The Neutrix Limit of the Hurwitz Zeta Function and Its Application

Zhongfeng Sun, Aijuan Li, and Huizeng Qin

***Abstract*—In this paper, the neutrix limit is used to extend the definition of the Hurwitz zeta function $\zeta(\alpha, x)$ and its partial derivatives to the whole complex plane except for non-positive integers α, in particular, the values of $\zeta(1, x)$ is obtained. This definition is equivalent to the Hermite's integral of $\zeta(\alpha, x)$ as $\alpha \neq 1, 0, -1, \ldots$. Moreover, some properties of $\zeta(1, x)$ are established and we find that $\zeta(1, x)$ is the inverse number of the digamma function. In addition, we pay our special attention to the closed forms of the certain integrals involving the Hurwitz zeta function, which can be expressed as a linear combination of the Riemann zeta functions and their derivatives.**

***Index Terms*—Hurwitz Zeta Function, Neutrix Limit, Riemann Zeta Function, Hermite's Integral, Abel-Plana Formula.**

I. INTRODUCTION

THE Hurwitz zeta function $\zeta(\alpha, x)$ is defined by the series

$$\zeta(\alpha, x) = \sum_{j=0}^{\infty} \frac{1}{(j+x)^{\alpha}}, \quad \alpha \in \mathbb{C}, \quad \Re(\alpha) > 1, \ x > 0, \quad (1)$$

where $\mathbb{C}$ denotes the complex set and $\Re(\alpha)$ denotes the real part of α. It can be continued analytically to $\mathbb{C} \setminus \{1\}$ by the Hermite's integral [1](pp.26)

$$\begin{aligned} \zeta(\alpha, x) = \ & \frac{x^{-\alpha}}{2} - \frac{x^{1-\alpha}}{1-\alpha} \\ & + 2\int_0^{\infty} \frac{\sin\left(\alpha \arctan \frac{y}{x}\right)}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)^{\frac{\alpha}{2}}} dy, \end{aligned} \quad (2)$$

where $\alpha \in \mathbb{C} \setminus \{1\}$ and $x > 0$. When $x = 1$, the Hurwitz zeta function reduces to the Riemann zeta function, *i.e.*, $\zeta(\alpha, 1) = \zeta(\alpha)$.

In recent years the issue of the Hurwitz zeta function have attracted much attention, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. Yue and Williams [2] applied the Hurwitz zeta function to evaluate certain integrals associated with the elementary functions. Miller and Adamchik [3], [4] established the closed form of the Hurwitz zeta function and its derivatives for rational arguments. Espinosa and Moll [5], [6] considered the evaluation of indefinite integrals involving the Hurwitz zeta function which can be represented by Hurwitz zeta function, Riemann zeta function and other special functions. Kanemitsu et al. expressed the infinite sums by the derivatives of the Hurwitz zeta function and the multiple gamma function in [7]. Furthermore, they discussed the integral representation and the asymptotic formula of the partial sum $L_u(x, a) = \sum_{0 \leq n \leq x}(n+a)^u$ in [8]. Coffey gave several infinite series representations for the Hurwitz zeta function in [9] and designed an efficient algorithm for computing the Hurwitz zeta function in [10]. Mező and Dil [11] proved that the hyperharmonic series can be transformed into the sums involving hyperharmonic numbers and the Hurwitz zeta function. Moreover, this result can be applied to obtain some identities for Riemann zeta and Hurwitz zeta function. With the help of the Hurwitz zeta function, Li et al. developed an algorithm for computing the beta function and its partial derivatives in [12].

Although the Hurwitz zeta function is well known for its applications in mathematics, the above-mentioned analytic continuation (2) to $\mathbb{C}$ has a simple pole of residue one. The main objective of this paper is to use the neutrix limit developed by van der Corput [13] to define $\zeta(1, x)$. The neutrix limit has been widely used to extend the definition of the special functions and their partial derivatives. For example, the lower incomplete gamma function $\gamma(\alpha, x)$ for the non-positive integer α [14], [15]; the beta function $B(x, y)$ [16] and the incomplete beta function $B(z; x, y)$ [17], [18] and their partial derivatives for negative integers x or y; the q-analogue of the gamma function $\Gamma_q(\alpha)$, the incomplete gamma function $\gamma_q(\alpha, x)$ and their derivatives for non-positive integer α [19], [20]; the q-beta function $B_q(x, y)$ for negative integers x or y [21]; the partial derivatives of the incomplete beta function $B(z; x, y)$ for all complex values of x and y [22].

