Orlicz Intersection Bodies

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***Abstract*—In this paper, we introduce the concept of Orlicz intersection body which is a extension of intersection body. Some basic properties are proved for the Orlicz intersection body. We also establish an Orlicz Busemann-Petty intersection inequality by using Steiner symmetrization.**

***Index Terms*—Convex bodies, L_p-intersection bodies, Orlicz intersection bodies, Busemann-Petty intersection inequality.**

I. Introduction

INTERSECTION bodies [8] played a crucial role for the solution of the celebrated Busemann-Petty problem. Many applications has been found in geometric tomography [2], affine isoperimetric inequalities [16] and the geometry of Banach spaces [7], [17].

Let K be a star body with respect to the origin in $\mathbf{R}^n$, the intersection body of K, IK, is the star body with radial function

$$\rho(IK;u)=\mathrm{vol}(K\cap u^{\perp}),\qquad u\in S^{n-1},$$

where $\mathrm{vol}(\cdot)$ denotes $(n-1)$-dimensional volume and $u^{\perp}$ is the hyperplane orthogonal to u.

Let K be a star body with respect to the origin in $\mathbf{R}^n$ and $0<p<1$. The L_p-intersection body of K, I_pK, was introduced by Gardner and Giannopoulos [1] as well as Yuan [18]:

$$\rho(I_pK;u)^p=\int_K |x\cdot u|^{-p}\mathrm{d}x,\qquad u\in S^{n-1},\tag{1}$$

where $x\cdot u$ is the usual inner product of $x,u\in\mathbf{R}^n$.

Haberl and Ludwig [6] gave the following definition of L_p-intersection body for convex polytopes. They also established a characterization for the new concept. There is a different constant between the L_p-intersection body of (1) and the following p-intersection body in [5]: Let K be a star body with respect to the origin in $\mathbf{R}^n$ and $0<p<1$, then the p-intersection body was defined by

$$\rho(I_pK;u)^p=\frac{1}{\Gamma(1-p)}\int_K |x\cdot u|^{-p}\mathrm{d}x,\qquad u\in S^{n-1}.\tag{2}$$

Since

$$\mathrm{vol}(K\cap u^{\perp})=\lim_{\varepsilon\to0^+}\frac{\varepsilon}{2}\int_K|x\cdot u|^{-1+\varepsilon}\mathrm{d}x$$

(cf. [7], p.9), it follows from (1) that

$$\rho(IK;u)=\lim_{p\to1^-}\frac{1-p}{2}\rho(I_pK;u)^p,$$

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namely, the intersection body of K is given by a limit of L_p-intersection bodies of K.

The Orlicz-Brunn-Minkowski theory for convex bodies was made by Lutwak, Yang and Zhang [9], [10]. This theory is far more general than the L_p-Brunn-Minkowski theory (We refer readers to [12], [13] for the recent development of research.). In this paper, we introduce the concept of Orlicz intersection body, and some basic properties are established. More importantly, we establish the Orlicz Busemann-Petty intersection inequality for the Orlicz intersection bodies. The technique we will use is that of the standard Steiner symmetrization argument developed by Lutwak, Yang and Zhang [9], [10].

Consider convex function $\phi:\mathbf{R}\backslash\{o\}\to(0,\infty)$ such that $\lim\limits_{t\to-\infty}\phi(t)=\lim\limits_{t\to+\infty}\phi(t)=0$ and $\lim\limits_{t\to0^-}\phi(t)=\lim\limits_{t\to0^+}\phi(t)=+\infty$. This means that ϕ must be increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. We will assume that ϕ is either strictly increasing on $(-\infty,0)$ or strictly decreasing on $(0,\infty)$ throughout this paper. Let Φ denote the class of all such convex functions ϕ.

Assume K is a star body with respect to the origin in $\mathbf{R}^n$ with volume $|K|$. If $\phi\in\Phi$, then we define the Orlicz intersection body $I_\phi K$ of K whose radial function at $x\in\mathbf{R}^n\backslash\{o\}$ is given by

$$\rho^{-1}_{I_\phi K}(x)=\sup\left\{\lambda>0:\frac{1}{|K|}\int_K\phi\left(\frac{x\cdot y}{\lambda}\right)\mathrm{d}y\le1\right\}.\tag{3}$$

When $\phi(t)=|t|^{-p}$, with $0<p<1$, then

$$I_\phi K=\frac{1}{|K|}I_pK.$$

We will establish the following affine isoperimetric inequality for the Orlicz intersection bodies.

Theorem 1.1. Suppose K is a convex body in $\mathbf{R}^n$ that contains the origin in its interior. If $\phi\in\Phi$, then the volume ratio

$$|I^*_\phi K|/|K|$$

is minimized if and only if K is an ellipsoid centered at the origin.

II. Notation and preliminaries

The setting is Euclidean n-space $\mathbf{R}^n$. We write $e_1,\cdots,e_n$ for the standard orthonormal basis of $\mathbf{R}^n$ and when we write $\mathbf{R}^n=\mathbf{R}^{n-1}\times\mathbf{R}$ we always assume that e_n is associated with the last factor. We will attempt to use x,y for vectors in $\mathbf{R}^n$ and x',y' for vectors in $\mathbf{R}^{n-1}$. Throughout, $B=\{x\in\mathbf{R}^n: x\cdot x\le1\}$ denotes the unit ball centered at the origin, and S^{n-1} denotes the surface of of B, and $\omega_n=|B|$ denotes n-dimensional volume of B. If Q is a Borel subset of $\mathbf{R}^n$ and Q is contained in an i-dimensional affine subspace of $\mathbf{R}^n$ but in no affine subspace of lower dimension, then $|Q|$ denotes the i-dimensional Lebesgue measure of Q. If $x\in\mathbf{R}^n$ then by abuse of notation we will write $|x|$ for the norm of x.

For $T \in \mathrm{GL}(n)$ write T^t for the transpose of T and T^{-t} for the inverse of the transpose (contra gradient) of T. Write $|\det T|$ for the absolute value of the determinant of T.

We say that a sequence $\{\phi_i\}$, of $\phi_i \in \Phi$, is such that $\phi_i \to \phi_0 \in \Phi$ provided

$$|\phi_i - \phi_0|_I := \max_{t\in I} |\phi_i(t) - \phi_0(t)| \to 0,$$

for every compact interval $I \subset \mathbf{R}$.

