Range-Based Threshold Spot Volatility Estimation for Jump Diffusion Models

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***Abstract*—We consider level-dependent volatility estimation for jump diffusion models and propose a range-based threshold spot volatility estimator with high frequency discrete observations. Under some weak conditions, the consistency and asymptotic normality of our estimator are provided. By our theoretical inferences, we find that the precision of our statistic is five times greater than that of pure threshold estimator.**

***Index Terms*—Range-based spot volatility estimation, Threshold, Precision, Consistency, Asymptotic normality, Jump diffusion models.**

I. Introduction

THE jump diffusion models are widely used in a variety of financial applications, such as interest rate modeling ([1], [2], [3]), bond pricing ([4]), derivative pricing ([5], [6]), risk management and hedging ([7], [8]), among others.

A state variable (an integrate rate, an exchange rate or a logarithmic asset price) $\{X_t\}_{t\geq 0}$ may be evolved by the following common jump diffusion process

$$X_t = X_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} J_i,\ t \in [0,T], \quad (1)$$

with initial condition $X_0 = x_0$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \Im, (\Im_t)_{t\geq 0}, P)$ and adapted to the filtration $(\Im_t)_{t\geq 0}$, namely, for any $0 \leq s \leq t < \infty$, it has $\Im_s \subseteq \Im_t \subseteq \Im$. Functions $\mu(\cdot)$ and $\sigma(\cdot)$ are, respectively, drift term and diffusion term of the process $\{X_t\}_{t\geq 0}$. $N = (N_t)_{t\geq 0}$ is a poisson process with constant intensity λ, jumping at times denoted by $(\tau_i)_{i=1,2,\cdots,N_T}$, and each J_i is the size of jump occurred at τ_i. The random variable J_i are $i.i.d.$ and independent of N.

Suppose the process $\{X_t\}_{t\geq 0}$ is observed discretely at equidistant time points $\{t = t_1, t_2, \cdots, t_n\}$ with $\delta = T/n = t_i - t_{i-1} (i = 1, 2, \cdots, n)$ which is a time distance between two consecutive observations. For ease of discussion, we denote the process $\{X_t\}_{t\geq 0}$ by $X = Y + J$, where Y and J are the continuous part and the jump part respectively.

In view of the fundamental role of the volatility term in financial applications, a great many scholars dedicate their interest and passion to estimate the volatility (integrated volatility or spot volatility) by disentangling the contributions due to the jumps and those due to the diffusion part from state variable with a exercisable mathematical or econometric technique, such as realized approach ([9]), maximum likelihood approach ([10]), power, bipower and multipower variation approach ([11], [12], [13]) and threshold-based approach ([2], [14], [15], [16]), among others.

Threshold method (or truncation-based method) for analyzing jump diffusion models was originally proposed in [17], it is a simple yet powerful methodology to identify jump. When the squared increment of the state variable is larger than a suitably defined threshold, the jump occurs. Using this technique, with low or moderate frequency data, [2] proposed an efficient spot volatility estimator with

$$\hat{\sigma}_n^2(x) = \frac{\sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}}-x}{h}\right)(\Delta_i X)^2 I_{\{(\Delta_i X)^2 \leq r(\delta)\}}}{\sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}}-x}{h}\right)\delta}, \quad (2)$$

where h is bandwidth, $K(\cdot)$ is kernel function, $\Delta_i X$ is denoted by the increment $(X_{t_i} - X_{t_{i-1}})$ and $r(\delta)$ is a deterministic function of the lag δ between two adjacent observations $(X_{t_{i-1}}, X_{t_i})$. Under a set of conditions, they obtain a stable convergence result. It is worth noting that this threshold technique is usually not suitable to be used in high frequency data directly due to well-known microstructure noise. In order to ignore the noise, people often sample sparsely at some lower frequency, which inevitably leads to the lack of information and efficiency.

Along with the rapid development of information technology and the increasing perfection of financial market, obtaining high frequency financial data (intraday data, hour data, minute data, even real time data) is becoming easier and easier. Exploring appropriate models and effective approach to analyze these high frequency data has become an important issue that mathematical scholars, statistical scholars and econometric scholars have to face and solve. The recent related works include but not limit to [18], [19], [20].

Realized range-based variance (developed in [21], [22], [23]) is formed from entire price process, so this technique for volatility estimation reveals more information than the realized method in which returns are sampled at fixed intervals. Its another advantage is that the high frequency financial data are not easy to be contaminated by the market microstructure noise. Unfortunately, there are few range-based works done on volatility estimation for jump diffusion models so far. Presenting a realized range-based multi-power variation theory, [24]drew a jump-robust inference about the diffusion volatility with high frequency data and reflected their estimators' significant efficiency.