The structure of this paper is as follows. In Section II, the neutrix limit is used to establish the definition of Hurwitz zeta function $\zeta(\alpha, x)$ and its partial derivatives for $x > 0$ and $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. In Section III, some properties of $\zeta(\alpha, x)$ for $\alpha = 1$ are discussed. In Section IV, the new definition of the Hurwitz zeta function and its properties are applied to evaluate certain integrals associated with the Hurwitz zeta function. A final conclusion is given in Section V.

II. DEFINITION OF THE HURWITZ ZETA FUNCTION BY THE NEUTRIX LIMIT

We introduce the following notations

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \ \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ \mathbb{N}_0^- := \{0, -1, -2, \ldots\},$$

and

$$\zeta^{(p)}(\alpha, x) = \frac{\partial^p}{\partial \alpha^p} \zeta(\alpha, x), \quad \zeta^{(p)}(\alpha) = \frac{d^p}{d\alpha^p} \zeta(\alpha),$$

where $\alpha \in \mathbb{C}, \ x > 0$ and $p \in \mathbb{N}_0$.

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Definition 2.1 For $x>0$, $p\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}\setminus\mathbb{N}_0^-$, we define

$$\begin{aligned}&\zeta^{(p)}(\alpha,x)\\=\;&(-1)^p\times N-\lim_{n\to\infty}\sum_{j=0}^{n}(j+x)^{-\alpha}\ln^p(j+x),\end{aligned}\tag{3}$$

where N is the neutrix having domain $N'=\mathbb{N}$ and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n,\quad \ln^r n\qquad(\lambda>0,r\in\mathbb{N})\tag{4}$$

and all functions which converge to zero in the usual sense as $n\to\infty$, see [13], [23], [24].

Lemma 2.2 (Abel-Plana formula [25] pp.290) Let $n\in\mathbb{N}$. Then

$$\begin{aligned}&\sum_{j=0}^{n}f(j)=\int_0^n f(t)dt+\frac{f(0)+f(n)}{2}+\\&i\int_0^\infty\frac{f(it)-f(n+it)-f(-it)+f(n-it)}{e^{2\pi t}-1}dt.\end{aligned}\tag{5}$$

In the following Theorems, we aim to establish the relation between (3) and the Hermite's integral (2) by using the Abel-Plana formula (5).

Theorem 2.3 1) Let $x>0$. Then

$$\begin{aligned}&N-\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j+x}\\=\;&-\ln x+\frac{1}{2x}+2\int_0^\infty\frac{y}{(e^{2\pi y}-1)(x^2+y^2)}dy.\end{aligned}\tag{6}$$

2) Let $\alpha\in\mathbb{C}$, $\alpha\neq-1,0,1,\dots$ and $x>0$. Then

$$\begin{aligned}&N-\lim_{n\to\infty}\sum_{j=0}^{n}(j+x)^{\alpha}=\frac{x^{\alpha}}{2}-\frac{x^{\alpha+1}}{\alpha+1}\\&-2\int_0^\infty\frac{\sin\left(\alpha\arctan\frac{y}{x}\right)}{(e^{2\pi y}-1)(x^2+y^2)^{-\frac{\alpha}{2}}}dy.\end{aligned}\tag{7}$$

Proof. 1) Setting $f(t)=(t+x)^{-1}$ in the Abel-Plana formula (5), we yield

$$\begin{aligned}&N-\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j+x}\\=\;&-\ln x+\frac{1}{2x}\\&+i\int_0^\infty\frac{1}{e^{2\pi y}-1}\left(\frac{1}{x+iy}-\frac{1}{x-iy}\right)dy\\&+N-\lim_{n\to\infty}\Bigg[\ln(n+x)+\frac{1}{2(n+x)}\\&+i\int_0^\infty\frac{1}{e^{2\pi y}-1}\\&\times\left(\frac{1}{n+x-iy}-\frac{1}{n+x+iy}\right)dy\Bigg],\end{aligned}\tag{8}$$

which means that

$$\begin{aligned}&N-\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j+x}\\=\;&-\ln x+\frac{1}{2x}+2\int_0^\infty\frac{y}{(e^{2\pi y}-1)(x^2+y^2)}dy\\&+N-\lim_{n\to\infty}\Bigg[\ln n+\ln(1+x/n)+\frac{1}{2(n+x)}\\&-2\int_0^\infty\frac{1}{e^{2\pi y}-1}\times\frac{y}{(n+x)^2+y^2}dy\Bigg].\end{aligned}\tag{9}$$

Combining (9) with the following inequality,

$$\begin{aligned}0\;&\le\int_0^\infty\frac{1}{e^{2\pi y}-1}\times\frac{y}{(n+x)^2+y^2}dy\\&\le\frac{1}{(n+x)^2}\int_0^\infty\frac{y}{e^{2\pi y}-1}dy\\&=\frac{\zeta(2)}{4\pi^2(n+x)^2},\end{aligned}\tag{10}$$

we obtain (6).

