Dynamical Behavior and Bifurcation Analysis of SEIR Epidemic Model and its Discretization

Wen-ju Du, Shuang Qin, Jian-gang Zhang and Jian-ning Yu

Abstract—The dynamical behaviors of a SEIR epidemic model are investigated in this paper. More precisely, we presented a new discrete-time SEIR epidemic model by using the Forward-Euler difference method. And the existence, stability and direction of Hopf bifurcation of the SEIR epidemic model and its discretization model are also studied. In addition, the numerical simulations were presented to illustrate the theoretical analysis. Finally, some comparisons of bifurcation between the continuous-time epidemic system and its discrete-time system are given.

***Index Terms*—Stability, Discretization, Hopf bifurcation, Center manifold theorem, Forward Euler scheme**

I. Introduction

The dynamical system refers to the dynamic system of change over time, which includes continuous dynamical systems and discrete dynamical systems. Despite the simplicity of dynamical systems, these systems have a rich dynamical behavior, ranging from stable equilibrium points to periodic and even chaotic oscillations. And Hopf bifurcation is an important dynamic bifurcation which closely related to some self-excited vibration phenomenon and has a high theoretical value in dynamic bifurcation and limited cycle research. Moreover, the research and application on bifurcation of autonomous systems has become a very popular topic [1-9]. But compared with the continuous systems, the discrete systems possess its unique dynamic characteristics. In the real life, many practical problems can be depicted by the discrete systems, and we can also to discretize the continuous systems. Therefore, the study of discrete system is very important and achieved great development in the field of mathematics, physics and engineering [10-13]. Hu et al. [14] presented a new epidemic model by using the Euler difference method, and discussed the Neimark-Sacker bifurcation of the system based on the center manifold theorem and the bifurcation theory. Elabbasy et al. [15] studied the Pitchfork bifurcation, Flip bifurcation and Neimark-Sacker bifurcation of a two-dimensional discrete Lorenz system. He et al. [16] focus on a third-order rational difference equation with positive parameters, and the existence and direction of the Neimark-Sacker bifurcation of the system are investigated in detail.

Manuscript received May 30, 2016; revised August 29, 2016. This work is supported by the National Natural Science Foundation (No.11161027, No.61364001).

Wen-ju Du is with the School of Traffic and Transportation, Lanzhou Jiaotong University, Lanzhou, China (phone: 086-9314956002; fax: 086-9314956002; e-mail: duwenjuok@126.com).

Shuang Qin, Jian-gang Zhang and Jian-ning Yu was with Department of Mathematics, Lanzhou Jiaotong University, Lanzhou, China (e-mail: qinshuangok@126.com, zhangjg7715776@126.com, yujn@mail.lzjtu.cn).

Epidemic is caused by the pathogen, which can be spread from human to human, human to animals and animals to animals. And the epidemic can makes a range of biological reduce or lose labor, death and spread rapidly in a certain period of time. Therefore, it has caused great attention of scientists and mathematicians. As early as 1927, Kermark and Mckendrick were established the mathematical model of infectious diseases by using the method of dynamics, and constructed the famous SIR bin model [17]. After the middle of the 20th century, the dynamics of infectious disease has been obtained rapid development, and the epidemic dynamical models have been widely investigated [18, 19]. At present, the bifurcation researches of discrete systems are mostly concentrated in the two-dimensional systems, and there are few studies focusing on the three dimensional discrete system. But compared with the two-dimensional model of infectious diseases, three-dimensional model can better reflect the mechanism of the spread of disease [20].

Wang [21] investigated an SEIR epidemic model with infectious force in latent period as follows:

$$\begin{cases} S' = A - \beta_1 ES - \beta_2 IS - (\mu+\rho)S, \\ E' = \beta_1 ES + \beta_2 IS - (\mu+v)E, \\ I' = vE - (\mu+\alpha+\delta+\gamma)I, \\ R' = \rho S + (\delta+\gamma)I - \mu R, \end{cases} \tag{1}$$

where S, E, I, R and N denote the numbers of susceptible, lurker, infective, recovered individuals and total numbers of the individuals, respectively. Assume that the susceptible crowd has a constant input rate A, and ρ is the effective vaccination rate. The proportional coefficient of Lurker become infected is v, and the lurker crowd possess bilinear incidence rate $\beta_1 ES$. The infected people have bilinear incidence rate $\beta_2 IS$, and α is the diseased death rate, δ is the cure rate, γ is the natural recovery rate, and $\mu > 0$ is the natural mortality rate. Here $N = S + E + I + R$, and all the coefficients are positive.

Due to the first three equations of model (1) is about (S, E, I) not including R, and the fourth equation is the linear equation of R. Therefore, the dynamical behavior of model (1) is equivalent to the dynamical behaviors of the following model:

$$\begin{cases} S' = A - \beta_1 ES - \beta_2 IS - (\mu+\rho)S \\ E' = \beta_1 ES + \beta_2 IS - (\mu+v)E \\ I' = vE - (\mu+\alpha+\delta+\gamma)I \end{cases} \tag{2}$$

Applying the Forward-Euler difference method to model (2), we obtained the following three-dimensional

(Advance online publication: 23 February 2017)

discrete-time SEIR epidemic model with infectious force in latent period:

$$\begin{cases} S_{n+1} = S_n + h\left[A - \beta_1 E_n S_n - \beta_2 I_n S_n - (\mu+\rho) S_n\right] \\ E_{n+1} = E_n + h\left[\beta_1 E_n S_n + \beta_2 I_n S_n - (\mu+v) E_n\right] \\ I_{n+1} = I_n + h\left[v E_n - (\mu+\alpha+\delta+\gamma) I_n\right] \end{cases} \tag{3}$$

where $h(0<h<1)$ is the step size.

The paper is organized as follows. In section 2, we discuss the stability of fixed points and the Hopf bifurcation by choosing α as the bifurcation parameter in model (2). The stability of fixed points and the existence, direction and stability of the Hopf bifurcation of the discretized system are investigated in section 3. In section 4, we present the numerical simulations illustrate our results with the theoretical analysis. We have given some comparisons of bifurcation between the continuous-time epidemic system and its discrete-time system in section 5. In section 6, conclusion of the paper is given.

II. Dynamical behaviors of the SEIR epidemic model

A. Linear Analysis of the Fixed Points and Existence of Hopf Bifurcation

Through a simple calculation, we can easily get the following two fixed points of model (2):

$$P_1 = \left(\frac{A}{\mu+\rho}, 0, 0\right), \quad P_2 = (S_0, E_0, I_0), \tag{4}$$

where

$$S_0 = \frac{(\mu+v)(\mu+\alpha+\delta+\gamma)}{\beta_1(\mu+\alpha+\delta+\gamma)+\beta_2 v},$$

$$E_0 = \frac{A\beta_2 v - (\mu+\alpha+\delta+\gamma)(\mu^2+v\rho+\mu v+\mu\rho+A\beta_1)}{(\mu+v)\left[\beta_1(\mu+\alpha+\delta+\gamma)+\beta_2 v\right]},$$

$$I_0 = \frac{v\left[A\beta_2 v - (\mu+\alpha+\delta+\gamma)(\mu^2+v\rho+\mu v+\mu\rho+A\beta_1)\right]}{(\mu+v)(\mu+\alpha+\delta+\gamma)\left[\beta_1(\mu+\alpha+\delta+\gamma)+\beta_2 v\right]}.$$

The Jacobian matrix of model (2) at the fixed point P_1 is given by

$$J_1 = \begin{bmatrix} -\mu-\rho & -\frac{A\beta_1}{\mu+\rho} & -\frac{A\beta_2}{\mu+\rho} \\ 0 & \frac{A\beta_1}{\mu+\rho}-(\mu+v) & \frac{A\beta_2}{\mu+\rho} \\ 0 & v & -(\mu+\alpha+\delta+\gamma) \end{bmatrix}, \tag{5}$$

and its characteristic polynomial is

$$\begin{aligned} p(\lambda) = (\lambda+\mu+\rho)\Bigg[&\lambda^2 + (2\mu+v+\alpha+\delta+\gamma-\frac{A\beta_1}{\mu+\rho})\lambda \\ &+\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)(\mu+\alpha+\delta+\gamma)-\frac{A\beta_2 v}{\mu+\rho}\Bigg], \end{aligned} \tag{6}$$

So, we can get the eigenvalues of J_1 as follows

$$\lambda_1 = -\mu-\rho, \quad \lambda_2 = \sqrt{\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)-\frac{A\beta_2 v}{\mu+\rho}}\,i,$$

$$\lambda_3 = -\sqrt{\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)-\frac{A\beta_2 v}{\mu+\rho}}\,i.$$

According to Routh-Hurwitz criterion, it is easy to obtain the following proposition.