Motivated by the literatures in [2], [23], [24], combining the range-based technique with the threshold idea, we

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present a range-based threshold volatility estimator for jump diffusion models when high frequency data are available. In contrast with integrate volatility estimation in [24], in this paper, the object we want to estimate is spot volatility. Our estimator proposed has the advantages of the two methods combined, such as the former's estimation precision and the latter's powerfulness.

II. Preliminaries

The following assumptions will be used in the work.

A1. The functions $\mu(\cdot), \sigma(\cdot)$ are time homogeneous and at least twice differentiable, and satisfy the local Lipschitz's condition

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \le C\,|x - y|\,.$$

A2. $\mu(\cdot), \sigma(\cdot), \mu'(\cdot), \sigma'(\cdot), \mu''(\cdot)$ and $\sigma''(\cdot)$ are all bounded.

A3. b satisfies $\frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{b} = o_P(1)$.

A4. $K(\cdot)$ is twice differential, symmetrical, nonnegative bounded function with support set $[-1, 1]$, satisfies

$$\int_{-1}^{1} K(x)dx = 1, \left|K^{(i)}(x)\right| < \infty \;\; (i = 0, 1, 2)$$

and

$$\int_{-1}^{1} |K(x)|\,|x|\,dx, \int_{-1}^{1} K^2(x)dx < \infty.$$

Define

$$L_Y(t, a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{[a, a+\varepsilon]}(Y_s)\, d[Y_s]\; (\forall a, t)$$

where $I(\cdot)$ is indicator function, $[Y]_t$ is the quadratic variation process of the process $\{Y_t\}_{t \ge 0}$, then we can define the chronological local time by

$$\begin{aligned}\overline{L}_Y(t, a) &= \frac{1}{\sigma^2(a)} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{[a, a+\varepsilon]}(Y_s)\sigma^2(Y_s)ds \\ &= \frac{1}{\sigma^2(a)} L_Y(t, a) \quad a.s. \quad (\forall a, t).\end{aligned}$$

Lemma 2.1. (The Occupation Time Formula) ([25]) Let Y_t be a continuous semi-martingale with quadratic variation process $[Y]_t$, and let $L_Y(t, a)$ be the local time at a , then

$$\int_0^t f(Y_s, s)d[Y]_s = \int_{-\infty}^{\infty} da \int_0^t f(a, s)dL_Y(s, a),$$

for every positive Borel measurable function f. Especially, if f is time homogeneous, then the expression can be simplified by

$$\int_0^t f(Y_s)d[Y]_s = \int_{-\infty}^{\infty} f(a)L_Y(t, a)da.$$

We extend Theorem 1 in [26] to the case for jump diffusion models.

Lemma 2.2. Under the conditions A3, A4, given $n \to \infty$, T fixed, and $b \to 0$ (as $n \to \infty$), then the quantity $\frac{\delta}{b} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right)$ converges to $\overline{L}_Y(T, x)$ with probability one.

Proof. For each n, define the following random sets

$$I_{0,n} = \{i \in \{1, 2, \cdots, n\} : \Delta_i N = 0\}\,,$$

and

$$I_{1,n} = \{i \in \{1, 2, \cdots, n\} : \Delta_i N \ne 0\}\,.$$

then

$$\begin{aligned}&\frac{\delta}{b} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) \\ &= \frac{\delta}{b} \sum_{i \in I_{0,n}} K\left(\frac{X_{t_i} - x}{b}\right) + \frac{\delta}{b} \sum_{i \in I_{1,n}} K\left(\frac{X_{t_i} - x}{b}\right).\end{aligned}$$

The second term above is dominated by $N_T \frac{\overline{K}\delta}{b} \left(\overset{a.s.}{\to} 0\right)$, where N_T is the jumps in [0,T]. For the first term, in view of Theorem 1 in [26], we have

$$\frac{\delta}{b} \sum_{i \in I_{0,n}} K\left(\frac{X_{t_i} - x}{b}\right) \overset{a.s.}{\to} \overline{L}_Y(T, x).$$