For $\phi \in \Phi$ define $\phi^\star \in \Phi$ by

$$\phi^\star(t) = \int_0^1 \phi(ts)\mathrm{d}s^n, \tag{4}$$

where $\mathrm{d}s^n = ns^{n-1}\mathrm{d}s$. Obviously, $\phi_i \to \phi_0 \in \Phi$ implies $\phi_i^\star \to \phi_0^\star$.

Associated with each $\phi \in \Phi$ is $c_\phi \in (0,\infty)$ defined by

$$c_\phi = \min\{c > 0 : \max\{\phi(c), \phi(-c)\} \le 1\}.$$

A set K in $\mathbf{R}^n$ is star-shaped at o in $o \in K$ and for each $x \in \mathbf{R}^n\backslash\{o\}$, the intersection $K \cap \{cx : c \ge 0\}$ is a (possibly degenerate) compact line segment. If K is star-shaped at o, we define its radial function ρ_K for $x \in \mathbf{R}^n\backslash\{o\}$ by

$$\rho_K(x) = \max\{c \ge 0 : cx \in K\}.$$

This definition is a slight modification of [2]; as defined here, the domain of ρ_K is always $\mathbf{R}^n\backslash\{o\}$. Radial functions are homogeneous of degree -1, that is,

$$\rho_K(rx) = r^{-1}\rho_K(x), \tag{5}$$

for all $x \in \mathbf{R}^n\backslash\{o\}$ and $r > 0$, and are therefore often regarded as functions on the unit sphere S^{n-1} in $\mathbf{R}^n$. Conversely, any nonnegative and homogeneous of degree -1 function on $\mathbf{R}^n\backslash\{o\}$ is the radial function of a unique subset of $\mathbf{R}^n$ that is star-shaped at o. A star set in $\mathbf{R}^n$ is a bounded Borel set that is star-shaped at o. If a set K in $\mathbf{R}^n$ is star-shaped at o, then K is a star set if and only if ρ_K, restricted to S^{n-1}, is a bounded Borel-measurable function. If a star sets K in $\mathbf{R}^n$ has a positive continuous radial function ρ_K, then a set K is called a star body. Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbf{R}^n$.

If $K \in \mathcal{S}_o^n$ and $c > 0$, then obviously for the dilate $cK = \{cx : x \in K\}$ we have

$$\rho_{cK}(x) = c\rho_K(x), \tag{6}$$

for all $x \in \mathbf{R}^n\backslash\{o\}$. Obviously, for $K, L \in \mathcal{S}_o^n$,

$$K \subseteq L \quad \text{if and only if} \quad \rho_K \le \rho_L.$$

More generally, from the definition of the radial function it follows immediately that for $T \in \mathrm{GL}(n)$ the radial function of the image $TK = \{Ty : y \in K\}$ of K is given by

$$\rho_{TK}(x) = \rho_K(T^{-1}x). \tag{7}$$

The radial distance $\widetilde{\delta}(K,L)$ between $K, L \in \mathcal{S}_o^n$ is

$$\widetilde{\delta}(K,L) = ||\rho_K - \rho_L||_\infty = \max_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

For $K \in \mathcal{S}_o^n$, define the real numbers R_K and r_K by

$$R_K = \max_{u\in S^{n-1}} \rho_K(u), \qquad r_K = \min_{u\in S^{n-1}} \rho_K(u). \tag{8}$$

Note that the definition of $\mathcal{S}_o^n$ is such that $0 < r_K \le R_K < \infty$, for all $K \in \mathcal{S}_o^n$.

Let $h_K = h(K;\cdot) : \mathbf{R}^n \to \mathbf{R}$ denote the support function of the convex body (compact convex subset) K in $\mathbf{R}^n$; i.e.,

$$h(K;x) = \max\{x \cdot y : y \in K\}.$$

The Hausdorff distance $\delta(K,L)$ between the convex bodies K and L is

$$\delta(K,L) = ||h_K - h_L||_\infty = \max_{u\in S^{n-1}} |h_K(u) - h_L(u)|.$$

We write $\mathcal{K}^n$ for the space of convex bodies of $\mathbf{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. On $\mathcal{K}_o^n$ the Haudorff metric and the radial metric induce the same topology.

For convex body containing the origin in their interiors, K, let K^* denote the polar of the body K; i.e.,

$$K^* = \{x \in \mathbf{R}^n : \ x \cdot y \le 1, \text{ for all } y \in K\}.$$

Obviously, we have $(K^*)^* = K$. From definition of the polar of the convex body K, we know that: If $K \in \mathcal{K}_o^n$, then the support and radial functions of K^*, the polar body of K, are defined respectively by

$$h_{K^*} = 1/\rho_K \quad \text{and} \quad \rho_{K^*} = 1/h_K. \tag{9}$$

For a convex body K and a direction $u \in S^{n-1}$, let K_u denote the image of the orthogonal projection of K onto $u^\perp$, the subspace of $\mathbf{R}^n$ orthogonal to u. The undergraph and overgraph functions, $\underline{l}_u(K;\cdot) : K_u \to \mathbf{R}$ and $\bar{l}_u(K;\cdot) \to \mathbf{R}$, of K in the direction u are given by

$$K = \{y' + tu : -\underline{l}_u(K;y') \le t \le \bar{l}_u(K;y') \text{ for } y' \in K_u\}.$$

Therefore, the Steiner symmetral S_uK of $K \in \mathcal{K}_o^n$ in direction u is defined by the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose undergraph and overgraph functions are

$$\begin{aligned} \underline{l}_u(S_uK;y') &= \bar{l}_u(S_uK;y') \\ &= \frac{1}{2}\big[\underline{l}_u(K;y') + \bar{l}_u(K;y')\big]. \end{aligned} \tag{10}$$

For $y' \in K_u$, define $m_{y'} = m_{y'}(u)$ by

$$m_{y'}(u) = \frac{1}{2}[\bar{l}_u(K;y') - \underline{l}_u(K;y')]$$

so that the midpoint of the chord $K \cap (y' + \mathbf{R}u)$ is $y' + m_{y'}(u)u$. The length $|K \cap (y' + \mathbf{R}u)|$ of this chord will be denoted by $\sigma_{y'} = \sigma_{y'}(u)$. Note that the midpoints of the chords of K in the direction u lie in a subspace if and only if there exists an $x_0' \in K_u$ such that

$$x_0' \cdot y' = m_{y'}, \qquad \text{for all } y' \in K_u.$$

In this case $\{y' - \underline{l}_u(K;y')u : y' \in \mathrm{relint}K_u\}$, the undergraph of K with respect to u, is mapped into the overgraph by the linear transformation

$$y' + tu \mapsto y' + [2(x_0' \cdot y') - t]u.$$

The following is a classical characterization of the ellipsoid: a convex body $K \in \mathcal{K}_o^n$ is an origin centered ellipsoid if and only if for each direction $u \in S^{n-1}$ all of the midpoints of the chords of K parallel to u lie in a subspace of $\mathbf{R}^n$. Gruber (see [10]) showed how the following Lemma is a consequence of the Gruber-Ludwig theorem [3]:

Lemma 2.1. A convex body $K \in \mathcal{K}_o^n$ is an origin centered ellipsoid if and only if there exists an $\varepsilon_K > 0$ such that for each direction $u \in S^{n-1}$ all of the chords of K that come within a distance of ε_K of the origin and are parallel to u, have midpoints that lie in a subspace of $\mathbf{R}^n$.