Proposition 1. If $(\mu+\alpha+\delta+\gamma)\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)-\frac{A\beta_2 v}{\mu+\rho}>0$, $\mu+\rho>0$, and $\alpha=\alpha_0=\frac{A\beta_1}{\mu+\rho}-v-\delta-\gamma-2\mu$, then the fixed point P_1 is asymptotically stable.

Proposition 2. Assume that $\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)^2-\frac{A\beta_2 v}{\mu+\rho}>0$. If characteristic polynomial (6) has a pair of purely imaginary eigenvalues $\lambda_{2,3}=\pm i\omega$, and $\mathrm{Re}\left(\lambda'_\alpha(\alpha_0)\right)\neq 0$, then the Hopf bifurcation occurs at the fixed point P_1 when the bifurcation parameter α pass through the critical value α_0.

Proof. Suppose that $\lambda = i\omega, (\omega>0)$ is a root of the equation (6), so we have

$$-\omega^2+\left(2\mu+v+\alpha+\delta+\gamma-\frac{A\beta_1}{\mu+\rho}\right)\omega i+\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)$$
$$(\mu+\alpha+\delta+\gamma)-\frac{A\beta_2 v}{\mu+\rho}=0,$$

then separating the real and imaginary parts of above equation, and we get

$$\begin{cases} -\omega^2+\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)(\mu+\alpha+\delta+\gamma)-\frac{A\beta_2 v}{\mu+\rho}=0, \\ (2\mu+v+\alpha+\delta+\gamma-\frac{A\beta_1}{\mu+\rho})\omega=0, \end{cases}$$

Through simple calculation, we have

$$\omega=\omega_0=\sqrt{\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)-\frac{A\beta_2 v}{\mu+\rho}},$$

$$\alpha=\alpha_0=\frac{A\beta_1}{\mu+\rho}-v-\delta-\gamma-2\mu,$$

Take the derivative of both sides of Eq. (6) with respect to α , we obtain

$$\frac{d\lambda}{d\alpha}=\frac{(\mu+\rho)\lambda^2+\left[(2\mu+\rho+v)(\mu+\rho)-A\beta_1\right]\lambda}{\Sigma}$$
$$+\frac{(\mu+\rho)\left[(\mu+v)(\mu+\rho)-A\beta_1\right]}{\Sigma},$$

and

$$\left.\frac{d\,\mathrm{Re}\,\lambda}{d\alpha}\right|_{\alpha=\alpha_0}\neq 0, \quad \left.\frac{dIm\lambda}{d\alpha}\right|_{\alpha=\alpha_0}\neq 0,$$

where

$$\Sigma = 3(\mu+\rho)\lambda^2+2(\mu+\rho)^2\lambda-A\beta+$$
$$(\mu+\gamma+\alpha+\delta)\left[(\mu+v)(\mu+\rho)-A\beta_1\right]_2.$$

According to the explicit criterion of Hopf bifurcation [22], we can get α_0 is the critical value of bifurcation, suppose that $\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)^2-\frac{A\beta_2 v}{\mu+\rho}>0$, when α pass through the critical value α_0, the model (2) occurs Hopf bifurcation at the fixed point P_1.

B. Direction and Stability of the Hopf Bifurcations

In this section, we study the direction and stability of Hopf bifurcation by using the normal form method and center manifold theory [23]. First, the model (2) can be written as

$$X_{n+1}=JX_n+\frac{1}{2}B(X_n,X_n)+\frac{1}{6}C(X_n,X_n,X_n)+O(X_n{}^4), \quad (7)$$

where J is the Jacobin matrix at the fixed point P_1, $O(X_n{}^4)$ is the 4 order indefinite small of X_n. And for $i=1,2,3$, we have

$$\begin{aligned} B_i(x,y)&=\sum_{j,k=1}^{3}\left.\frac{\partial^2 X_i(\xi,0)}{\partial\xi_j\partial\xi_k}\right|_{\xi=0} x_j y_k,\\ C_i(x,y,z)&=\sum_{j,k,l=1}^{3}\left.\frac{\partial^3 X_i(\xi,0)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0} x_j y_k z_l, \end{aligned} \quad (8)$$

Let $p,q\in C^3$ be vectors such that:

$$Jq=i\omega_0 q,\ J^T p=-i\omega_0 p,\langle p,q\rangle=\sum_{i=1}^{3}\bar{p}_i q_i=1, \quad (9)$$

where J^{T} is the transpose of the J. For model (2), we can get

$$\begin{aligned} B(x,y)&=\left(-\beta_1(x_1y_2+x_2y_1)-\beta_2(x_1y_3+x_3y_1),\beta_1(x_1y_2\right.\\ &\left.+x_2y_1)+\beta_2(x_1y_3+x_3y_1),0\right)^{\mathrm{T}},C(x,y,z)=(0,0,0)^{\mathrm{T}}, \end{aligned} \quad (10)$$

$$q=\left(\varepsilon_1+\varepsilon_2 i,\varepsilon_3+\varepsilon_4 i,1\right)^{\mathrm{T}},\quad p=(0,\varepsilon_5+\varepsilon_6 i,\varepsilon_7+\varepsilon_8 i)^{\mathrm{T}}, \quad (11)$$

where

$$\varepsilon_1=\frac{A\beta_1\left[A\beta_1-(\mu+\nu)(\mu+\rho)+A\beta_2\nu+\omega_0^2\right]}{\nu(\mu+\rho)\left[(\mu+\rho)^2+\omega_0^2\right]},$$

$$\varepsilon_2=\frac{A\beta_1\omega_0\left[(\mu+\rho)^2-A\beta_1+(\mu+\nu)(\mu+\rho)-A\beta_2\nu\right]}{\nu(\mu+\rho)^2\left[(\mu+\rho)^2+\omega_0^2\right]},$$

$$\varepsilon_3=\frac{A\beta_1-(\mu+\nu)(\mu+\rho)}{\nu(\mu+\rho)},$$

$$\varepsilon_4=\frac{\sqrt{-\left[A\beta_1-(\mu+\nu)(\mu+\rho)\right]^2-A\beta_2\nu(\mu+\rho)}}{\nu(\mu+\rho)},$$

$$\begin{aligned}\varepsilon_5&=\frac{\left[\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)^2+(\mu+\rho)^2\omega_0^2\right]\varepsilon_3}{\Omega}\\ &+\frac{\left[A\beta_1-(\mu+\nu)(\mu+\rho)\right]\left[2\omega_0(\mu+\rho)\varepsilon_4+A\beta_2\right]}{\Omega},\end{aligned}$$

$$\begin{aligned}\varepsilon_6&=\frac{\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)^2\varepsilon_4}{\Omega}-\\ &\frac{(\mu+\rho)^2\omega_0^2\varepsilon_4-A\beta_2\omega_0(\mu+\rho)}{\Omega},\end{aligned}$$

$$\begin{aligned}\varepsilon_7&=\frac{A\beta_2\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)\varepsilon_3}{\Omega}\\ &+\frac{\left[\omega_0(\mu+\rho)\varepsilon_4+A\beta_2\right]A\beta_2}{\Omega},\end{aligned}$$

$$\varepsilon_8=\frac{A\beta_2\left[\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)\varepsilon_4+\omega_0(\mu+\rho)\varepsilon_3\right]}{\Omega},$$