2) Taking $f(t)=(t+x)^{\alpha}$ in the Abel-Plana formula (5), we have

$$\begin{aligned}&\sum_{j=0}^{n}(j+x)^{\alpha}\\=\;&\int_0^n(t+x)^{\alpha}dt+\frac{x^{\alpha}+(n+x)^{\alpha}}{2}\\&+i\int_0^\infty\Big[(x+iy)^{\alpha}-(n+x+iy)^{\alpha}\\&-(x-iy)^{\alpha}+(n+x-iy)^{\alpha}\Big]\frac{1}{e^{2\pi y}-1}dy\\=\;&-\frac{x^{\alpha+1}}{\alpha+1}+\frac{x^{\alpha}}{2}+\frac{(n+x)^{\alpha+1}}{\alpha+1}+\frac{(n+x)^{\alpha}}{2}\\&+i\int_0^\infty\frac{(x+iy)^{\alpha}-(x-iy)^{\alpha}}{e^{2\pi y}-1}dy\\&+i\int_0^\infty\frac{(n+x-iy)^{\alpha}-(n+x+iy)^{\alpha}}{e^{2\pi y}-1}dy.\end{aligned}\tag{11}$$

Using the following formula

$$\begin{aligned}&i\left[(x+iy)^{\alpha}-(x-iy)^{\alpha}\right]\\=\;&-2(x^2+y^2)^{\frac{\alpha}{2}}\sin\left(\alpha\arctan\frac{y}{x}\right),\end{aligned}\tag{12}$$

we have

$$\begin{aligned}&i\int_0^\infty\frac{(x+iy)^{\alpha}-(x-iy)^{\alpha}}{e^{2\pi y}-1}dy\\=\;&-\int_0^\infty\frac{2\left(x^2+y^2\right)^{\frac{\alpha}{2}}\sin\left(\alpha\arctan\frac{y}{x}\right)}{(e^{2\pi y}-1)}dy.\end{aligned}\tag{13}$$

Denote

$$\begin{aligned}C_\beta^0\;&:=1,\\C_\beta^k\;&:=\frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}\\&=\frac{(\beta-k+1)_k}{k!},\end{aligned}\tag{14}$$

where $\beta\in\mathbb{C}$ and $k\in\mathbb{N}$.

Using Taylor series expansions and (14), we obtain

$$\begin{aligned}&i\left[(n+x-iy)^{\alpha}-(n+x+iy)^{\alpha}\right]\\=\;&2\sum_{s=1}^{m}(-1)^{s-1}C_\alpha^{2s-1}(n+x)^{\alpha-2s+1}y^{2s-1}\\&+C_\alpha^{2m}(-1)^m y^{2m}\Big[(n+x-\theta_1(iy))^{\alpha-2m}\\&-(n+x+\theta_2(iy))^{\alpha-2m}\Big],\end{aligned}\tag{15}$$

where $m \in \mathbb{N}$ satisfies $\Re(\alpha - 2m) < 0$ and

$$(n+x)^\beta = n^\beta(1+x/n)^\beta = \sum_{u=0}^{m_1} C_\beta^u n^{\beta-u} x^u + O\left(n^{\beta-1-m_1}\right), \tag{16}$$

where $m_1 \in \mathbb{N}$ satisfies $\Re(\beta - 1 - m_1) < 0$.

From (15) and (16), we obtain

$$\begin{aligned} & i\int_0^\infty \frac{(n+x-iy)^\alpha - (n+x+iy)^\alpha}{e^{2\pi y}-1}dy \\ = & \ i\int_0^{n+x} \frac{(n+x-iy)^\alpha - (n+x+iy)^\alpha}{e^{2\pi y}-1}dy \\ & +i\int_{n+x}^\infty \frac{(n+x-iy)^\alpha - (n+x+iy)^\alpha}{e^{2\pi y}-1}dy \\ = & \ R_m + 2\sum_{s=1}^m (-1)^{s-1} C_\alpha^{2s-1}(n+x)^{\alpha-2s+1} \\ & \times \int_0^\infty \frac{y^{2s-1}}{e^{2\pi y}-1}dy \\ = & \ R_m + 2\sum_{s=1}^m (-1)^{s-1} C_\alpha^{2s-1} \frac{\Gamma(2s)\zeta(2s)}{(2\pi)^{2s}} \\ & \times \left[\sum_{u=0}^{m_1} C_{\alpha-2s+1}^u n^{\alpha-2s+1-u} x^u \right. \\ & \left. +O\left(n^{\alpha-2s-m_1}\right)\right], \end{aligned} \tag{17}$$