III. Main results

For any partition sequence $0 = t_0 < t_1 < \cdots < t_n = T$, we define the range of the process $\{X_t\}_{t \ge 0}$ between the sampling time t_{i-1} and t_i as

$$y_{X_{t_i,\delta_i}} = \sup_{t_{i-1} \le s, t \le t_i} \{X_t - X_s\}.$$

Similarly, we define the range of a standard Brownian motion between the sampling time t_{i-1} and t_i as

$$y_{W_{t_i,\delta_i}} = \sup_{t_{i-1} \le s, t \le t_i} \{W_t - W_s\}.$$

[22] derived the moment generating function of the range of a scaled Brownian motion $X_t = \sigma W_t$. The rth moment generating function can be expressed as

$$E[y^r_{X_{t_i,\delta_i}}] = \lambda_r \delta_i^{r/2} \sigma^r \quad (r \ge 1),$$

where $\lambda_r = E[y^r_{W_{1,1}}]$. In the situation of equidistant sampling, we abbreviate the range $y_{X_{t_i,\delta_i}}$ to $y_{X_{t_i}}$ ($y_{W_{t_i,\delta_i}}$ to $y_{W_{t_i}}$).

We propose a range-based spot volatility threshold estimator as

$$\hat{\sigma}^2(x) = \frac{\sum\limits_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) y^2_{X_{t_i}} I_{\{y^2_{X_{t_i}} \le r(\delta)\}}}{\delta \lambda_2 \sum\limits_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right)}, \tag{3}$$

where $r(\delta)$ is a deterministic function of the lag between two adjacent observations $(X_{t_i}, X_{t_{i-1}})$, such that $\lim\limits_{\delta \to 0} r(\delta) = 0$ and $\lim\limits_{\delta \to 0} (\delta \log \frac{1}{\delta})/r(\delta) = 0$.

Let

$$\frac{\delta}{b} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) = D(T, x),$$

then the estimator $\hat{\sigma}^2(x)$ can also be denoted by

$$\hat{\sigma}^2(x) = \frac{\sum\limits_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) y^2_{X_{t_i}} I_{\{y^2_{X_{t_i}} \le r(\delta)\}}}{\lambda_2 b D(T, x)}. \tag{4}$$

Lemma 3.1. ([27]) Suppose that $\sum_{i=1}^{N_t} J_i$ is a finite activity jump process, where N is a non-explosive counting process and the random variable J_i satisfy, $\forall t \in [0,1]$, $P\{\Delta N_t \neq 0, J_{N_t} = 0\} = 0$. $r(\delta)$ is a deterministic function of the lag between two adjacent observations $(X_{t_i}, X_{t_{i-1}})$, such that $\lim_{\delta\to 0} r(\delta) = 0$ and $\lim_{\delta\to 0} (\delta \log \frac{1}{\delta})/r(\delta) = 0$. Then for P-almost all ω, $\exists \bar{\delta}(\omega) > 0$, $\forall \delta \leq \bar{\delta}(\omega)$, we have

$$I_{\{y^2_{X_{t_i}} \leq r(\delta)\}}(\omega) = I_{\{\Delta_i N = 0\}}(\omega) \quad (\forall i = 1, ..., n). \tag{5}$$

Theorem 3.2. Under the conditions A1-A4, given $n \to \infty$, T fixed and $b \to 0$ (as $n \to \infty$), then

$$\hat{\sigma}^2(x) \xrightarrow{P} \sigma^2(x).$$

Proof. To prove this theorem, it is enough to prove the following two results:

$$(I)\, \frac{1}{\lambda_2 b D(T,x)} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) \sigma^2(X_{t_{i-1}})\, y^2_{W_{t_i}} \xrightarrow{P} \sigma^2(x);$$

$$(II)\, \hat{\sigma}^2(x) \xrightarrow{P} \frac{1}{\lambda_2 b D(T,x)} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) \sigma^2(X_{t_{i-1}}) y^2_{W_{t_i}}.$$

Next, we prove the above two results in sequence.

Result (I) Define

$$\varsigma_i = \frac{1}{\lambda_2 \delta} \sigma^2(X_{t_{i-1}}) y^2_{W_{t_i}}, \tag{6}$$

$$\xi_i = \frac{\delta}{bD(T,x)} K\left(\frac{X_{t_i} - x}{b}\right) \varsigma_i, \tag{7}$$

$$V_n = \frac{\delta}{bD(T,x)} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) \varsigma_i. \tag{8}$$