$$\begin{aligned}\Omega&=\left[\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)\varepsilon_4+\omega_0(\mu+\rho)\varepsilon_3\right]^2+\\ &\left[\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)\varepsilon_3+\omega_0(\mu+\rho)\varepsilon_4+A\beta_2\right]^2.\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}B(q,q)&=\left(-2\left[\beta_1(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)+\beta_2\varepsilon_1\right]-2\left[\beta_1(\varepsilon_1\varepsilon_4+\varepsilon_2\varepsilon_3)+\beta_2\varepsilon_1\right]i,\right.\\ &\left.2\left[\beta_1(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)+\beta_2\varepsilon_1\right]+2\left[\beta_1(\varepsilon_1\varepsilon_4+\varepsilon_2\varepsilon_3)+\beta_2\varepsilon_1\right]i,0\right)^{\mathrm{T}},\end{aligned} \quad (12)$$

$$\begin{aligned}B(q,\bar{q})&=\left(-2\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)-2\beta_2\varepsilon_1,\right.\\ &\left.2\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+2\beta_2\varepsilon_1,0\right)^{\mathrm{T}},\end{aligned} \quad (13)$$

$$\begin{aligned}h_{11}&=-A^{-1}B(q,\bar{q})^{\mathrm{T}}=\left(\varepsilon_9,\varepsilon_{10},\varepsilon_{11}\right)^{\mathrm{T}},\\ h_{20}&=(2i\omega_0 I_3-A)^{-1}B(q,q)^{\mathrm{T}}=\left(k_1+k_2 i,k_3+k_4 i,k_5+k_6 i\right)^{\mathrm{T}},\end{aligned} \quad (14)$$

$$\begin{aligned}H_{21}&=C(q,q,\bar{q})+B(\bar{q},h_{20})+2B(q,h_{11})\\ &=\left(\theta_3+2\theta_1+(\theta_4+2\theta_2)i,-\theta_3-2\theta_1-(\theta_4+2\theta_2)i,0\right)^{\mathrm{T}},\end{aligned} \quad (15)$$

where

$$\begin{aligned}\varepsilon_9&=-2(\mu+\rho)\left[\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right]-\\ &\frac{2A\left[\beta_2\nu-\beta_1(\mu+\alpha+\delta+\gamma)\right]\left[\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right]}{(\mu+\rho)\left[(\mu+\alpha+\delta+\gamma)\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)+A\beta_2\nu\right]},\end{aligned}$$

$$\varepsilon_{10}=-\frac{2\left[\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right](\mu+\rho)(\mu+\alpha+\delta+\gamma)}{(\mu+\alpha+\delta+\gamma)\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)+A\beta_2\nu},$$

$$\varepsilon_{11}=-\frac{2\nu\left[\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right](\mu+\rho)}{(\mu+\alpha+\delta+\gamma)\left(A\beta_1-\mu^2-\nu\mu-\rho\mu-\nu\rho\right)+A\beta_2\nu},$$

$$k_1=2\left[\beta_1(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right]\left(\mu+\rho+\frac{A\beta_1}{\mu+\rho}\right)-\frac{2\omega_0 k_6}{\nu},$$

$$k_2=\frac{2\omega_0 k_5}{\nu}-\frac{k_6}{\nu}\left(\mu+\rho+\frac{A\beta_1}{\mu+\rho}\right),$$

$$k_3=\frac{2\omega_0 k_6}{\nu}-\frac{k_5}{\nu}\left(\mu+\nu-\frac{A\beta_1}{\mu+\rho}\right),$$

$$k_4=-\frac{k_6}{\nu}\left(\mu+\nu-\frac{A\beta_1}{\mu+\rho}\right)-\frac{2\omega_0 k_5}{\nu},$$

$$k_5=-2\nu\left[\beta_1(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)+\varepsilon_1\beta_2\right],$$

$$k_6=-2\nu\left[\beta_1(\varepsilon_1\varepsilon_4+\varepsilon_2\varepsilon_3)+\varepsilon_2\beta_2\right],$$

$$\theta_1=-\beta_1(\varepsilon_1\varepsilon_{10}+\varepsilon_3\varepsilon_9)-\beta_2(\varepsilon_1\varepsilon_{11}+\varepsilon_9),$$

$$\theta_2=-\beta_1(\varepsilon_2\varepsilon_{10}+\varepsilon_4\varepsilon_9)-\beta_2\varepsilon_2\varepsilon_{11},$$

$$\theta_3=-\beta_1(\varepsilon_1k_3+\varepsilon_2k_4+\varepsilon_3k_1+\varepsilon_4k_2)-\beta_2(\varepsilon_1k_5+\varepsilon_2k_6+k_1),$$

$$\theta_4=-\beta_1(\varepsilon_1k_4-\varepsilon_2k_3+\varepsilon_3k_2-\varepsilon_4k_1)-\beta_2(\varepsilon_1k_6-\varepsilon_2k_5+k_2).$$

Through direct calculation, we also has

$$G_{21}=\langle p,H_{21}\rangle=\varepsilon_5(2\theta_1+\theta_3)+\varepsilon_6(2\theta_2+\theta_4)i, \quad (16)$$

Theorem 1. Consider model (2), the first Lyapunov coefficient associated to the fixed point P_1 is given by

$$l_1=\frac{1}{2}\mathrm{Re}\,G_{21}=-\frac{1}{2}\varepsilon_5(2\theta_1+\theta_3), \quad (17)$$

If l_1 is different from zero, then model (2) has a transversal Hopf point at P_1. More precisely, if $l_1<0$, the Hopf bifurcation at the fixed point P_1 is supercritical and there exists a stable periodic orbit near the asymptotically stable fixed point P_1; if $l_1>0$, the Hopf bifurcation at the fixed point P_1 is subcritical.

III. Dynamical Behaviors of the Discretized SEIR Epidemic Model

A. Stability of the Fixed Points of the Discretized Model

Next, we study the asymptotic stability of model (3). The fixed points of model (3) satisfy:

$$\begin{cases} S_n = S_n + h\left[A - \beta_1 E_n S_n - \beta_2 I_n S_n - (\mu+\rho) S_n\right] \\ E_n = E_n + h\left[\beta_1 E_n S_n + \beta_2 I_n S_n - (\mu+v) E_n\right] \\ I_n = I_n + h\left[vE_n - (\mu+\alpha+\delta+\gamma) I_n\right] \end{cases} \quad (18)$$

The Jacobian matrix J^* of model (3) at the fixed point $P^* = (S^*, E^*, I^*)$ is given by

$$\begin{bmatrix} 1 - h\beta_1 E^* - h\beta_2 I^* - h(\mu+\rho) & -h\beta_1 S^* & -h\beta_2 S^* \\ h\beta_1 E^* + h\beta_2 I^* & 1 + h\beta_1 S^* - h(\mu+v) & h\beta_2 S^* \\ 0 & hv & 1 - h(\mu+\alpha+\delta+\gamma) \end{bmatrix}, \quad (19)$$

For simplification, we only consider the fixed point P_1. And the characteristic equation of model (3) at the fixed point P_1 has the form

$$p(\lambda) = \left[\lambda - 1 + h(\mu+\rho)\right]\left(\lambda^2 + a_1\lambda + a_2\right), \quad (20)$$

where

$$a_1 = h(2\mu + v + \alpha + \delta + \gamma) - 2 - \frac{A\beta_1}{\mu+\rho},$$

$$a_2 = \left[1 + \frac{A\beta_1}{\mu+\rho} - h(\mu+v)\right]\left[1 - h(\mu+\alpha+\delta+\gamma)\right] - \frac{A\beta_2 v h^2}{\mu+\rho}.$$

From the local stability theory of fixed point, it is easy to obtain the following Proposition:

Proposition 3. (1) P_1 is local asymptotically stable if one of the following conditions holds:

(a) $a_1 - 4a_2 \ge 0,\ 0 < h(\mu+\rho) < 2,\ -1 - a_1 < a_2 < \min\{1 + a_1, 1 - a_1\}$;

(b) $a_1 - 4a_2 < 0,\ 0 < h(\mu+\rho) < 2,\ a_2 < 1$.