If $K \subset \mathbf{R}^{n-1} \times \mathbf{R}$ for $(x', t) \in \mathbf{R}^{n-1} \times \mathbf{R}$, then we will usually write $h(K; x', t)$ rather than $h(K; (x', t))$.

Lemma 2.2. ([10], Lemma 1.2) Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. For $y' \in \operatorname{relint} K_u$, the overgraph and undergraph functions of K in direction u are given by

$$\bar{l}_u(K; y') = \min_{x' \in u^\perp} \{h_K(x', 1) - x' \cdot y'\},$$

and

$$\underline{l}_u(K; y') = \min_{x' \in u^\perp} \{h_K(x', -1) - x' \cdot y'\}.$$

The following crude estimate will be required.

Lemma 2.3. ([10], Lemma 1.3) Suppose $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$. If $y' \in (r_K/2)B \cap u^\perp$ and $x_1', x_2' \in u^\perp$ are such that

$$\bar{l}_u(K; y') = h_K(x_1', 1) - x_1' \cdot y'$$

and

$$\underline{l}_u(K; y') = h_K(x_2', -1) - x_2' \cdot y',$$

then both

$$|x_1'|, |x_2'| \leq \frac{2R_K}{r_K}.$$

We also need the following lemma.

Lemma 2.4. ([15]) Let $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbf{R}^n$. If

$$0 < m_1 \leq x_k \leq M_1, \quad 0 < m_2 \leq y_k \leq M_2, \quad k = 1, \cdots, n,$$

then

$$\left(\sum_{k=1}^n x_k^2\right)\left(\sum_{k=1}^n y_k^2\right) \leq \left(\frac{\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}}{2}\right)^2 \left(\sum_{k=1}^n x_k y_k\right)^2.$$

Lemma 2.4 implies that if $x, y \in \mathbf{R}^n$, then there exist a constant $c_0 \in (0,1)$ such that

$$|x \cdot y| \geq c_0 |x||y|. \tag{11}$$

III. Definition and basic properties of Orlicz intersection bodies

We give the concept of Orlicz intersection body in the introduction: If $\phi \in \Phi$ and $K \in \mathcal{S}_o^n$, then

$$\rho_{I_\phi K}^{-1}(x) = \sup\left\{\lambda > 0 : \int_{S^{n-1}} \phi^\star\left(\frac{(x \cdot u)}{\lambda}\rho_K(u)\right) \mathrm{d}V_K^*(u) \leq 1\right\}, \tag{12}$$

where $\phi^\star$ is defined by (4), and $\mathrm{d}V_K^*$ is the volume-normalized dual conical measure of K, defined by

$$|K| \mathrm{d}V_K^* = \frac{1}{n}\rho_K^n \mathrm{d}S,$$

where S is Lebesgue measure on S^{n-1} (i.e., $(n-1)$-dimensional Hausdorff measure). We shall make use of the fact that the volume-normalized dual conical measure

$$V_K^* \text{ is a probability measure on } S^{n-1}. \tag{13}$$

From the following fact, we obtain the equivalence of the two definitions (3) and (12).

$$\int_K \phi\left(\frac{x \cdot y}{\lambda}\right) \mathrm{d}y = \frac{1}{n}\int_{S^{n-1}} \phi^\star\left(\frac{(x \cdot v)}{\lambda}\rho_K(v)\right)\rho_K(v)^n \mathrm{d}S(v). \tag{14}$$

(14) is given by the following:

$$\begin{aligned}
&\int_K \phi\left(\frac{x \cdot y}{\lambda}\right) \mathrm{d}y \\
= &\int_{S^{n-1}} \int_0^{\rho_K(v)} \phi\left(\frac{(x \cdot v)}{\lambda} r\right) r^{n-1} \mathrm{d}r \mathrm{d}S(v) \\
= &\int_{S^{n-1}} \left(\int_0^1 \phi\left(\frac{(x \cdot v)}{\lambda}\rho_K(v) t\right) t^{n-1} \mathrm{d}t\right) \rho_K(v)^n \mathrm{d}S(v) \\
= &\frac{1}{n}\int_{S^{n-1}} \phi^\star\left(\frac{(x \cdot v)}{\lambda}\rho_K(v)\right)\rho_K(v)^n \mathrm{d}S(v).
\end{aligned}$$

It follows from definition (12) and (6) that,

$$\rho_{I_\phi K}(rx) = r^{-1}\rho_{I_\phi K}(x),$$

for $x \in \mathbf{R}^n \backslash \{o\}$ and $r > 0$, and $I_\phi K$ has a positive continuous radial function $\rho_{I_\phi K}(x)$, then $I_\phi K \in \mathcal{S}_o^n$.

Since $\phi^\star$ is strictly decreasing on $(0, \infty)$ or strictly increasing on $(-\infty, 0)$, we obtain the function

$$\lambda \mapsto \int_{S^{n-1}} \phi^\star\left(\frac{(x \cdot v)}{\lambda}\rho_K(v)\right) \mathrm{d}V_K^*(v)$$

is strictly increasing in $(0, \infty)$, and it is also continuous. Thus, we have:

Lemma 3.1. Suppose $K \in \mathcal{S}_o^n$ and $u_0 \in S^{n-1}$. Then

$$\int_{S^{n-1}} \phi^\star\left(\frac{(u_0 \cdot v)}{\lambda_0}\rho_K(v)\right) \mathrm{d}V_K^*(v) = 1$$

if and only if

$$\rho_{I_\phi K}^{-1}(u_0) = \lambda_0.$$

(5) and (6) shows that Lemma 3.1 holds for all $u_0 \in \mathbf{R}^n \backslash \{o\}$.