We can easily obtain

$$E[\varsigma_i | \Im_{i-1}] = \sigma^2(X_{t_{i-1}}),$$

then

$$\begin{aligned}
&\sum_{i=1}^{n} E[\xi_i | \Im_{i-1}] \\
&= \frac{\delta}{bD(T,x)} \sum_{i=1}^{n} E\left[K\left(\frac{X_{t_i} - x}{b}\right) \varsigma_i \middle| \Im_{i-1}\right] \\
&= \frac{\delta}{bD(T,x)} \sum_{i=1}^{n} K\left(\frac{X_{t_i} - x}{b}\right) \sigma^2(X_{t_{i-1}}) \\
&= \frac{1}{bD(T,x)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K\left(\frac{X_{u-} - x}{b}\right) \sigma^2(X_u)\, du \\
&+ \frac{1}{bD(T,x)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K\left(\frac{X_{u-} - x}{b}\right) (\sigma^2(X_{t_{i-1}}) - \sigma^2(X_u)) du \\
&+ \frac{1}{bD(T,x)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(K\left(\frac{X_{t_i} - x}{b}\right) - K\left(\frac{X_{u-} - x}{b}\right)\right) du \sigma^2(X_{t_{i-1}}) \\
&= A_1 + A_2 + A_3.
\end{aligned} \tag{9}$$

The third term A_3 in (9) coincides with

$$\begin{aligned}
&\frac{1}{bD(T,x)} \sum_{i\in I_{0,n}} \int_{t_{i-1}}^{t_i} \left(K\left(\frac{X_{t_i} - x}{b}\right) - K\left(\frac{X_{u-} - x}{b}\right)\right) du \sigma^2(X_{t_{i-1}}) \\
&+ \frac{1}{bD(T,x)} \sum_{i\in I_{1,n}} \int_{t_{i-1}}^{t_i} \left(K\left(\frac{X_{t_i} - x}{b}\right) - K\left(\frac{X_{u-} - x}{b}\right)\right) du \sigma^2(X_{t_{i-1}}) \\
&= A_{3,1} + A_{3,2}.
\end{aligned} \tag{10}$$

For $A_{3,1}$ in (10), it is

$$\frac{1}{bD(T,x)} \sum_{i\in I_{0,n}} \int_{t_{i-1}}^{t_i} \left| K'\left(\frac{\tilde{X}_{iu} - x}{b}\right) \right| \left| \frac{X_{t_i} - X_u}{b} \right| du \sigma^2(X_{t_{i-1}}),$$

where $\tilde{X}_{iu}$ is a value on the line segment connecting $X_{t_{i-1}}$ to X_u.

Using the property of uniform boundedness of the increments of X paths when $J \equiv 0$ (UBI property for short), we have

$$\max_{i\leq n} \sup_{t_{i-1}\leq u\leq t_i} |X_{t_i} - X_u| = O_{a.s.}\left((\delta \log(1/\delta))^{\frac{1}{2}}\right), \tag{11}$$

and then

$$\begin{aligned}
K'\left(\frac{\tilde{X}_{iu} - x}{b}\right) &= K'\left(\frac{X_{u-} - x}{b} + \frac{\tilde{X}_{iu} - X_{u-}}{b}\right) \\
&= K'\left(\frac{X_{u-} - x}{b} + O_{a.s.}\left(\frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{b}\right)\right) \\
&= K'\left(\frac{X_{u-} - x}{b} + o_P(1)\right),
\end{aligned} \tag{12}$$

it follows from (11) and (12) that A_{31} is bounded by

$$\begin{aligned}
&\frac{1}{bD(T,x)} \frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{b} \int_0^T \left| K'\left(\frac{X_{u-} - x}{b} + o_P(1)\right) \right| \left| \sigma^2(X_u + o_P(1)) \right| du \\
&= \frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{bD(T,x)} \frac{1}{b} \int_{-\infty}^{+\infty} \left| K'\left(\frac{p - x}{b} + o_P(1)\right) \right| \sigma^2(p + o_P(1)) \bar{L}_Y(T,x) dp \\
&= \frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{bD(T,x)} \int_{-\infty}^{+\infty} \left| K'(c + o_P(1)) \right| \sigma^2(cb + x + o_P(1)) \bar{L}_Y(T, cb + x) dc \\
&= O_P\left(\frac{(\delta \log(1/\delta))^{\frac{1}{2}}}{b}\right) \xrightarrow{P} 0,
\end{aligned}$$

the first equality above uses Lemma 2.1, the last equality above uses Lemma 2.2 and boundness of K' and σ. Using Taylor expansion for K, UBI property and boundness of K', we obtain

$$A_{3,2} = o_P\left(N_T \frac{\bar{K}'\delta}{bD(T,x)}\right) \xrightarrow{P} 0.$$

Similarly, using the assumption A1, UBI property and Lemma 2.1, 2.2, it is easy to see that

$$A_2 \leq O_P\left((\delta \log(1/\delta))^{\frac{1}{2}}\right) \xrightarrow{P} 0.$$