(2) P_1 is unstable if one of the following conditions holds:

(a) $a_1 - 4a_2 \ge 0,\ h(\mu+\rho) > 2,\ a_2 < \min\{-1 - a_1, a_1 - 1\}$ or $a_2 > \max\{-1 - a_1, a_1 - 1\}$;

(b) $a_1 - 4a_2 < 0,\ h(\mu+\rho) > 2,\ a_2 > 1$.

(3) P_1 is non-hyperbolic point if one of the following conditions holds:

(a) $a_1 - 4a_2 \ge 0,\ h(\mu+\rho) = 2,\ a_2 = -1 - a_1$ or $a_2 = a_1 - 1$;

(b) $a_1 - 4a_2 < 0,\ h(\mu+\rho) = 2,\ a_2 = 1$.

B. Existence of Hopf Bifurcation

When $a_1 - 4a_2 < 0$, the eigenvalues of model (3) at fixed point $P_1 = (A/\mu+\rho, 0, 0)$ can be written as

$$\lambda_1 = 1 - h(\mu+\rho),\ \ \lambda_{2,3} = -\frac{1}{2}a_1 \pm \frac{\sqrt{4a_2 - a_1^2}}{2}i. \quad (21)$$

Assume that

$$\alpha_0 = \frac{A\beta_1 - (\mu+v)(\mu+\rho) - A\beta_2 vh}{\mu+\rho+A\beta_1 h - h(\mu+v)(\mu+\rho)} - (\mu+\delta+\gamma),$$

and $h(\mu+\rho) \ne 0, 2,\ \mu+\rho+A\beta_1 h \ne h(\mu+v)(\mu+\rho),$ we can get:

$$\left|\lambda_1(\alpha_0)\right| = \left|1 - h(\mu+\rho)\right|,\ \ \left|\lambda_{2,3}(\alpha_0)\right| = a_2 = 1, \quad (22)$$

$$\left.\frac{d\left(\left|\lambda_{2,3}\right|\right)}{d\alpha}\right|_{\alpha=\alpha_0} = \frac{h\left[h(\mu+v)(\mu+\rho) - (\mu+\rho+A\beta_1 h)\right]}{\mu+\rho} \ne 0, \quad (23)$$

$$\lambda_1(\alpha_0) = 1 - h(\mu+\rho), \lambda_{2,3}(\alpha_0) = -\frac{a_1}{2} \pm \frac{\sqrt{4 - a_1^2}}{2}i. \quad (24)$$

and by calculation we can get $\lambda_{2,3}^{\ m}(\alpha_0) \ne 1, 2, 3, 4$. According to bifurcation theory [24], the Hopf bifurcation occurs at the fixed point $P_1 = (A/\mu+\rho, 0, 0)$.

C. Direction and Stability of the Hopf Bifurcation

Next, we investigate the stability and direction of the Hopf bifurcation by using the Kuznetsov's normal form method and center manifold theory [25]. First, the model (3) can be written as

$$X_{n+1} = JX_n + \frac{1}{2}B(X_n, X_n) + \frac{1}{6}C(X_n, X_n, X_n) + O(X_n^{\ 4}), \quad (25)$$

where J is the Jacobin matrix at the fixed point P_1, $O(X_n^{\ 4})$ is the 4 order indefinite small of X_n, and for $i = 1, 2, 3$, we can get:

$$\begin{aligned} B_i(x,y) &= \sum_{j,k=1}^{3} \left.\frac{\partial^2 X_i(\xi,0)}{\partial\xi_j \partial\xi_k}\right|_{\xi=0} x_j y_k, \\ C_i(x,y,z) &= \sum_{j,k,l=1}^{3} \left.\frac{\partial^3 X_i(\xi,0)}{\partial\xi_j \partial\xi_k \partial\xi_l}\right|_{\xi=0} x_j y_k z_l, \end{aligned} \quad (26)$$

Let $p, q \in C^3$ be vectors such that:

$$Jq = \lambda_2 q,\ \ J^{\mathrm{T}} p = \lambda_3 p,\ \ \langle p, q\rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1, \quad (27)$$

where J^{T} is the transpose of the J, and λ_2, λ_3 is a pair of complex conjugate eigenvalues at the fixed point P_1. For the model (3), we can get

$$\begin{aligned} &B(x,y) = (-\beta_1(x_1 y_2 + x_2 y_1) - \beta_2(x_1 y_3 + x_3 y_1), \beta_1(x_1 y_2 \\ &+ x_2 y_1) + \beta_2(x_1 y_3 + x_3 y_1), 0)^{\mathrm{T}},\ \ C(x,y,z) = (0,0,0)^{\mathrm{T}}, \end{aligned} \quad (28)$$

and

$$q = (\varphi_1 + \varphi_2 i, \varphi_3 + \varphi_4 i, 1)^{\mathrm{T}},\ \ p = (0, \varphi_5 + \varphi_6 i, \varphi_7 + \varphi_8 i)^{\mathrm{T}}, \quad (29)$$

where

$$\varphi_1 = \frac{(A\beta_1 h\varphi_3 + A\beta_2 h)\left[2 - 2h(\mu+\rho) + a_1\right] - A\beta_1 h\varphi_4\sqrt{4 - a_1^2}}{2 + 2a_1 + 2h(\mu+\rho)\left[2h(\mu+\rho) - 2 - a_1\right]},$$

$$\varphi_2 = \frac{2A\beta_1 h\varphi_4 + \sqrt{4 - a_1^2}}{2 + 2a_1 + 2h(\mu+\rho)\left[2h(\mu+\rho) - 2 - a_1\right]},$$

$$\varphi_3 = \frac{2 - 2h(\mu+\alpha+\delta+\gamma) + a_1}{2hv}, \qquad \varphi_4 = -\frac{\sqrt{4 - a_1^2}}{\mu+\rho},$$

$$\varphi_5 = \frac{(\mu+\rho)\left[2h(\mu+\alpha+\delta+\gamma) - 2 - a_1\right]\varphi_7 + (\mu+\rho)\varphi_8\sqrt{4 - a_1^2}}{2A\beta_2 h},$$

$$\varphi_6 = \frac{(\mu+\rho)\left[2h(\mu+\alpha+\delta+\gamma) - 2 - a_1\right]\varphi_8 - (\mu+\rho)\varphi_7\sqrt{4 - a_1^2}}{2A\beta_2 h},$$

$$\varphi_7 = \frac{2A\beta_2 h(\mu+\rho)\left[2h(\mu+\alpha+\delta+\gamma) - 2 - a_1\right]\varphi_3}{\Delta} + \frac{4A^2\beta_2^{\ 2}h^2 - 2A\beta_2 h(\mu+\rho)\varphi_4\sqrt{4 - a_1^2}}{\Delta},$$

$$\varphi_8=\frac{2A\beta_2 h(\mu+\rho)\left[2h(\mu+\alpha+\delta+\gamma)-2-a_1\right]\varphi_3}{\Delta}+\frac{\varphi_4\sqrt{4-a_1^2}\,2A\beta_2 h(\mu+\rho)}{\Delta},$$

$$\Delta=\left\{(\mu+\rho)\left[2h(\mu+\alpha+\delta+\gamma)-2-a_1\right]\varphi_4+\varphi_3\sqrt{4-a_1^2}\right\}^2+\left\{\left[2h(\mu+\alpha+\delta+\gamma)-2-a_1\right]\varphi_3(\mu+\rho)\varphi_4\sqrt{4-a_1^2}+2A\beta_2 h\right\}^2.$$

