_{\Omega_r} \middle| \; v \in X_h, \; v \equiv 0 \; in \; \Omega \setminus \Omega_r \right\}, \tag{28}$$

$$Q_h(\Omega_r) : = \left\{ q|_{\Omega_r} \middle| \; q \in Q_h \right\}, \tag{29}$$

$$M_h(\Omega_r) : = Q_h(\Omega_r) \cap L^2(\Omega_r), \tag{30}$$

$$\bar{M}_h : = \left\{ q \in L^2(\Omega) \middle| \; q|_{\Omega_r} = const, \; r = 1, \cdots, R \right\}. \tag{31}$$

IV. Convergence analysis of the mixed finite element scheme

Firstly, based on the conforming finite element spaces defined in Section III, we have

$$\begin{aligned} &a_1(\psi_h, \psi_h) \\ &= \sum_{K \in J_h} \left[\int_K A^{-1} \psi_h \cdot \psi_h dx + \int_K div\psi_h \cdot div\psi_h dx \right] \\ &\geqslant C \sum_{J \in J_h} \int_J (\psi_h^2 + div\psi_h^2) dx \\ &= C \parallel \psi_h \parallel_H^2 . \end{aligned} \tag{32}$$

This is that $a_1(\cdot,\cdot)$ satisfy the strong ellipticity property on $X_h \times M_h$.

Lemma 4.1 There exists a positive number $\lambda_1 > 0$, the following $B.B$ conditions holds on $(\bar{X}_h, \bar{M}_h)$.

$$\sup_{v_h \in \bar{X}_h} \frac{(divv_h, q_h)}{|v_h|_1} \geqslant \lambda_1 \parallel q_h \parallel_0, \quad \forall q_h \in \bar{M}_h. \tag{33}$$

Proof. For any $\phi \in H^2(\Omega)^2$, we introduce the operator $\Pi_h \phi$, where $\Pi_h \phi$ equals to ϕ at the four vertices, and equals to the average value of ϕ on the four edges, $\Pi_h|_K \circ F_K^{-1} \in Q_2^-$. Then, $\forall v_h \in \bar{M}_h$, we have

$$\begin{aligned} b(\phi - \Pi_h \phi, v_h) &= \sum_{K \in J_h} \int_K v_h div(\phi - \Pi_h \phi) dx \\ &= \sum_K v_h \int_K div(\phi - \Pi_h \phi) dx \\ &= \sum_K v_h \int_{\partial K} (\phi - \Pi_h \phi) \cdot \overrightarrow{n} ds = 0, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \parallel \Pi_h \phi \parallel_1 &\leqslant \parallel I_h \phi - \phi \parallel_1 + \parallel \phi \parallel_1 \\ &\leqslant \parallel \phi \parallel_1 + \parallel \phi \parallel_1 \\ &\leqslant \lambda_1 \parallel \phi \parallel_1 . \end{aligned} \tag{35}$$

By use of the Fortin principle [13], the $B.B$ condition holds in $(\bar{X}_h, \bar{M}_h)$. $\square$

Lemma 4.2 There exists a positive number $\lambda > 0$, the following $B.B$ conditions holds on Ω_r.

$$\sup_{v_h \in X_h(\Omega_r)} \frac{(divv_h, q_h)}{|v_h|_{1,\Omega_r}} \geqslant \lambda \parallel q_h \parallel_{0,\Omega_r}, \quad \forall q_h \in M_h(\Omega_r). \tag{36}$$

Proof. $\forall q \in M_h(K)$, we define

$$\widehat{v} : \widehat{K} \to R^2, \quad \widehat{v}(\widehat{x}) := -B_K^{-T} (\widehat{\nabla}(q \circ F_k))(\widehat{x}) \widehat{b}(\widehat{x}),$$

where

$$\widehat{\nabla} = (\frac{\partial}{\partial \widehat{x}_1}, \frac{\partial}{\partial \widehat{x}_2})^T, \ \widehat{b}(\widehat{x}) = (1-\widehat{x}_1^2)(1-\widehat{x}_2^2). \qquad (37)$$

Obviously, $\widehat{b}(\widehat{x}) > 0$ in the $\widehat{K}$, then $\widehat{v} = 0$ on the $\partial\widehat{K}$. Because of $\widehat{\nabla}(q \circ F_K) \in P_{k-2}^2$, we have $\widehat{v} \in Q_k^2$.