We shall require more than $\rho_{I_\phi K} > 0$. Specifically,

Lemma 3.2. Suppose $\phi \in \Phi$, and $K \in \mathcal{S}_o^n$. Then there exists a real $c_0 \in (0,1)$ such that

$$\frac{c_{\phi^\star}}{R_K} \leq \rho_{I_\phi K} \leq \frac{c_{\phi^\star}}{c_0 r_K}.$$

Proof. Let $u_0 \in S^{n-1}$ and $\rho_{I_\phi K}^{-1}(u_0) = \lambda_0$. Then

$$\int_{S^{n-1}} \phi^\star\left(\frac{(u_0 \cdot v)}{\lambda_0}\rho_K(v)\right) \mathrm{d}V_K^*(v) = 1. \tag{15}$$

We first obtain the lower estimate. From the definition of $\phi^\star$, either $\phi^\star(c_{\phi^\star}) = 1$ or $\phi^\star(-c_{\phi^\star}) = 1$. If $\phi^\star(c_{\phi^\star}) = 1$, then from the fact that $\phi^\star$ is non-negative, Jensen's inequality, and Lemma 3.1 together with the fact that $\phi^\star$ is monotone decreasing on $(0,\infty)$ and the probability measure (13), we have

$$\begin{aligned}\phi^\star(c_{\phi^\star}) &= 1\\ &= \int_{S^{n-1}}\phi^\star\Big(\frac{(u_0\cdot v)}{\lambda_0}\rho_K(v)\Big)\mathrm{d}V_K^*(v)\\ &\geq \phi^\star\Big(\int_{S^{n-1}}\frac{(u_0\cdot v)\rho_K(v)}{\lambda_0}\mathrm{d}V_K^*(v)\Big)\\ &\geq \phi^\star\Big(\int_{S^{n-1}}\frac{R_K}{\lambda_0}\mathrm{d}V_K^*(v)\Big)\\ &= \phi^\star\Big(\frac{R_K}{\lambda_0}\Big).\end{aligned}$$

Since $\phi^\star$ is monotone decreasing on $(0,\infty)$ so we conclude

$$\lambda_0 \leq \frac{c_{\phi^\star}}{R_K}.$$

Then we obtain the lower bound for $\rho_{I_\phi K}$:

$$\rho_{I_\phi K} \geq \frac{c_{\phi^\star}}{R_K}.$$

The case $\phi^\star(-c_{\phi^\star})=1$ is handled the same way and gives the same result.

To give the upper estimate, we see that from the definition of $c_{\phi^\star}$, the fact that $\phi^\star$ is strictly increasing on $(-\infty,0)$ and strictly decreasing on $(0,\infty)$, together with the fact that the function $t\mapsto\max\{\phi^\star(t),\phi^\star(-t)\}$ is monotone decreasing on $(0,\infty)$, Lemma 3.1, the inequality (11) and the probability measure (13), it follows that

$$\begin{aligned}&\max\{\phi^\star(c_{\phi^\star}),\phi^\star(-c_{\phi^\star})\}\\ &=1\\ &=\int_{S^{n-1}}\phi^\star\Big(\frac{u_0\cdot v}{\lambda_0}\rho_K(v)\Big)\mathrm{d}V_K^*(v)\\ &\leq\int_{S^{n-1}}\max\Big\{\phi^\star\Big(\frac{|u_0\cdot v|\rho_K(v)}{\lambda_0}\Big),\\ &\quad\phi^\star\Big(-\frac{|u_0\cdot v|\rho_K(v)}{\lambda_0}\Big)\Big\}\mathrm{d}V_K^*(v)\\ &\leq\int_{S^{n-1}}\max\Big\{\phi^\star\Big(\frac{c_0\rho_K(v)}{\lambda_0}\Big),\\ &\quad\phi^\star\Big(-\frac{c_0\rho_K(v)}{\lambda_0}\Big)\Big\}\mathrm{d}V_K^*(v)\\ &\leq\int_{S^{n-1}}\max\Big\{\phi^\star\Big(\frac{c_0r_K}{\lambda_0}\Big),\\ &\quad\phi^\star\Big(-\frac{c_0r_K}{\lambda_0}\Big)\Big\}\mathrm{d}V_K^*(v)\\ &=\max\Big\{\phi^\star\Big(\frac{c_0r_K}{\lambda_0}\Big),\phi^\star\Big(-\frac{c_0r_K}{\lambda_0}\Big)\Big\}.\end{aligned}$$

Since the even function $t\mapsto\max\{\phi^\star(t),\phi^\star(-t)\}$ is monotone decreasing on $(0,\infty)$ so we conclude

$$\lambda_0 \geq \frac{c_0 r_K}{c_{\phi^\star}}.$$

Then the upper bound is obtained for $\rho_{I_\phi K}$:

$$\rho_{I_\phi K} \leq \frac{c_{\phi^\star}}{c_0 r_K}.$$

For $c>0$, (3) and (5) immediately get that

$$\rho_{I_\phi cK} = c^{-1}\rho_{I_\phi K}. \tag{16}$$

The following shows that the Orlicz intersection body operator $I_\phi:\mathcal{K}_o^n\to\mathcal{S}_o^n$ is continuous.

Lemma 3.3. Suppose $\phi\in\Phi$. If $K_i\in\mathcal{S}_o^n$ and $K_i\to K\in\mathcal{S}_o^n$, then $I_\phi K_i\to I_\phi K$.

Proof. If $\phi\in\Phi$ and ϕ is continuous, convex, and either strictly increasing in $(-\infty,0)$ or strictly decreasing in $(0,\infty)$, then we will show that for $u_0\in S^{n-1}$

$$\rho_{I_\phi K_i}(u_0)\to\rho_{I_\phi K}(u_0).$$

Suppose

$$\rho^{-1}_{I_\phi K_i}(u_0)=\lambda_i.$$

From Lemma 3.2, we have that there exists a real $c_0\in(0,1)$ such that

$$\frac{c_0 r_K}{c_{\phi^*}}\leq\lambda_i\leq\frac{R_K}{c_{\phi^*}}.$$

Since $K_i\to K\in\mathcal{K}_o^n$, we have $r_{K_i}\to r_K>0$ and $R_{K_i}\to R_K<\infty$, and thus there exist a,b such that $0<a\leq\lambda_i\leq b<\infty$, for all i. To show that the bounded sequence $\{\lambda_i\}$ converges to $\rho^{-1}_{I_\phi K}(u_0)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $\rho^{-1}_{I_\phi K}(u_0)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have