So the coefficients of the normal of model (3) can be computed by the following formulas:

$$g_{20}=\langle p,B(q,q)\rangle=\varphi_9+\varphi_{10}i,\quad g_{11}=\langle p,B(q,\bar{q})\rangle=\varphi_{11}+\varphi_{12}i,\quad g_{02}=\langle p,B(\bar{q},\bar{q})\rangle=\varphi_{13}+\varphi_{14}i, \tag{30}$$

where

$$\varphi_9=2\beta_1 h(\varphi_2\varphi_4-\varphi_1\varphi_3)(\varphi_5-\varphi_6)-\beta_2 h(\varphi_1\varphi_5+\varphi_2\varphi_6),$$
$$\varphi_{10}=2\beta_1 h(\varphi_2\varphi_4-\varphi_1\varphi_3)(\varphi_6-\varphi_5)-\beta_2 h(\varphi_1\varphi_6+\varphi_2\varphi_5),$$
$$\varphi_{11}=2h\varphi_5\left[\beta_1(\varphi_1\varphi_3+\varphi_2\varphi_4)+\beta_2\varphi_1\right],$$
$$\varphi_{12}=2h\varphi_6\left[\beta_1(\varphi_1\varphi_3+\varphi_2\varphi_4)+\beta_2\varphi_1\right],$$
$$\varphi_{13}=2\beta_1 h(\varphi_1\varphi_3+\varphi_2\varphi_4)(\varphi_5+\varphi_6)+2\beta_2 h(\varphi_1\varphi_5+\varphi_2\varphi_6),$$
$$\varphi_{14}=2\beta_1 h(\varphi_1\varphi_3+\varphi_2\varphi_4)(\varphi_6-\varphi_5)+2\beta_2 h(\varphi_1\varphi_6-\varphi_2\varphi_5),$$

and

$$\begin{aligned}g_{21}&=\langle p,C(q,q,\bar{q})\rangle+2\left\langle p,B\left(q,(I_n-J)^{-1}B(q,\bar{q})\right)\right\rangle\\&+\left\langle p,B\left(\bar{q},(\lambda_2^2 I_n-J)^{-1}B(q,q)\right)\right\rangle\\&+\frac{\lambda_3(1-2\lambda_2)}{1-\lambda_2}g_{20}g_{11}+\frac{2}{1-\lambda_2}|g_{11}|^2+\frac{\lambda_2}{\lambda_2^3-1}|g_{02}|^2\\&=2M_4+N_1+N_3+N_5+(2M_5+N_2+N_4+N_6)i,\end{aligned}\tag{31}$$

where

$$M_1=\frac{\varphi_{12}(\mu+\alpha+\delta+\gamma)\left[A\beta_1+(\mu+v)(\mu+\rho)-2A\beta_2 v\right]}{\varphi_6 h(\mu+\rho)\left[(\mu+v)(\mu+\rho)(\mu+\alpha+\delta+\gamma)-A\beta_2 v\right]},$$
$$M_2=\frac{\varphi_{12}(\mu+\rho)(\mu+\alpha+\delta+\gamma)}{\varphi_6 h\left[(\mu+v)(\mu+\rho)(\mu+\alpha+\delta+\gamma)-A\beta_2 v\right]},$$
$$M_3=\frac{\varphi_{12}v(\mu+\rho)}{\varphi_6 h\left[(\mu+v)(\mu+\rho)(\mu+\alpha+\delta+\gamma)-A\beta_2 v\right]},$$
$$M_4=h\left[\beta_2 M_3-\beta_1(M_1-M_2)\right](\varphi_1\varphi_5+\varphi_2\varphi_6)-\beta_2 hM_1\varphi_5,$$
$$M_5=h\left[\beta_2 M_3-\beta_1(M_1-M_2)\right](\varphi_2\varphi_5-\varphi_1\varphi_6)-\beta_2 hM_1\varphi_6,$$
$$\begin{aligned}N_1=&-\varphi_5 h\left[\beta_1(\varphi_1\eta_9+\varphi_2\eta_{10}+\varphi_3\eta_7+\varphi_4\eta_8)+\beta_2(\varphi_1\eta_{11}+\varphi_2\eta_{12})\right]\\&-\varphi_6 h\left[\beta_1(\varphi_1\eta_{10}-\varphi_2\eta_9+\varphi_3\eta_8-\varphi_4\eta_7)+\beta_2(\varphi_1\eta_{12}-\varphi_2\eta_{11})\right],\end{aligned}$$
$$\begin{aligned}N_2=&\varphi_6 h\left[\beta_1(\varphi_1\eta_9+\varphi_2\eta_{10}+\varphi_3\eta_7+\varphi_4\eta_8)+\beta_2(\varphi_1\eta_{11}+\varphi_2\eta_{12})\right]\\&+\varphi_5 h\left[\beta_1(\varphi_1\eta_{10}-\varphi_2\eta_9+\varphi_3\eta_8-\varphi_4\eta_7)+\beta_2(\varphi_1\eta_{12}-\varphi_2\eta_{11})\right],\end{aligned}$$
$$N_3=\frac{(a_1+1)(a_1+2)(\varphi_{10}\varphi_{12}-\varphi_9\varphi_{11})}{4+2a_1}+\frac{(3+2a_1)\sqrt{4-a_1^2}(\varphi_{10}\varphi_{11}+\varphi_9\varphi_{12})}{4+2a_1},$$
$$N_4=\frac{(3+2a_1)\sqrt{4-a_1^2}(\varphi_{10}\varphi_{12}-\varphi_9\varphi_{11})}{4+2a_1}-\frac{(a_1+1)(a_1+2)(\varphi_{10}\varphi_{11}+\varphi_9\varphi_{12})}{4+2a_1},$$
$$N_5=\frac{(8a_1^4+2a_1^3-32a_1^2-8a_1+16)(\varphi_{13}^2+\varphi_{14}^2)}{a_1^3(4a_1^2-a_1+20)-4(a_1-2)^2}+\varphi_{11}^2+\varphi_{12}^2,$$
$$N_6=\frac{2a_1\sqrt{4-a_1^2}(4a_1^2+a_1-4)(\varphi_{13}^2+\varphi_{14}^2)}{a_1^3(4a_1^2-a_1+20)-4(a_1-2)^2}+\frac{(\varphi_{11}^2+\varphi_{12}^2)\sqrt{4-a_1^2}}{a_1+2},$$
$$\eta_1=\frac{1}{2}a_1^2-2+h(\mu+\rho),$$
$$\eta_2=\frac{1}{2}a_1^2-2-\frac{A\beta_1 h}{\mu+\rho}+h(\mu+v),$$
$$\eta_3=\frac{1}{2}a_1^2-2+h(\mu+\alpha+\delta+\gamma),\eta_4=\frac{a_1\sqrt{4-a_1^2}}{2},$$
$$\eta_5=(\mu+\rho)\left[\eta_1(\eta_2\eta_3-\eta_4^2)-\eta_4^2(\eta_2+\eta_3)\right]-\eta_1 A\beta_2 vh^2,$$
$$\eta_6=\eta_4(\mu+\rho)\left[(\eta_2\eta_3-\eta_4^2)+\eta_1(\eta_2+\eta_3)\right]-\eta_4 A\beta_2 vh^2,$$
$$\eta_7=\frac{(f_1\eta_3-f_2\eta_4)\left[(\mu+\rho)(\eta_2\eta_5+\eta_4\eta_6)+A\beta_1 h\eta_5\right]}{\eta_5^2+\eta_6^2}-\frac{(f_2\eta_3+f_1\eta_4)\left[(\mu+\rho)(\eta_4\eta_5-\eta_2\eta_6)-A\beta_1 h\eta_6\right]}{\eta_5^2+\eta_6^2},$$
$$\eta_8=\frac{(f_2\eta_3+f_1\eta_4)\left[(\mu+\rho)(\eta_1\eta_5+\eta_4\eta_6)+A\beta_1 h\eta_5\right]}{\eta_5^2+\eta_6^2}-\frac{(f_1\eta_3-f_2\eta_4)\left[(\mu+\rho)(\eta_4\eta_5-\eta_2\eta_6)-A\beta_1 h\eta_6\right]}{\eta_5^2+\eta_6^2},$$
$$\eta_9=\frac{(\mu+\rho)(f_2\eta_4-f_1\eta_3)\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2}+\frac{\eta_4(f_2\eta_3+f_1\eta_4)(\eta_2+\eta_3)(\mu+\rho)^2}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2},$$
$$\eta_{10}=\frac{\eta_4(f_1\eta_3-f_2\eta_4)(\eta_2+\eta_3)(\mu+\rho)^2}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2}-\frac{(\mu+\rho)(f_1\eta_4+f_2\eta_3)\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2},$$
$$\eta_{11}=\frac{(\mu+\rho)vh\left[f_1 A\beta_2 vh^2-f_1(\eta_2\eta_3-\eta_4^2)(\mu+\rho)\right]}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2}-\frac{vhf_2\eta_4(\mu+\rho)^2(\eta_2+\eta_3)}{\left[(\eta_2\eta_3-\eta_4^2)(\mu+\rho)-A\beta_2 vh^2\right]^2+\eta_4^2(\eta_2+\eta_3)^2(\mu+\rho)^2},$$