For the first term A_1 in (9), using Lemma 2.1 and Taylor expansion for σ^2, we have

$$\begin{aligned}
A_1 &= \frac{1}{bD(T,x)} \int_0^T K\left(\frac{X_{u-} - x}{b}\right) \sigma^2(X_u)\, du \\
&= \frac{1}{bD(T,x)} \int_0^T K\left(\frac{X_{u-} - x}{b}\right) \left(\sigma^2(x) + \left(\sigma^2(X_u) - \sigma^2(x)\right)\right) du \\
&= \frac{1}{bD(T,x)} \int_{-\infty}^{+\infty} K\left(\frac{p - x}{b}\right) \left(\sigma^2(x) + \left(\sigma^2(p) - \sigma^2(x)\right)\right) \bar{L}_Y(T,p) dp \\
&= \sigma^2(x) + \frac{1}{bD(T,x)} \int_{-\infty}^{+\infty} K\left(\frac{p - x}{b}\right) (\sigma^2)'(\varepsilon)(p - x) \bar{L}_Y(T,p)\, dp,
\end{aligned} \tag{13}$$

where ε is a value on the line segment connecting p to x, the second term of the last equality above is

$$O_{a.s.}\left(\frac{b}{D(T,x)}\int_{-\infty}^{+\infty}|K(c)||c|\bar{L}_Y(T,bc+x)\,dc\right),$$

using the assumption A4 and Lemma 2.2, it tends to Zero, therefore,

$$\mathrm{A}_1\to\sigma^2(x).$$

Combing with A_1, A_2 and A_3, we have

$$\sum_{i=1}^{n}E\left[\xi_i|\Im_{i-1}\right]\xrightarrow{P}\sigma^2(x).$$

By setting

$$\theta_i=\frac{\delta}{bD(T,x)}K\left(\frac{X_{t_i}-x}{b}\right)\left(\varsigma_i-E\left[\varsigma_i|\Im_{i-1}\right]\right),$$

we obtain

$$\mathrm{E}\left[\theta_i^2\middle|\Im_{i-1}\right]=\frac{\delta^2}{b^2D^2(T,x)}K^2\left(\frac{X_{t_i}-x}{b}\right)\Lambda_2\sigma^4(X_{t_i}),\qquad(14)$$

where $\Lambda_2=\left(\lambda_4-\lambda_2^2\right)/\lambda_2^2$.
Using the same steps to (9), we obtain

$$\begin{aligned}\sum_{i=1}^{n}E\left[\theta_i^2\middle|\Im_{i-1}\right]&=\frac{\Lambda_2\delta^2}{b^2D^2(T,x)}\sum_{i=1}^{n}K^2\left(\frac{X_{t_i}-x}{b}\right)\sigma^4(X_{t_i})\\&=O_P\left(\frac{\delta}{bD(T,x)}\right)\\&\xrightarrow{P}0,\end{aligned}$$

then

$$\begin{aligned}&\frac{1}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_{i-1}}\right)y_{W_{t_i}}^2\\&=V_n\xrightarrow{P}\sigma^2(x).\end{aligned}$$

Result (II) Using Lemma 3.1, write

$$\begin{aligned}&\hat{\sigma}^2(x)\\&=\frac{\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)y_{X_{t_i}}^2I_{\{\Delta_iN=0\}}}{\delta\lambda_2\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)}\\&=\frac{\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sup\limits_{t_{i-1}\le s,t\le t_i}\left(\int_s^t\mu(Y_u)\,du+\int_s^t\sigma(Y_u)\,dW_u\right)^2}{\delta\lambda_2\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)}\\&-\frac{\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sup\limits_{t_{i-1}\le s,t\le t_i}\left(\int_s^t\mu(Y_u)\,du+\int_s^t\sigma(Y_u)dW_u\right)^2I_{\{\Delta_iN\neq0\}}}{\delta\lambda_2\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)}\\&=\frac{\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)y_{Y_{t_i}}^2}{\delta\lambda_2\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)}-\frac{\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)y_{Y_{t_i}}^2I_{\{\Delta_iN\neq0\}}}{\delta\lambda_2\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)}\\&=D_1+D_2.\end{aligned}\qquad(15)$$

For $y_{Y_{t_i}}^2$ in D_2, using Triangle inequality, it is dominated by

$$\begin{aligned}&2\left(\sup_{t_{i-1}\le s,t\le t_i}\int_s^t\mu(X_u)\,du\right)^2+2\left(\sup_{t_{i-1}\le s,t\le t_i}\int_s^t\sigma(X_u)\,dW_u\right)^2\\&=E_1+E_2.\end{aligned}$$