$$\lambda_i\to\lambda_*.$$

Clearly, $a\leq\lambda_*\leq b$. Let $\overline{K}_i=\lambda_iK_i$. Since $K_i\to K$, we have

$$\overline{K}_i\to\lambda_*K.$$

Now (16), and the fact that $\rho^{-1}_{I_\phi K_i}(u_0)=\lambda_i$, shows that $\rho_{I_\phi\overline{K}_i}(u_0)=1$. Namely,

$$\int_{S^{n-1}}\phi^\star\Big(\frac{u_0\cdot u}{\rho_{\overline{K}_i}(u)}\Big)\mathrm{d}V^*_{\overline{K}_i}(u)=1$$

for all i. It follows from $\overline{K}_i\to\lambda_*K$ and the continuity of $\phi^\star$ that

$$\int_{S^{n-1}}\phi^\star\Big(\frac{u_0\cdot u}{\rho_{\lambda_*K}(u)}\Big)\mathrm{d}V^*_{\lambda_*K}(u)=1,$$

which by Lemma 3.1 yields

$$\rho_{I_\phi\lambda_*K}(u_0)=1.$$

Thus, combining (16) and (5), we have

$$\rho_{I_\phi K}(u_0)=\lambda_*.$$

This shows the desired result, i.e., $\rho_{I_\phi K_i}(u_0)\to\rho_{I_\phi K}(u_0)$.

But for radial functions on S^{n-1} pointwise and uniform convergence are equivalent (see, e.g., Schneider [16], p.54). Thus, the pointwise convergence $\rho_{I_\phi K_i}\to\rho_{I_\phi K}$ on S^{n-1} completes the proof.

The following result shows that the Orlicz intersection body operator is also continuous in ϕ.

Lemma 3.4. If $\phi_i\to\phi\in\Phi$, then $I_{\phi_i}K\to I_\phi K$, for each $K\in\mathcal{S}_o^n$.

Proof. Let $K\in\mathcal{K}_o^n$ and $u_0\in S^{n-1}$. We will show that

$$\rho_{I_{\phi_i}K}(u_0)\to\rho_{I_\phi K}(u_0).$$

If $\phi\in\Phi$ with ϕ is continuous, convex, and either strictly increasing in $(-\infty,0)$ or strictly decreasing in $(0,\infty)$, and let

$$\rho^{-1}_{I_{\phi_i}K}(u_0)=\lambda_i,$$

then rom Lemma 3.2 we have that there exists a real $c_0 \in (0,1)$ such that

$$\frac{c_0 r_K}{c_{\phi^*}} \le \lambda_i \le \frac{R_K}{c_{\phi^*}}.$$

Since $\phi_i^\star \to \phi^\star \in \Phi$, we have $c_{\phi_i^\star} \to c_{\phi^\star} \in (0,\infty)$ and thus there exist a, b such that $0 < a \le \lambda_i \le b < \infty$, for all i.

To show that the bounded sequence $\{\lambda_i\}$ converges to $\rho^{-1}_{I_\phi K}(u_0)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $\rho^{-1}_{I_\phi K}(u_0)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence we have

$$\lambda_i \to \lambda_*.$$

Obviously, $0 < a \le \lambda_* \le b$. Since $\rho^{-1}_{I_{\phi_i} K}(u_0) = \lambda_i$,

$$1 = \int_{S^{n-1}} \phi_i^\star\Big(\frac{u_0 \cdot u}{\lambda_i}\rho_K(u)\Big) \mathrm{d}V_K^*(u).$$

Combining $\phi_i^\star \to \phi^\star \in \Phi_1$ with $\lambda_i \to \lambda_*$, we have

$$1 = \int_{S^{n-1}} \phi^\star\Big(\frac{u_0 \cdot u}{\lambda_*}\rho_K(u)\Big) \mathrm{d}V_K^*(u).$$

Thus, Lemma 3.1 gives

$$\rho^{-1}_{I_\phi K}(u_0) = \lambda_*.$$

This shows that $\rho_{I_{\phi_i} K}(u_0) \to \rho_{I_\phi K}(u_0)$ as desired.

Since the radial functions $\rho_{I_{\phi_i} K} \to \rho_{I_\phi K}$ pointwise (on S^{n-1}) they converge uniformly and hence

$$I_{\phi_i} K \to I_\phi K.$$

The operator I_ϕ intertwines with elements of $\mathrm{SL}(n)$:

Lemma 3.5. Suppose $\phi \in \Phi$. For a star body $K \in \mathcal{S}_o^n$ and a linear transformation $T \in \mathrm{GL}(n)$,

$$I_\phi(TK) = T^{-t}(I_\phi K). \tag{17}$$

Proof. Suppose $x_0 \in \mathbf{R}^n$ and

$$\rho(I_\phi TK; x_0)^{-1} = \lambda_0.$$

From Lemma 3.1 and (14), the substitution $z = Ty$ and the facts that $|TK| = |\det T||K|$ and $\mathrm{d}z = |\det T|\mathrm{d}y$, we have

$$\begin{aligned} 1 &= \frac{1}{|TK|}\int_{TK} \phi\Big(\frac{x_0 \cdot z}{\lambda_0}\Big)\mathrm{d}z \\ &= \frac{1}{|\det T||K|}\int_K \phi\Big(\frac{x_0 \cdot Ty}{\lambda_0}\Big)|\det T|\mathrm{d}y \\ &= \frac{1}{|K|}\int_K \phi\Big(\frac{T^t x_0 \cdot y}{\lambda_0}\Big)\mathrm{d}y. \end{aligned}$$

Lemma 3.1, (12) and (7) imply that

$$\lambda_0 = \rho(I_\phi K; T^t x_0)^{-1} = \rho(T^{-t} I_\phi K; x_0)^{-1},$$

giving $\rho(I_\phi TK; x_0) = \rho(T^{-t} I_\phi K; x_0)$.

IV. Proof of the Orlicz Busemann-Petty intersection inequality

Lemma 4.1. Suppose $\phi \in \Phi$, and $K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$ and $x_1', x_2' \in u^\perp$, then

$$\begin{aligned} &\rho\Big(I_\phi(S_u K); \frac{1}{2}x_1' + \frac{1}{2}x_2', 1\Big)^{-1} \\ &\le \frac{1}{2}\rho(I_\phi K; x_1', 1)^{-1} + \frac{1}{2}\rho(I_\phi K; x_2', -1)^{-1}. \end{aligned} \tag{18}$$

Equality in the inequality implies that all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K}{2\max\{1, |x_1'|, |x_2'|\}},$$

have midpoints that lie in a subspace.