$$\eta_{12}=\frac{(\mu+\rho)vh\left[f_1\eta_4(\eta_2+\eta_3)(\mu+\rho)+f_2A\beta_2vh^2\right]}{\left[(\eta_2\eta_3-\eta_4^{\ 2})(\mu+\rho)-A\beta_2vh^2\right]^2+\eta_4^{\ 2}(\eta_2+\eta_3)^2(\mu+\rho)^2}-\frac{f_2vh(\eta_2\eta_3-\eta_4^{\ 2})(\mu+\rho)^2}{\left[(\eta_2\eta_3-\eta_4^{\ 2})(\mu+\rho)-A\beta_2vh^2\right]^2+\eta_4^{\ 2}(\eta_2+\eta_3)^2(\mu+\rho)^2},$$

$$f_1=h\left[2\beta_1(\varphi_2\varphi_4-\varphi_1\varphi_3)-\beta_2\varphi_1\right],$$

$$f_2=-h\left[2\beta_1(\varphi_1\varphi_3-\varphi_2\varphi_4)-\beta_2\varphi_2\right].$$

Then, by calculate one has

$$\begin{aligned}l_1(\alpha_0)&=\mathrm{Re}\left(\frac{\lambda_3 g_{21}}{2}\right)-\mathrm{Re}\left(\frac{\lambda_3^{\ 2}(1-2\lambda_2)}{2(1-\lambda_2)}g_{20}g_{11}\right)-\frac{1}{2}|g_{11}|^2-\frac{1}{4}|g_{02}|^2\\&=-\frac{1}{4}\Big[a_1(2M_4+N_1+2N_3+N_5)+2(\varphi_{11}^{\ 2}+\varphi_{12}^{\ 2})\\&\quad-\sqrt{4-a_1^{\ 2}}(2M_5+N_2+2N_4+N_6)+\varphi_{13}^{\ 2}+\varphi_{14}^{\ 2}\Big],\end{aligned}\tag{32}$$

Theorem 2. The direction and stability of Hopf bifurcation at the fixed point P_1 is determined by $l_1(\alpha_0)$. If $l_1(\alpha_0)<0(>0)$, the Hopf bifurcation of model (3) at $\alpha_0=\dfrac{A\beta_1-(\mu+v)(\mu+\rho)-A\beta_2vh}{\mu+\rho+A\beta_1h-h(\mu+v)(\mu+\rho)}-(\mu+\delta+\gamma)$ is supercritical (subcritical), and the unique closed invariant curve bifurcating from P_1 is asymptotically stable (unstable).

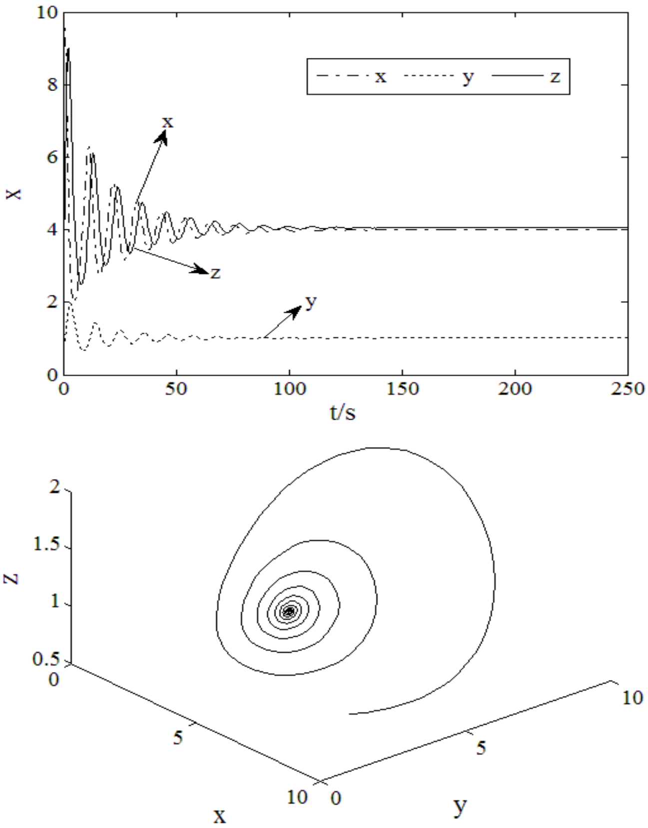


Fig. 1. Time history and phase diagram of model (2) with $\alpha = 0.5$

IV. NUMERICAL SIMULATIONS

A. Hop bifurcation simulation of continuous Model

First, we give a numerical example of model (2). Let $A=10,\mu=0.2,\rho=0.5,\delta=0.1,\beta_1=0.05,\beta_2=0.014,\gamma=0.03,$ $v=0.13$, and through compute we get the critical value $\alpha_0=0.543$. The fixed point P_1 is stable when $\alpha=0.5<\alpha_0$, and unstable when $\alpha=0.6>\alpha_0$, as show in Fig. 1 and 2, respectively. From the formulas in previous section, we can get $l_1(\alpha_0)=0.3164>0$. Thus, the periodic solutions bifurcating from the fixed point P_1 are subcritical and unstable.

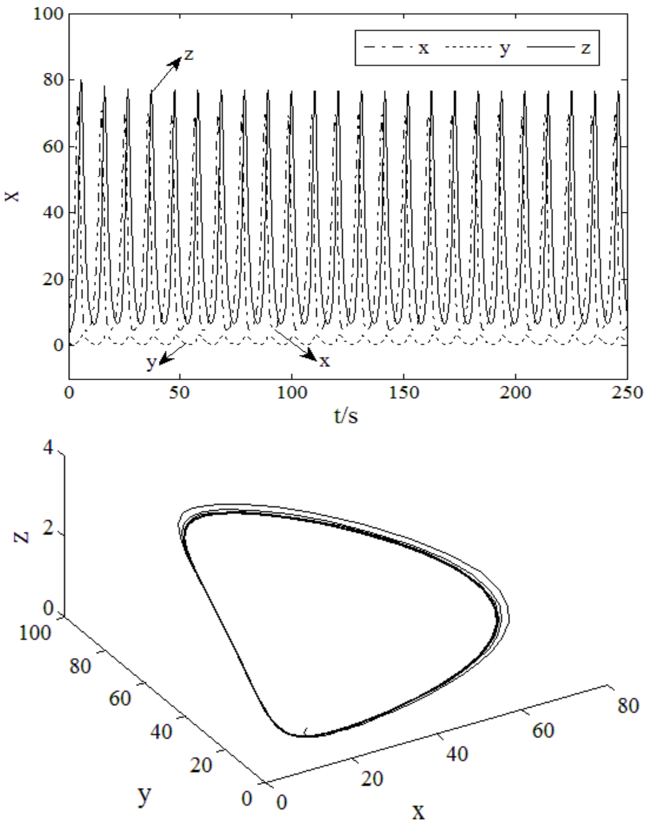
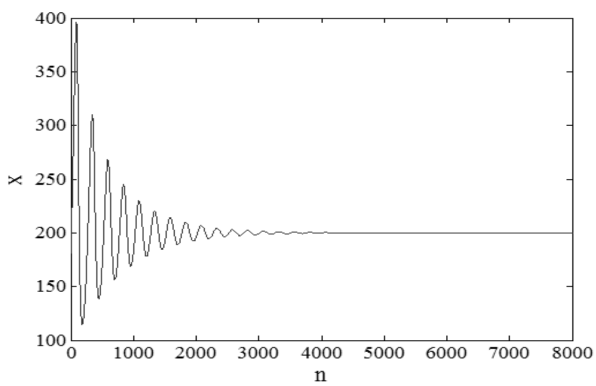