Obviously,

$$E_1=O_{a.s.}\left(\delta^2\right).$$

For E_2, using Burkholder-Davis-Gundy inequality (hereafter indicated as BDG inequality), there exists a constant $C\ (>0)$ to make

$$E_2\le C\int_{t_{i-1}}^{t_i}\sigma^2(X_u)\,du=O_{a.s.}(\delta).$$

Therefore,

$$D_2=O_{a.s.}\left(\frac{\bar{K}N_T\delta}{bD(T,x)}\right)\xrightarrow{P}0.\qquad(16)$$

By (15) and (16), we have

$$\begin{aligned}&\hat{\sigma}^2(x)-\frac{1}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_{i-1}}\right)y_{W_{t_i}}^2\\&=D_1-\frac{1}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_{i-1}}\right)y_{W_{t_i}}^2-D_2\\&=\frac{1}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(y_{Y_{t_i}}^2-\sigma^2\left(X_{t_{i-1}}\right)y_{W_{t_i}}^2\right)+o_P(1).\end{aligned}\qquad(17)$$

For the main part in the second equality of (17), it can be decomposed as F_1+F_2, where

$$F_1=\frac{2}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma\left(X_{t_{i-1}}\right)y_{W_{t_i}}\left(y_{Y_{t_i}}-\sigma\left(X_{t_{i-1}}\right)y_{W_{t_i}}\right),$$

and

$$F_2=\frac{1}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(y_{Y_{t_i}}-\sigma\left(X_{t_{i-1}}\right)y_{W_{t_i}}\right)^2.$$

By the define of $y_{Y_{t_i}}$ and $y_{W_{t_i}}$, we have

$$\begin{aligned}&F_2\le\frac{1}{\lambda_2bD(T,x)}\times\\&\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sup_{t_{i-1}\le s,t\le t_i}\left|\int_s^t\mu(X_u)du+\int_s^t(\sigma(X_u)-\sigma(X_{t_i}))dW_u\right|\right)^2\\&\quad\le\frac{2}{\lambda_2bD(T,x)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sup_{t_{i-1}\le s,t\le t_i}\left|\int_s^t\mu(X_u)\,du\right|\right)^2\\&\quad+\frac{2}{\lambda_2bD(T,x)}\times\\&\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sup_{t_{i-1}\le s,t\le t_i}\left|\int_s^t(\sigma(X_u)-\sigma(X_{t_i}))\,dW_u\right|\right)^2\\&\quad=F_{21}+F_{22},\end{aligned}$$

the second inequality above uses Triangle inequality. It is easy to see $F_{21}=O_{a.s.}(\delta)$. For F_{22}, using UDG inequality, the assumption A1, UBI property and Lemma 2.1, 2.2, there exists a constant $C(>0)$ to make

$$E\left[F_{22}\right]\leq\frac{2C}{\lambda_2 bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\int_{t_{i-1}}^{t_i}\left(\sigma\left(X_u\right)-\sigma\left(X_{t_i}\right)\right)^2du$$
$$=O_{a.s.}\left(\delta\log\left(1/\delta\right)\right),$$

thus, $F_2=o_{a.s.}\left(1\right)$. For F_1, we have

$$F_1\leq\frac{2}{\lambda_2 bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma\left(X_{t_i}\right)y_{W_{t_i}}G_i,$$

where

$$G_i=\sup_{t_{i-1}\leq s,t\leq t_i}\left|\int_s^t\mu\left(X_u\right)du+\int_s^t\left(\sigma\left(X_u\right)-\sigma\left(X_{t_i}\right)\right)dW_u\right|,$$

using Hölder's inequality with counting measure, we obtain

$$F_1\leq\frac{2}{\lambda_2 D\left(T,x\right)}\left(\frac{1}{b}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_i}\right)y_{W_{t_i}}^2\right)^{\frac{1}{2}}\times$$
$$\left(\frac{1}{b}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)G_i^2\right)^{\frac{1}{2}},$$

using Hölder's inequality with probability measure, we have

$$E\left[F_1\right]\leq\frac{2}{\lambda_2 D\left(T,x\right)}\left(E\left[\frac{1}{b}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_i}\right)y_{W_{t_i}}^2\right]\right)^{\frac{1}{2}}\times$$
$$\left(E\left[\frac{1}{b}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)G_i^2\right]\right)^{\frac{1}{2}}$$
$$\leq\frac{2C}{\lambda_2 D\left(T,x\right)}O_{a.s.}\left(\sqrt{D\left(T,x\right)}\right)\times$$
$$\left(O_{a.s.}\left(\delta D\left(T,x\right)\right)+O_{a.s.}\left(\delta\log\left(1/\delta\right)D\left(T,x\right)\right)\right)^{\frac{1}{2}}$$
$$=O_{a.s.}\left(\left(\delta\log\left(1/\delta\right)\right)^{\frac{1}{2}}\right),$$

the second inequality above uses the similar decomposition and method to F_2. Combining with F_1, F_2 and (17), it has

$$\hat{\sigma}^2\left(x\right)-\frac{1}{\lambda_2 bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\sigma^2\left(X_{t_{i-1}}\right)y_{W_{t_i}}^2\xrightarrow{P}0.$$

These complete the proof of the whole theorem 3.2.