Proof. Suppose that $K' = K_u$ denotes the image of the projection of K onto the subspace $u^\perp$. For each $y' \in K'$, let $\sigma_{y'}(u) = \sigma_{y'} = |K \cap (y' + \mathbf{R}u)|$ be the length of the chord $K \cap (y' + \mathbf{R}u)$, and let $m_{y'} = m_{y'}(u)$ be defined such that $y' + m_{y'}u$ is the midpoint of the chord $K \cap (y' + \mathbf{R}u)$. If $\lambda_1, \lambda_2 > 0$, then we obtain

$$\begin{aligned} &\int_K \phi\Big(\frac{(x_1',1)\cdot y}{\lambda_1}\Big)\mathrm{d}y \\ =& \int_K \phi\Big(\frac{(x_1',1)\cdot(y',s)}{\lambda_1}\Big)\mathrm{d}y'\mathrm{d}s \\ =& \int_{K'} \mathrm{d}y' \int_{m_{y'}-\sigma_{y'}/2}^{m_{y'}+\sigma_{y'}/2} \phi\Big(\frac{x_1'\cdot y' + s}{\lambda_1}\Big)\mathrm{d}s \\ =& \int_{K'} \mathrm{d}y' \int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \phi\Big(\frac{x_1'\cdot y' + t + m_{y'}}{\lambda_1}\Big)\mathrm{d}t \\ =& \int_{S_u K} \phi\Big(\frac{x_1'\cdot y' + t + m_{y'}(u)}{\lambda_1}\Big)\mathrm{d}y'\mathrm{d}t. \end{aligned} \tag{19}$$

Let $t = -m_{y'} + s$, and

$$\begin{aligned} &\int_K \phi\Big(\frac{(x_2',-1)\cdot y}{\lambda_2}\Big)\mathrm{d}y \\ =& \int_K \phi\Big(\frac{(x_2',-1)\cdot(y',s)}{\lambda_2}\Big)\mathrm{d}y'\mathrm{d}s \\ =& \int_{K'} \mathrm{d}y' \int_{m_{y'}-\sigma_{y'}/2}^{m_{y'}+\sigma_{y'}/2} \phi\Big(\frac{x_2'\cdot y' - s}{\lambda_2}\Big)\mathrm{d}s \\ =& \int_{K'} \mathrm{d}y' \int_{-\sigma_{y'}/2}^{\sigma_{y'}/2} \phi\Big(\frac{x_2'\cdot y' + t - m_{y'}}{\lambda_2}\Big)\mathrm{d}t \\ =& \int_{S_u K} \phi\Big(\frac{x_2'\cdot y' + t - m_{y'}(u)}{\lambda_2}\Big)\mathrm{d}y'\mathrm{d}t, \end{aligned} \tag{20}$$

by making the change of variables $t = m_{y'} - s$. Abbreviate

$$x_0' = \frac{1}{2}x_1' + \frac{1}{2}x_2' \quad \text{and} \quad \lambda_0 = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2.$$

It follows from the convexity of ϕ that

$$\begin{aligned} &2\phi\Big(\frac{(x_0'\cdot y' + t)}{\lambda_0}\Big) \\ \le\;& \frac{\lambda_1}{\lambda_0}\phi\Big(\frac{x_1'\cdot y' + t + m_{y'}}{\lambda_1}\Big) + \frac{\lambda_2}{\lambda_0}\phi\Big(\frac{x_2'\cdot y' + t - m_{y'}}{\lambda_2}\Big). \end{aligned} \tag{21}$$

From (19), (20) and (21), it follows that

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_0}\int_K \phi\Big(\frac{(x_1',1)\cdot y}{\lambda_1}\Big)\mathrm{d}y \\
&+\frac{\lambda_2}{\lambda_0}\int_K \phi\Big(\frac{(x_2',-1)\cdot y}{\lambda_2}\Big)\mathrm{d}y \\
= \;& \frac{\lambda_1}{\lambda_0}\int_{S_uK} \phi\Big(\frac{x_1'\cdot y'+t+m_{y'}(u)}{\lambda_1}\Big)\mathrm{d}y'\mathrm{d}t \\
&+\frac{\lambda_2}{\lambda_0}\int_{S_uK} \phi\Big(\frac{x_2'\cdot y'+t-m_{y'}(u)}{\lambda_2}\Big)\mathrm{d}y'\mathrm{d}t \\
\geq \;& 2\int_{S_uK}\phi\Big(\frac{x_0'\cdot y'+t}{\lambda_0}\Big)\mathrm{d}y'\mathrm{d}t \qquad (22)\\
= \;& 2\int_{S_uK}\phi\Big(\frac{(x_0',1)\cdot (y',t)}{\lambda_0}\Big)\mathrm{d}y'\mathrm{d}t \\
= \;& 2\int_{S_uK}\phi\Big(\frac{(x_0',1)\cdot y}{\lambda_0}\Big)\mathrm{d}y.
\end{aligned}$$

Let

$$\lambda_1=\rho(I_\phi K;x_1',1)^{-1} \quad\text{and}\quad \lambda_2=\rho(I_\phi K;x_2',-1)^{-1}.$$

From Lemma 3.1, we obtain

$$\frac{1}{|K|}\int_K\phi\Big(\frac{(x_1',1)\cdot y}{\lambda_1}\Big)\mathrm{d}y=1$$

and

$$\frac{1}{|K|}\int_K\phi\Big(\frac{(x_2',-1)\cdot y}{\lambda_2}\Big)\mathrm{d}y=1.$$

Combining with the fact that $|K|=|S_uK|$ shows that

$$\frac{1}{|S_uK|}\int_{S_uK}\phi\Big(\frac{(\frac12 x_1'+\frac12 x_2',1)\cdot y}{\frac12\lambda_1+\frac12\lambda_2}\Big)\mathrm{d}y\leq 1,$$

which together with definition (12) yields

$$\frac{1}{\rho_{S_uK}(\frac12 x_1'+\frac12 x_2',1)}\leq \frac12\lambda_1+\frac12\lambda_2,$$

with equality forcing (in light of the continuity of ϕ) equality in (21) for all $y'\in K'$ and all $t\in[-\sigma_{y'}/2,\sigma_{y'}/2]$. This establishes the desired inequality (18).