Fig. 2. Time history and phase diagram of model (2) with $\alpha = 0.6$

B. Hop bifurcation simulation of discrete Model

Next, we choose one group parameters: $A=10,\mu=0.4,$ $\rho=0.12,\delta=0.2,\beta_1=0.1,\beta_2=0.1,\gamma=0.3,v=0.1,h=0.1$ and by compute we get the critical value $\alpha_0=0.329$. The fixed point P_1 is stable when $\alpha=0.31<\alpha_0$, and unstable when $\alpha=0.34>\alpha_0$, as shown in Fig. 3 and Fig. 4, respectively. Based on the previous conclusions and through complex calculations, we have $l_1(\alpha_0)=0.0764>0$. Therefore the Hopf bifurcation of model (3) at the fixed point P_1 is subcritical, and the unique invariant curve which is resulting from the bifurcation at fixed point is unstable.



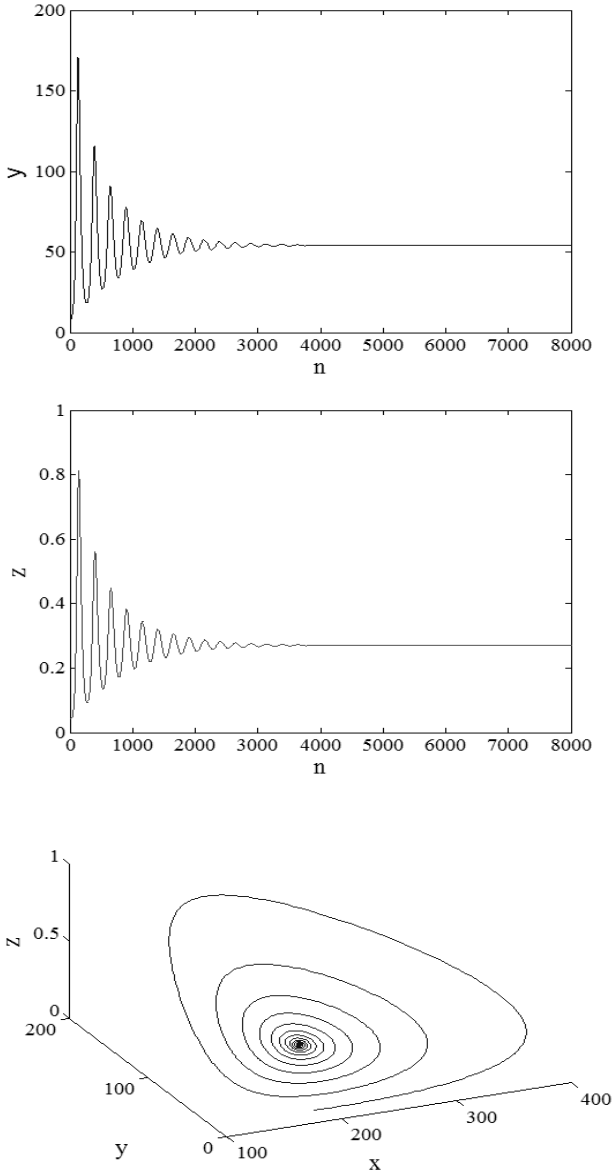


Fig. 3. Time history and phase diagram of model (3) with $\alpha = 0.31$

V. Comparisons

For the continuous-time model (2) at the fixed point P_1 with $\left(\mu+v-\frac{A\beta_1}{\mu+\rho}\right)^2+\frac{A\beta_2 v}{\mu+\rho}>0$, we know that the critical value of Hopf bifurcation as follows:

$$\alpha_0^1=\frac{A\beta_1}{\mu+\rho}-v-\delta-\gamma-2\mu, \tag{33}$$

For the discrete-time model (3) at the fixed point P_1 with $a_1-4a_2<0$, let $h=0.4, \beta_2=0.5$, we obtain the critical value of Hopf bifurcation as follows:

$$\alpha_0^2=\frac{A\beta_1-(\mu+v)(\mu+\rho)-0.2Av}{\mu+\rho+0.4\left[A\beta_1-(\mu+v)(\mu+\rho)\right]}-(\mu+\delta+\gamma), \tag{34}$$

By simple calculation, we can get the following conclusions.

Proposition 4. Hopf bifurcations of continuous-time model (2) and discrete-time model (3) occur simultaneously when

$$\frac{0.4A^2\beta_1^2+(\mu+\rho)\left[(\mu+v)(\mu+\rho-0.4A\beta_1)-0.2Av\right]}{(\mu+\rho)\left[\mu+\rho+0.4A\beta_1-0.4(\mu+v)(\mu+\rho)\right]}=v+\mu,$$

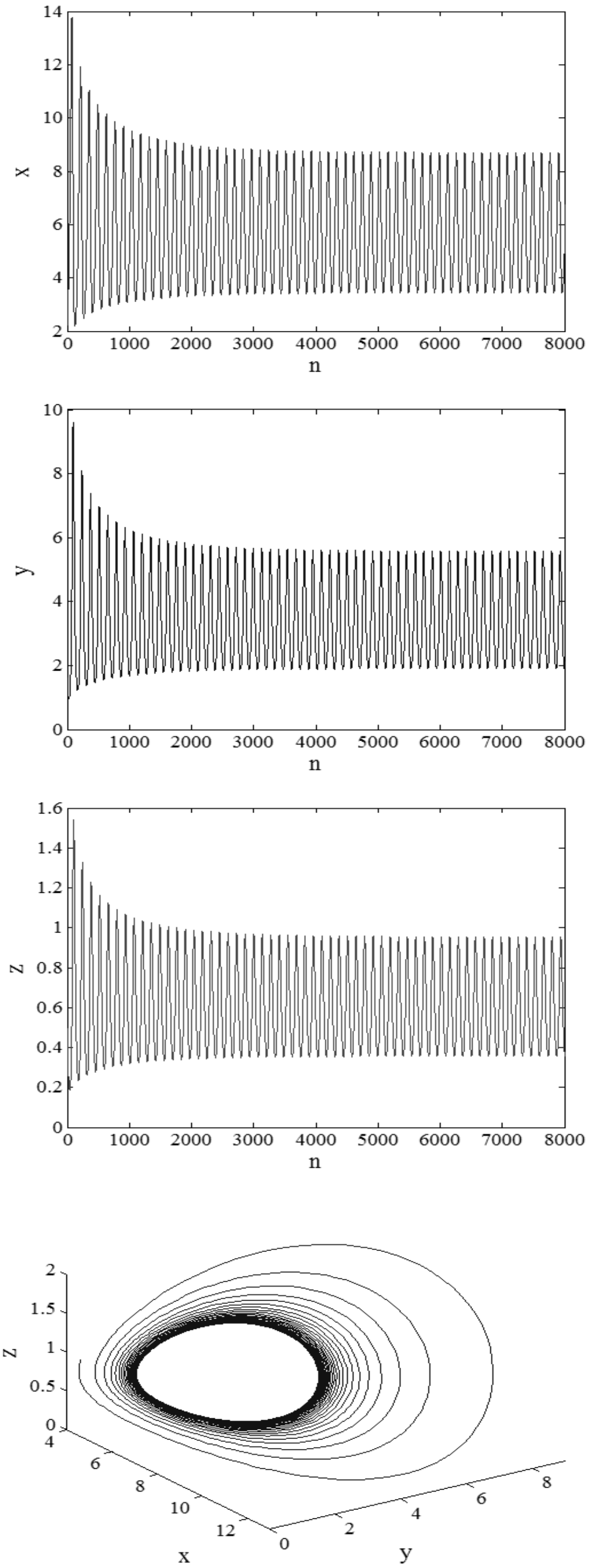


Fig. 4. Time history and phase diagram of model (3) with $\alpha = 0.34$

Proposition 5. The continuous-time model (2) undergoes Hopf bifurcation earlier than the discrete-time model (3) when

$$\frac{0.4A^2\beta_1^2+(\mu+\rho)\left[(\mu+v)(\mu+\rho-0.4A\beta_1)-0.2Av\right]}{(\mu+\rho)\left[\mu+\rho+0.4A\beta_1-0.4(\mu+v)(\mu+\rho)\right]}<v+\mu,$$