Theorem 3.3. Assume that the conditions of Theorem 3.2 hold and

$$\frac{b^3D\left(T,x\right)}{\delta}=o_P(1),$$

then the asymptotic distribution of the volatility estimator $\hat{\sigma}^2\left(x\right)$ is of the form

$$\sqrt{\frac{bD\left(T,x\right)}{\delta}}\left(\hat{\sigma}^2\left(x\right)-\sigma^2\left(x\right)\right)\xrightarrow{d}N\left(0,\Lambda_2\sigma^4\left(x\right)\int_{-1}^{1}K^2\left(c\right)dc\right),$$

where $\Lambda_2=\left(\lambda_4-\lambda_2^2\right)/\lambda_2^2$.

Proof. $\hat{\sigma}^2\left(x\right)-\sigma^2\left(x\right)$ can be decomposed as

$$\hat{\sigma}^2(x)-V_n+\sum_{i=1}^{n}\theta_i+\left(\frac{\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)E\left[\varsigma_i\right|\Im_{i-1}]-\sigma^2\left(x\right)\right).$$

First, by the proof in Theorem 3.2, we have

$$\hat{\sigma}^2\left(x\right)-V_n=O_P\left(\left(\delta\log\left(1/\delta\right)\right)^{\frac{1}{2}}\right).$$

Secondly,

$$\frac{\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)E\left[\varsigma_i\right|\Im_{i-1}]-\sigma^2\left(x\right)$$
$$=\frac{\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sigma^2\left(X_{t_{i-1}}\right)-\sigma^2\left(x\right)\right)$$
$$=\frac{\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sigma^2\left(X_{t_{i-1}}\right)-\sigma^2\left(X_{t_i}\right)\right)$$
$$+\frac{\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K\left(\frac{X_{t_i}-x}{b}\right)\left(\sigma^2\left(X_{t_i}\right)-\sigma^2\left(x\right)\right)$$
$$=O_{a.s.}\left(\left(\delta\log\left(1/\delta\right)\right)^{\frac{1}{2}}\right)+O_{a.s.}\left(b\right)$$
$$=O_P\left(b\right).$$

then

$$\sqrt{\frac{bD\left(T,x\right)}{\delta}}\left(\hat{\sigma}^2\left(x\right)-\sigma^2\left(x\right)\right)$$
$$=\sqrt{\frac{bD\left(T,x\right)}{\delta}}\left(\sum_{i=1}^{n}\theta_i+O_P\left(\left(\delta\log\left(1/\delta\right)\right)^{\frac{1}{2}}\right)+O_P\left(b\right)\right)$$
$$=\sqrt{\frac{bD\left(T,x\right)}{\delta}}\sum_{i=1}^{n}\theta_i+o_P\left(1\right). \quad (18)$$

Next, we consider the distribution of $\sqrt{\frac{bD(T,x)}{\delta}}\sum_{i=1}^{n}\theta_i$. The sum of the conditional second moment can be calculated as

$$\frac{bD\left(T,x\right)}{\delta}\sum_{i=1}^{n}E\left[\theta_i^2\right|\Im_{i-1}]$$
$$=\frac{bD\left(T,x\right)}{\delta}\frac{\delta^2}{b^2D^2\left(T,x\right)}\sum_{i=1}^{n}K^2\left(\frac{X_{t_i}-x}{b}\right)\Lambda_2\sigma^4\left(X_{t_{i-1}}\right)$$
$$=\frac{\Lambda_2\delta}{bD\left(T,x\right)}\sum_{i=1}^{n}K^2\left(\frac{X_{t_i}-x}{b}\right)\sigma^4\left(X_{t_{i-1}}\right)$$
$$=\frac{\Lambda_2\delta}{bD\left(T,x\right)}\sum_{i\in I_{0,n}}K^2\left(\frac{X_{t_i}-x}{b}\right)\sigma^4\left(X_{t_{i-1}}\right)$$
$$+\frac{\Lambda_2\delta}{bD\left(T,x\right)}\sum_{i\in I_{1,n}}K^2\left(\frac{X_{t_i}-x}{b}\right)\sigma^4\left(X_{t_{i-1}}\right)$$
$$=H_1+H_2.$$