Considering that $v = \widehat{v} \circ F_K^{-1} \in X_h(K)$, then

$$(\widehat{\nabla}\widehat{q})(\widehat{x}) = B_K^T(\nabla q)(F_K(\widehat{x})), \qquad (38)$$

where $\widehat{q} = q \circ F_K$. We have

$$\begin{aligned} & (divv, q)_K \\ = \ & -(v, \nabla q)_K = -\int_K v\nabla q dx \\ = \ & -\int_{\widehat{K}} (\widehat{v})(\widehat{x})(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x} \\ = \ & -\int_{\widehat{K}} -B_K^{-T}\widehat{\nabla}\widehat{q}(\widehat{x})\widehat{b}(\widehat{x})(\nabla q)(F_K(\widehat{x}))|h_1h_2|d\widehat{x} \\ \overset{(38)}{=} \ & h_1h_2\int_{\widehat{K}} \widehat{b}(\widehat{x})(\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))d\widehat{x} \end{aligned} \qquad (39)$$

Next, we prove that there exist $C > 0$, such that

$$\begin{aligned} & \int_{\widehat{K}} \widehat{b}(\widehat{x})(\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x} \\ & \geqslant C\int_{\widehat{K}} (\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x}. \end{aligned} \qquad (40)$$

By use of (38), we get that

$$\begin{aligned} & \frac{(\widehat{\nabla}\widehat{q})^T(\widehat{x})(\widehat{\nabla}\widehat{q})(\widehat{x})}{\| B_k \|^2} \\ = \ & \frac{(B_K(\nabla q)^T(F_K(\widehat{x})), B_K^T(\nabla q)(F_K(\widehat{x})))}{\| B_K \|^2} \\ \leqslant \ & (\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x})) \\ \leqslant \ & \| B_K^{-1} \|^2 \ (\widehat{\nabla}\widehat{q})^T(\widehat{x})(\widehat{\nabla}\widehat{q})(\widehat{x}) \end{aligned} \qquad (41)$$

$$\begin{aligned} & \frac{\int_{\widehat{K}} \widehat{b}(\widehat{x})(\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x}}{\int_{\widehat{K}} (\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x}} \\ \geqslant \ & \frac{1}{\| B_K \|^2 \| B_K^{-1} \|^2} \frac{\int_{\widehat{K}} \widehat{b}(\widehat{x})(\widehat{\nabla}\widehat{q})^T(\widehat{x})(\widehat{\nabla}\widehat{q})(\widehat{x})d\widehat{x}}{\int_{\widehat{K}} (\widehat{\nabla}\widehat{q})^T(\widehat{x})(\widehat{\nabla}\widehat{q})(\widehat{x})d\widehat{x}}, \end{aligned} \qquad (42)$$

Then, we have

$$\begin{aligned} & (divv, q) \\ \geqslant \ & C\int_{\widehat{K}} (\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x} \\ \geqslant \ & C_1|q|_{1,K}^2 \end{aligned} \qquad (43)$$

Noticing that $|\widehat{b}(\widehat{x})| \leqslant 1$ and

$$\begin{aligned} \widehat{v}(\widehat{x}) &= -B_K^T(\widehat{\nabla}(q \circ F_K))(\widehat{x})(\widehat{b})(\widehat{x}) \\ &\overset{(38)}{=} -(\nabla q)(F_K(\widehat{x}))(\widehat{b})(\widehat{x}) \\ &= -\widehat{b}(\widehat{x})B_K^{-T}(\widehat{\nabla}\widehat{q})(\widehat{x}) \\ &= -\widehat{b}(\widehat{x})(\nabla q)(F_K(\widehat{x})), \end{aligned} \qquad (44)$$

Then, we have

$$\begin{aligned} \| v \|_{0,K}^2 &= \int_{\widehat{K}} \widehat{v}^T(\widehat{x})\widehat{v}(\widehat{x})|detDF_k(\widehat{x})|d\widehat{x} \\ &\leqslant \int_{\widehat{K}} (\nabla q)^T(F_K(\widehat{x}))(\nabla q)(F_K(\widehat{x}))|detDF_K(\widehat{x})|d\widehat{x} \\ &\leqslant |q|_{1,K}^2. \end{aligned} \qquad (45)$$