where $m_1 \in \mathbb{N}$ satisfies $\Re(\alpha - m_1) < 0$ and

$$\begin{aligned} & R_m \\ = & \ i\int_{n+x}^\infty \frac{(n+x-iy)^\alpha - (n+x+iy)^\alpha}{e^{2\pi y}-1}dy \\ & +(-1)^m C_\alpha^{2m} \int_0^{n+x} \Big[(n+x-\theta_1(iy))^{\alpha-2m} \\ & -(n+x+\theta_2(iy))^{\alpha-2m}\Big] \frac{y^{2m}}{e^{2\pi y}-1}dy \\ & +2\sum_{s=1}^m (-1)^s C_\alpha^{2s-1}(n+x)^{\alpha-2s+1} \\ & \times \int_{n+x}^\infty \frac{y^{2s-1}dy}{e^{2\pi y}-1} \\ = & \ R_m^1 + R_m^2 + R_m^3, \end{aligned} \tag{18}$$

where $\theta_1, \theta_2 \in (0,1)$.

Next, we consider properties of R_m^1, R_m^2 and R_m^3 when the parameter n large enough.

$$\begin{aligned} & |R_m^1| \\ = & \left| i\int_{n+x}^\infty \frac{(n+x-iy)^\alpha - (n+x+iy)^\alpha}{e^{2\pi y}-1}dy \right| \\ \leq & \ B_1(\alpha)\int_{n+x}^\infty \frac{e^{\pi y/2}}{e^{2\pi y}-1}dy \\ \leq & \ B_1(\alpha)\int_{n+x}^\infty e^{-\pi y/2}dy \\ = & \ \frac{2}{\pi}B_1(\alpha)e^{-\pi(n+x)/2}, \end{aligned} \tag{19}$$

where $B_1(\alpha)$ is the positive constant only depends on α, and

$$\begin{aligned} & |R_m^2| \\ = & \left| (-1)^m C_\alpha^{2m} \int_0^{n+x} \Big[(n+x-\theta_1(iy))^{\alpha-2m} \right. \\ & \left. -(n+x+\theta_2(iy))^{\alpha-2m}\Big] \frac{y^{2m}}{e^{2\pi y}-1}dy \right| \\ \leq & \ B_2(\alpha,m)(n+x)^{\Re(\alpha)-2m} \int_0^{n+x} \frac{y^{2m}}{e^{2\pi y}-1}dy \\ \leq & \ B_2(\alpha,m)\int_0^\infty \frac{y^{2m}}{e^{2\pi y}-1}dy \times n^{\Re(\alpha)-2m} \\ = & \ O\left(n^{\Re(\alpha)-2m}\right), \end{aligned} \tag{20}$$

where $\Re(\alpha - 2m) < 0$ and $B_2(\alpha, m)$ is the positive constant which depend on α and m, and

$$\begin{aligned} |R_m^3| = & \left| 2\sum_{s=1}^m (-1)^s C_\alpha^{2s-1}(n+x)^{\alpha-2s+1} \right. \\ & \left. \times \int_{n+x}^\infty \frac{y^{2s-1}dy}{e^{2\pi y}-1} \right| \\ \leq & \ 2\sum_{s=1}^m \left|C_\alpha^{2s-1}\right| (n+x)^{\Re(\alpha)-2s+1} B_3(s) \\ & \times \int_{n+x}^\infty \frac{e^{\pi y/2}}{e^{2\pi y}-1}dy \\ \leq & \ 2\sum_{s=1}^m B_3(s)\left|C_\alpha^{2s-1}\right| (n+x)^{\Re(\alpha)-2s+1} \\ & \times \int_{n+x}^\infty e^{-\pi y/2}dy \\ \leq & \ \frac{4}{\pi}\sum_{s=1}^m B_3(s)\left|C_\alpha^{2s-1}\right| \\ & (n+x)^{\Re(\alpha)-2s+1} e^{-\pi(n+x)/2}, \end{aligned} \tag{21}$$

where $B_3(s)$ is the positive constant only depends on s.

Using (18), (19), (20) and (21), we get

$$\lim_{n\to\infty} R_m = \lim_{n\to\infty}\left(R_m^1 + R_m^2 + R_m^3\right) = 0. \tag{22}$$

Combining (11), (16), (13), (17) and (22), we conclude that (7) holds. ■

Theorem 2.4 Let $p \in \mathbb{N}$ and $x > 0$.