Suppose that equality holds. Hence there is equality in (21) for all $y'\in K'$ and all $t\in[-\sigma_{y'}/2,\sigma_{y'}/2]$.

From definition (8) of r_K we see that if $|y'|<r_K/2$ then

$$\Big(-\frac{r_K}{2},\frac{r_K}{2}\Big)\subset\Big(m_{y'}-\frac{\sigma_{y'}}{2},m_{y'}+\frac{\sigma_{y'}}{2}\Big) \qquad (23)$$

and

$$\Big(-\frac{r_K}{2},\frac{r_K}{2}\Big)\subset\Big(-m_{y'}-\frac{\sigma_{y'}}{2},-m_{y'}+\frac{\sigma_{y'}}{2}\Big). \qquad (24)$$

Suppose y' with

$$|y'|<\frac{r_K}{2\max\{1,|x_1'|,|x_2'|\}}.$$

Therefore,

$$x_1'\cdot y'\in\Big(-\frac{r_K}{2},\frac{r_K}{2}\Big)$$

and

$$x_2'\cdot y'\in\Big(-\frac{r_K}{2},\frac{r_K}{2}\Big).$$

It follows from (23) and (24) that

$$x_1'\cdot y'+m_{y'}\in\Big(-\frac{\sigma_{y'}}{2},\frac{\sigma_{y'}}{2}\Big)$$

and

$$x_2'\cdot y'-m_{y'}\in\Big(-\frac{\sigma_{y'}}{2},\frac{\sigma_{y'}}{2}\Big).$$

Thus, the linear functions

$$t\mapsto x_1'\cdot y'+t+m_{y'} \quad\text{and}\quad t\mapsto x_2'\cdot y'+t-m_{y'}$$

both have their root in $(-\sigma_{y'}/2,\sigma_{y'}/2)$. Thus, they either (1) have their root at the same $t=t_{y'}\in(-\sigma_{y'}/2,\sigma_{y'}/2)$ or (2) there will exist a $t=t^\star_{y'}\in(-\sigma_{y'}/2,\sigma_{y'}/2)$ at which these functions have opposite signs.

From the case (2). combining with the fact that

$$x_1'\cdot y'+t^\star_{y'}+m_{y'} \quad\text{and}\quad x_2'\cdot y'+t^\star_{y'}-m_{y'}$$

have opposite signs tells us that

$$x_1'\cdot y'+t+m_{y'} \quad\text{and}\quad x_2'\cdot y'+t-m_{y'}$$

have opposite signs for all $t\in(t^*_{y'}-\delta_{y'},t^*_{y'}+\delta_{y'})$ for some $\delta_{y'}>0$. This and the fact that there is equality in (21) together with the fact that ϕ can not be linear in a neighborhood of the origin gives

$$\frac{x_1'\cdot y'+t+m_{y'}}{\lambda_1}=\frac{x_2'\cdot y'+t-m_{y'}}{\lambda_2},$$

for all $t\in(t^*_{y'}-\delta_{y'},t^*_{y'}+\delta_{y'})$ which contradicts the assumption that the linear functions have opposite signs.

In case (1) the linear functions

$$t\mapsto x_1'\cdot y'+t+m_{y'} \quad\text{and}\quad t\mapsto x_2'\cdot y'+t-m_{y'}$$

have a root at the same $t=t_{y'}\in(-\sigma_{y'}/2,\sigma_{y'}/2)$ and this immediately yields

$$(x_2'-x_1')\cdot y'=2m_{y'}.$$

But this means that for $|y'|<r_K/\max\{2,2|x_1'|,2|x_2'|\}$, the midpoints

$$\{(y',m_{y'}):y'\in K'\}$$

of the chords of K parallel to u lie in the subspace

$$\Big\{(y',\frac12(x_2'-x_1')\cdot y'):y'\in K'\Big\}$$

of $\mathbf{R}^n$.

From the inequality (18) of Lemma 4.1, we have

$$\begin{aligned}
&\rho\Big(I_\phi(S_uK);\frac12 x_1'+\frac12 x_2',-1\Big)^{-1}\\
\leq\;&\frac12\rho(I_\phi K;x_1',1)^{-1}+\frac12\rho(I_\phi K;x_2',-1)^{-1}.
\end{aligned}$$

If ϕ is assumed to be strictly convex, then the equality conditions of the inequality in Lemma 4.1 are simple.

Lemma 4.2. Suppose $\phi\in\Phi$ is strictly convex on $\mathbf{R}$, and $K\in\mathcal{K}^n_o$. If $u\in S^{n-1}$ and $x_1',x_2'\in u^\perp$, then

$$\begin{aligned}
&\rho\Big(I_\phi(S_uK);\frac12 x_1'+\frac12 x_2',1\Big)^{-1}\\
\leq\;&\frac12\rho(I_\phi K;x_1',1)^{-1}+\frac12\rho(I_\phi K;x_2',-1)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
&\rho\Big(I_\phi(S_uK);\frac12 x_1'+\frac12 x_2',-1\Big)^{-1}\\
\leq\;&\frac12\rho(I_\phi K;x_1',1)^{-1}+\frac12\rho(I_\phi K;x_2',-1)^{-1}.
\end{aligned}$$

Equality in either inequality, implies

$$\rho(I_\phi K; x_1', 1) = \rho(I_\phi K; x_2', -1)$$

and that all of the midpoints of the chords of K parallel to u lie in a subspace.

Proof. Observe that equality forces equality in (21) for all $y' \in K'$ and all $t \in [-\sigma_{y'}/2, \sigma_{y'}/2]$. Since ϕ is strictly convex, this means that we must have ϕ can not be linear in a neighborhood of the origin given by

$$\frac{x_1' \cdot y' + t + m_{y'}}{\lambda_1} = \frac{x_2' \cdot y' + t - m_{y'}}{\lambda_2}, \tag{25}$$

for all $t \in (-\sigma_{y'}/2, \sigma_{y'}/2)$. Choosing $\lambda_1 = \rho(I_\phi K; x_1', 1)^{-1}$ and $\lambda_2 = \rho(I_\phi K; x_2', -1)^{-1}$, (25) immediately yields

$$\rho(I_\phi K; x_1', 1)^{-1} = \lambda_1 = \lambda_2 = \rho(I_\phi K; x_2', -1)^{-1},$$

and

$$(x_2' - x_1') \cdot y' = 2m_{y'},$$

for all $y' \in K'$. But this means that the midpoints $\{(y', m_{y'}) : y' \in K'\}$ of the chords of K parallel to u lie in the subspace

$$\Big\{(y', \frac{1}{2}(x_2' - x_1') \cdot y') : y' \in K'\Big\}$$

of $\mathbf{R}^n$.