For the term H_2, it tends as to zero obviously. For H_1 then, we have

$$H_1=\frac{\Lambda_2\delta}{bD\left(T,x\right)}\times$$
$$\sum_{i\in I_{0,n}}K^2\left(\frac{X_{t_i}-x}{b}\right)\left(\left(\sigma^4(X_{t_i})-\sigma^4(x)\right)+\left(\sigma^4\left(X_{t_{i-1}}\right)-\sigma^4\left(X_{t_i}\right)\right)+\sigma^4(x)\right), \quad (19)$$

by the assumption A1, UBI property and Lemma 2.1, 2.2, we obtain that H_1 converges to $\Lambda_2\sigma^4\left(x\right)\int_{-1}^{1}K^2\left(c\right)dc$ in probability. We verify Liapounov's condition as follows

$$\frac{b^{\frac{3}{2}}\left(D\left(T,x\right)\right)^{\frac{3}{2}}}{\delta^{\frac{3}{2}}}\sum_{i=1}^{n}E\left[\theta_i^3\middle|\Im_{i-1}\right]$$
$$=\frac{\delta^{\frac{3}{2}}}{b^{\frac{3}{2}}\left(D\left(T,x\right)\right)^{\frac{3}{2}}}\sum_{i\in I_{0,n}}K^3\left(\frac{X_{t_i}-x}{b}\right)E\left[\left(\varsigma_i-E\left[\varsigma_i\middle|\Im_{i-1}\right]\right)^3\middle|\Im_{i-1}\right]$$
$$+\frac{\delta^{\frac{3}{2}}}{b^{\frac{3}{2}}\left(D\left(T,x\right)\right)^{\frac{3}{2}}}\sum_{i\in I_{1,n}}K^3\left(\frac{X_{t_i}-x}{b}\right)E\left[\left(\varsigma_i-E\left[\varsigma_i\middle|\Im_{i-1}\right]\right)^3\middle|\Im_{i-1}\right]$$
$$=L_1+L_2.$$

Obviously, $L_2\to 0$. For L_1, we have

$$L_1=\frac{\delta^{\frac{3}{2}}}{b^{\frac{3}{2}}\left(D\left(T,x\right)\right)^{\frac{3}{2}}}\sum_{i\in I_{0,n}}K^3\left(\frac{X_{t_i}-x}{b}\right)\left(\frac{\lambda^6}{\lambda_2^3}-\frac{3\lambda_4}{\lambda_2^2}+2\right)\sigma^6\left(X_{t_{i-1}}\right)$$
$$=O_P\left(\frac{\delta^{\frac{1}{2}}}{b^{\frac{1}{2}}\left(D\left(T,x\right)\right)^{\frac{1}{2}}}\right)$$
$$\xrightarrow{P}0.$$

If the Liapounov's condition is satisfied, then the Lindeberg's condition is also satisfied, hence, by the Corollary 3.1 in [28], we obtain

$$\sqrt{\frac{bD\left(T,x\right)}{\delta}}\sum_{i=1}^{n}\theta_i\xrightarrow{d}N\left(0,\Lambda_2\sigma^4\left(x\right)\int_{-1}^{1}K^2\left(c\right)dc\right),$$

further, we have

$$\sqrt{\frac{bD\left(T,x\right)}{\delta}}\left(\hat{\sigma}^2\left(x\right)-\sigma^2\left(x\right)\right)\xrightarrow{d}N\left(0,\Lambda_2\sigma^4\left(x\right)\int_{-1}^{1}K^2\left(c\right)dc\right). \tag{20}$$

Remark 3.1. The equation $\frac{b^3D(T,x)}{\delta}=o_P\left(1\right)$ in Theorem 3.3 guarantees that the term $\sqrt{\frac{bD(T,x)}{\delta}}O_P\left(b\right)$ in (18) is asymptotically negligible.

Remark 3.2. Given the different definition of the local time of X, [2] proposed a similar asymptotic variance with (20). Notice that the constant appearing in [2] is 2, in contrast, the number in (20) is $\Lambda_2\approx 0.4$, therefore, the precision of our estimator is five times greater than that of the estimator in [2].

IV. Conclusions

This paper proposes range-based threshold spot volatility estimation for jump diffusion models. It is formed from entire high frequency data and reduce the influence of microstructure noise effectively. Comparing with the general pure threshold estimator, our method improves the estimation precision by 5 times.

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