Next, we will obtain two positive numbers C_2 and C_3, such that

$$|v|_{1,K} \leqslant C_2h \| v \|_{0,K}, \quad v \in X_h(K) \qquad (46)$$

$$\| q \|_{0,K} \leqslant C_3h|q|_{1,K}, \quad q \in M_h(K) \qquad (47)$$

By using the result of (38),

$$(\nabla v)(F_K(\widehat{x})) = B_K^{-T}(\widehat{\nabla}\widehat{v})(\widehat{x}). \qquad (48)$$

we have

$$\begin{aligned} |v|_{1,K}^2 &= \| B_K^{-T} \|^2 \int_{\widehat{K}} |(\widehat{\nabla}\widehat{v})(\widehat{x})|^2|detDF_K(\widehat{x})|d\widehat{x} \\ &\leqslant h_1h_2 \| B_K^{-T} \|^2 \int_{\widehat{K}} |(\widehat{\nabla}\widehat{v})(\widehat{x})|^2 d\widehat{x}. \end{aligned} \qquad (49)$$

By the equivalence property of the norms on the finite element spaces, there exists $\widetilde{C}$ such that $|\widehat{v}|_{1,\widehat{K}} \leqslant \widetilde{C} \| \widehat{v} \|_{0,\widehat{K}}$. On the other hand,

$$\begin{aligned} \| \widehat{v} \|_{0,\widehat{K}}^2 &= \int_K |v(x)|^2|detDF_K^{-1}(\widehat{x})|d\widehat{x} \\ &\leqslant \frac{\| v \|_{0,K}^2}{2h_1h_2}. \end{aligned} \qquad (50)$$

Thus the proof of equation (46) is completed.

Similarly,

$$\begin{aligned} \| q \|_{0,K}^2 &= \int_K q^2dx = \int_{\widehat{K}} \widehat{q}^2det(DF_K(\widehat{x}))d\widehat{x} \\ &\leqslant \int_{\widehat{K}} \widehat{q}^2d\widehat{x} \cdot C \quad meas(S_K) \\ &\leqslant C\int_{\widehat{K}} |\nabla\widehat{q}|^2d\widehat{x} \cdot C \quad meas(S_K) \\ &\leqslant C|\widehat{q}|_{1,\widehat{K}}^2 \end{aligned} \qquad (51)$$

and

$$\begin{aligned} & \int_{\widehat{K}} [(\frac{\partial\widehat{q}}{\partial\widehat{x}_1})^2 + (\frac{\partial\widehat{q}}{\partial\widehat{x}_2})^2]d\widehat{x} \\ = \ & \int_K [h_1^2(\frac{\partial q}{\partial x_1})^2 + h_2^2(\frac{\partial q}{\partial x_2})^2]\frac{1}{h_1h_2}dx \\ \leqslant \ & C|q|_{1,K}^2, \end{aligned} \qquad (52)$$

The proof of equation (47) is also completed.

Then, we have

$$\begin{aligned}&|v|_{1,K} \parallel q \parallel_{0,K} \overset{(46)(47)}{\leqslant} C_2C_3 \parallel v \parallel_{0,K} |q|_{1,K} \\ &\leqslant C_2C_3|q|^2_{1,K} \overset{(43)}{\leqslant} C_4(divv, q)_K\end{aligned} \tag{53}$$

The $B.B$ condition is satisfied. □

The following lemma given in [1] shows the relationship between the global and local $B.B$ conditions:

Lemma 4.3 The equations (33) and (36) hold, and there exists a positive β, such that

$$\sup_{v_h\in X_h} \frac{(divv_h, q_h)}{|v_h|_1} \geqslant \beta \parallel q_h \parallel_0, \quad \forall q_h \in M_h. \tag{54}$$

Lemma 4.4 From Lemmas 4.3, $b(\cdot,\cdot)$ satisfy the $B.B$ condition on $X_h \times M_h$, the problem (23) has a unique solution by the MFEM theory [13].

Finally, from Lemma 4.1–4.4, the optimal error estimate of this new mixed finite element scheme can be given as

Theorem 2.1 Let (ψ, u) and (ψ_h, u_h) be the solutions of (6) and (23), respectively, then we have the following error estimate:

$$\parallel \psi - \psi_h \parallel_H + \parallel u - u_h \parallel_M \leqslant Ch^2(|\psi|_{3,\Omega} + |u|_{2,\Omega}). \quad \square \tag{55}$$

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