Lemma 4.3. Suppose $\phi \in \Phi, K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$, then

$$I_\phi^*(S_u K) \subseteq S_u(I_\phi^* K).$$

If the inclusion is an identity then all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K r_{I_\phi^* K}}{4R_{I_\phi^* K}},$$

have midpoints that lie in a subspace.

Proof. Let $y' \in \mathrm{relint}(I_\phi^* K)_u$. According to Lemma 2.2 there exist $x_1' = x_1'(y')$ and $x_2' = x_2'(y')$ in $u^\perp$ such that

$$\bar{l}_u(I_\phi^* K, y') = h(I_\phi^* K; x_1', 1) - x_1' \cdot y', \tag{26}$$
$$\bar{l}_u(I_\phi^* K, y') = h(I_\phi^* K; x_2', -1) - x_2' \cdot y'. \tag{27}$$

From (10), (26) and (27) as well as Lemma 4.1 and Lemma 2.2, it follows that

$$\begin{aligned}
&\bar{l}_u(S_u(I_\phi^* K); y')\\
=\ & \frac{1}{2}\bar{l}_u(I_\phi^* K; y') + \frac{1}{2}\underline{l}_u(I_\phi^* K; y')\\
=\ & \frac{1}{2}(h(I_\phi^* K; x_1', 1) - x_1' \cdot y')\\
&+\frac{1}{2}(h(I_\phi^* K; x_2', -1) - x_2' \cdot y')\\
=\ & \frac{1}{2}h(I_\phi^* K; x_1', 1) + \frac{1}{2}h(I_\phi^* K; x_2', -1)\\
&-\big(\frac{1}{2}x_1' + \frac{1}{2}x_2'\big) \cdot y'\\
\geq\ & h_{I_\phi^*(S_u K)}\big(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1\big) - \big(\frac{1}{2}x_1' + \frac{1}{2}x_2'\big) \cdot y'\\
\geq\ & \min_{x' \in u^\perp} \Big\{h_{I_\phi^*(S_u K)}(x', 1) - x' \cdot y'\Big\}\\
=\ & \bar{l}_u(I_\phi^*(S_u K); y'),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\underline{l}_u(S_u(I_\phi^* K); y')\\
=\ & \frac{1}{2}\bar{l}_u(I_\phi^* K; y') + \frac{1}{2}\underline{l}_u(I_\phi^* K; y')\\
=\ & \frac{1}{2}(h(I_\phi^* K; x_1', 1) - x_1' \cdot y')\\
&+\frac{1}{2}(h(I_\phi^* K; x_2', -1) - x_2' \cdot y')\\
=\ & \frac{1}{2}h(I_\phi^* K; x_1', 1) + \frac{1}{2}h(I_\phi^* K; x_2', -1)\\
&-\big(\frac{1}{2}x_1' + \frac{1}{2}x_2'\big) \cdot y'\\
\geq\ & h_{I_\phi^*(S_u K)}\big(\frac{1}{2}x_1' + \frac{1}{2}x_2', -1\big) - \big(\frac{1}{2}x_1' + \frac{1}{2}x_2'\big) \cdot y'\\
\geq\ & \min_{x' \in u^\perp} \Big\{h_{I_\phi^*(S_u K)}(x', -1) - x' \cdot y'\Big\}\\
=\ & \underline{l}_u(I_\phi^*(S_u K); y').
\end{aligned}$$

This gives the inclusion. Now let

$$I_\phi^*(S_u K) = S_u(I_\phi^* K).$$

Together with Lemma 2.2, for each $y' \in (I_\phi^* K)_u \cap (r_{I_\phi^* K}/2)B$, there exist $x_1' = x_1'(y')$ and $x_2' = x_2'(y')$ in $u^\perp$ such that

$$\bar{l}_u(I_\phi^* K, y') = h(I_\phi^* K; x_1', 1) - x_1' \cdot y', \tag{29}$$
$$\bar{l}_u(I_\phi^* K, y') = h(I_\phi^* K; x_2', -1) - x_2' \cdot y'. \tag{30}$$

Since $I_\phi^*(S_u K) = S_u(I_\phi^* K)$, it follows from (28) that

$$\begin{aligned}
&h_{I_\phi^*(S_u K)}\big(\frac{1}{2}x_1' + \frac{1}{2}x_2', 1\big)\\
=\ & \frac{1}{2}h_{I_\phi^* K}(x_1', 1) + \frac{1}{2}h_{I_\phi^* K}(x_2', -1).
\end{aligned} \tag{31}$$

From Lemma 2.3, (29) and (30), we have that both

$$|x_1'|, |x_2'| \leq \frac{2R_{I_\phi^* K}}{r_{I_\phi^* K}}.$$

(31) and the equality conditions in Lemma 4.1 show that all of the chords of K parallel to u, whose distance from the origin is less than

$$\frac{r_K r_{I_\phi^* K}}{4R_{I_\phi^* K}},$$

have midpoints that lie in a subspace.

The following is a direct consequence of Lemma 2.1:

Corollary 4.4. Suppose $\phi \in \Phi$ and $K \in \mathcal{K}_o^n$. If $u \in S^{n-1}$, then

$$I_\phi^*(S_u K) \subseteq S_u(I_\phi^* K). \tag{32}$$

If the inclusion is an identity for all u, then K is an ellipsoid centered at the origin.

Theorem 4.5. If $\phi \in \Phi$ and $K \in \mathcal{K}_o^n$, then

$$\frac{|I_\phi^* K|}{|K|} \geq \frac{|I_\phi^* B|}{|B|},$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof. Combining with the Steiner symmetrization argument, there is a sequence of directions $\{u_i\}$, such that the sequence $\{K_i\}$ converges to cB, where the sequences $\{K_i\}$ is defined by

$$K_i = S_{u_i} \cdots S_{u_i} K,$$

with $|K| = |K_i|$ and thus $|K| = |cB| = c^n|B|$. Since the Steiner symmetrization keeps the volume, by Corollary 4.4 we have

$$|I^*_\phi K| \geq |I^*_\phi(cB)| = |cI^*_\phi B| = c^n|I^*_\phi B|,$$

namely,

$$\frac{|I^*_\phi K|}{|K|} \geq \frac{|I^*_\phi B|}{|B|},$$

with equality if and only if K is an ellipsoid centered at the origin.

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