Proposition 6. The discrete-time model (3) undergoes Hopf bifurcation earlier than the continuous-time model (2) when

$$\frac{0.4A^2\beta_1^2+(\mu+\rho)\left[(\mu+v)(\mu+\rho-0.4A\beta_1)-0.2Av\right]}{(\mu+\rho)\left[\mu+\rho+0.4A\beta_1-0.4(\mu+v)(\mu+\rho)\right]}>v+\mu.$$

VI. CONCLUSIONS

In this paper, we introduced a SEIR epidemic model and obtained a new discrete-time epidemic model by using the Forward-Euler difference method. A necessary and sufficient condition for existence of the solution of the SEIR epidemic model are obtained, and we also investigated the local stability of the fixed point of the epidemic model and its discretized counterpart. Besides, the stability and direction of Hopf bifurcation are proved by using the Kuznetsov's normal form method and center manifold theory. And the numerical simulations were presented to illustrate the above main results. Finally we give some comparisons of bifurcation between the discrete-time epidemic model and its continuous-time model..

REFERENCES

[1] G. Licsko, A. Champneys, C. Hos, "Nonlinear analysis of a single stage pressure relief valve," *IAENG International Journal of Applied Mathematics*, vol. 39, no. 4, pp. 286-299, 2009.

[2] D. Jana, S. Chakraborty, N. Bairagi, "Stability, nonlinear oscillations and bifurcation in a delay-induced predator-prey system with harvesting," *Engineering Letters*, vol. 20, no. 3, pp. 238-246, 2012.

[3] Y. L. Song, Y. Y. Han, T. H. Zhang, "Stability and Hopf bifurcation in a model of gene expression with distributed time delays," *Applied Mathematics and Computation*, vol. 243, no. 15, pp. 398-412, Sep. 2014.

[4] L. P. Li, L. H. Huang, "Concurrent homoclinic bifurcation and Hopf bifurcation for a class of planar Filippov systems," *Journal of Mathematical Analysis and Applications*, vol. 411, no. 1, pp. 83-94, Mar. 2014.

[5] Z. Balanov, Q. W. Hu, W. Krawcewicz, "Global Hopf bifurcation of differential equations with threshold type state-dependent delay," *J. Differential Equations*, vol. 257, no. 7, pp. 2622-2670, Oct. 2014.

[6] F. X. Wang, "Bifurcations of nonlinear normal modes via the configuration domain and the time domain shooting methods," *Commun Nonlinear Sci Numer Simulat*, vol. 20, no. 2, pp. 614-628, Feb. 2015.

[7] V. Pandey, S. Singh, "Detailed bifurcation analysis with a simplified model for advance heavy water reactor system," *Commun Nonlinear Sci Numer Simulat*, vol. 20, no. 1, pp. 186-198, Jan. 2015.

[8] J. Llibre, C. Valls, "Hopf bifurcation of a generalized Moon-Rand system," *Commun Nonlinear Sci Numer Simulat*, vol. 20, no. 2, pp. 1070-1077, Mar. 2015.

[9] W. J. Du, J. G. Zhang, S. Qin, "Hopf Bifurcation and Sliding Mode Control of Chaotic Vibrations in a Four-dimensional Hyperchaotic System," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp247-255, 2016.

[10] F. Gurcan, S. Kartal, I. Ozturk, F. Bozkurt, "Stability and bifurcation analysis of a mathematical model for tumor-immune interaction with piecewise constant arguments of delay," *Chaos, Solitons & Fractals*, vol. 68, no. 1, pp. 169-179, Nov. 2014.

[11] G. D. Zhang, Y. Shen, B. S. Chen. "Bifurcation analysis in a discrete differential-algebraic predator-prey system," *Applied Mathematical Modelling*, vol. 38, no. 1, pp. 4835-4848, Oct. 2014.

[12] C. Wang, X. Y. Li, "Further investigations into the stability and bifurcation of a discrete predator–prey model," *Journal of Mathematical Analysis and Applications*, vol. 422, no. 2, pp. 920-939, Feb. 2015.

[13] A. A. Elsadany, "Dynamical complexities in a discrete-time food chain," *Computational Ecology and Software*, vol. 2, no. 2, pp. 124-139, Jun. 2012.

[14] Z. Y. Hu, Z. D. Teng, L. Zhang, "Stability and bifurcation analysis in a discrete SIR epidemic model," *Mathematics and Computers in Simulation*, vol. 97, no. 1, pp. 80-93, Mar. 2014.

[15] E. M. Elabbasy, A. A. Elsadany, Y. Zhang, "Bifurcation analysis and chaos in a discrete reduced Lorenz system," *Applied Mathematics and Computation*, vol. 228, no. 1, pp. 184-194, Feb. 2014.

[16] Z. M. He, J. Qiu. "Neimark-Sacker Bifurcation of a third-order rational difference equation," *Journal of Difference Equations and Applications*, vol. 19, no. 9, pp. 1513-1522, Mar. 2013.

[17] M. D. Kermark, A. G. Mckendrick, "Contributions to the Mathematical Theory of Epidemics. III. Further Studies of the Problem of Endemicity," *roceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 115, no. 5, pp. 700-721, Jul. 1933.

[18] O. N. Bjørnstad, B. F. Finkenstaedt, B. T. Greenfell, "Dynamics of measles epidemics: estimating scaling of transmission rates using a time series SIR model," *Ecol. Monogr.*, vol. 72, no. 2, pp. 169-184, May 2002.

[19] L. Stone, R. Olinky, A. Huppert, "Seasonal dynamics of recurrent epidemics," *Nature*, vol. 446, no. 7135, pp. 533-536, Mar. 2007.

[20] D. Xiao, S. Ruan, "Global analysis of an epidemic model with nonmonotone incidence rate," *Mathematical Biosciences*, vol. 208, no. 2, pp. 419-429, Aug. 2007.

[21] B. J. Wang, "Control strategies of an SEIR epidemic model with infectious force in latent period," *Journal of Northeast Normal University*, vol.46, no. 1, pp. 28-32, Mar. 2014.

[22] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, "Theory and Applications of Hopf Bifurcation," in *Cambridge University Press*, 1st ed., vol.41, The Pitt Building, Trumpington Street, Cambridge, United Kingdom, 1981, pp. 14-71.

[23] S. A. A. Hosseini, "Dynamic stability and bifurcation of a nonlinear in-extensional rotating shaft with internal damping," *Nonlinear Dynamics*, vol.74, no. 1, pp. 345-358, Jun. 2013.

[24] B. G. Xin, T. Chen, J. H. Ma, "Neimark-Sacker Bifurcation in a Discrete-Time Financial System," *Discrete Dynamics in Nature and Society*, vol. 2010, no. 1, pp. 1-13, Aug. 2010.

[25] B. G. Xin, Z. H. Wu, "Neimark-Sacker Bifurcation Analysis and 0–1 Chaos Test of an Interactions Model between Industrial Production and Environmental Quality in a Closed Area," *Sustainability*, vol.7, no. 8, pp. 10191-10209, Jul. 2015.