1) Then

$$\begin{aligned} & N - \lim_{n\to\infty}\sum_{j=0}^n \frac{\ln^p(j+x)}{j+x} \\ = & \ \frac{1}{2x}\ln^p x - \frac{1}{p+1}\ln^{p+1} x + \sum_{j=0}^{\left[\frac{p}{2}\right]} \frac{(-1)^j C_p^{2j}}{2^{p-2j-1}} \\ & \times \int_0^\infty \frac{y\left(\arctan\frac{y}{x}\right)^{2j} \ln^{p-2j}\left(x^2+y^2\right)}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)}dy \\ & -x\sum_{j=0}^{\left[\frac{p-1}{2}\right]} \frac{(-1)^j C_p^{2j+1}}{2^{p-2j-2}} \\ & \times \int_0^\infty \frac{\left(\arctan\frac{y}{x}\right)^{2j+1} \ln^{p-2j-1}\left(x^2+y^2\right)}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)}dy. \end{aligned} \tag{23}$$

2) If $\alpha \in \mathbb{C}$ and $\alpha \neq -1, 0, 1, \ldots$, then

$$\begin{aligned}
& N - \lim_{n\to\infty} \sum_{j=0}^{n} (j+x)^{\alpha} \ln^p (j+x) \\
= & -p! x^{\alpha+1} \sum_{j=0}^{p} \frac{(-1)^{p-j} \ln^j x}{j! \, (\alpha+1)^{p-j+1}} + \frac{x^{\alpha} \ln^p x}{2} \\
& - \sum_{j=0}^{p} 2^{1+j-p} C_p^j \int_0^{\infty} \sin\left(\alpha \arctan \frac{y}{x} + \frac{j\pi}{2}\right) \\
& \times \frac{\left(x^2+y^2\right)^{\frac{\alpha}{2}} \ln^{p-j}\left(x^2+y^2\right) \left(\arctan \frac{y}{x}\right)^j}{e^{2\pi y}-1} dy.
\end{aligned} \tag{24}$$

Proof. 1) Replacing $f(t)$ by $\frac{\ln^p(t+x)}{t+x}$ in the Abel-Plana formula (5), we have

$$\begin{aligned}
& \sum_{j=0}^{n} \frac{\ln^p(j+x)}{j+x} \\
= & \int_0^n \frac{\ln^p(t+x)}{t+x} dt + \frac{\ln^p(n+x)}{2(n+x)} + \frac{\ln^p x}{2x} \\
& + i \int_0^{\infty} \frac{1}{e^{2\pi y}-1} \left(\frac{\ln^p(x+iy)}{x+iy} - \frac{\ln^p(x-iy)}{x-iy} \right. \\
& \left. + \frac{\ln^p(n+x-iy)}{n+x-iy} - \frac{\ln^p(n+x+iy)}{n+x+iy} \right) dy \\
= & \frac{\ln^{p+1}(n+x)}{p+1} + \frac{\ln^p(n+x)}{2(n+x)} - \frac{\ln^{p+1} x}{p+1} \\
& + \frac{\ln^p x}{2x} + \int_0^{\infty} \left[\left(x^2+y^2\right)^{-1} A_0(x,y) \right. \\
& \left. - \left((n+x)^2+y^2\right)^{-1} A_n(x,y) \right] \frac{1}{e^{2\pi y}-1} dy,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
& \ln(k+x+iy) = \frac{1}{2}\psi_k + i\phi_k, \\
& \psi_k = \ln\left((k+x)^2+y^2\right), \quad \phi_k = \arctan \frac{y}{k+x},
\end{aligned} \tag{26}$$

for $k = 0$ or n, and

$$\begin{aligned}
& A_k(x,y) \\
= & [y + i(k+x)] \ln^p((k+x)+iy) \\
& + [y - i(k+x)] \ln^p((k+x)-iy) \\
= & [y + i(k+x)] \Big(\frac{1}{2}\psi_k + i\phi_k\Big)^p \\
& + [y - i(k+x)] \Big(\frac{1}{2}\psi_k - i\phi_k\Big)^p \\
= & [y + i(k+x)] \sum_{q=0}^{p} C_p^q \Big(\frac{1}{2}\psi_k\Big)^{p-q} \Big(i\phi_k\Big)^q \\
& + [y - i(k+x)] \sum_{q=0}^{p} C_p^q \Big(\frac{1}{2}\psi_k\Big)^{p-q} \Big(-i\phi_k\Big)^q \\
= & \sum_{q=0}^{p} C_p^q 2^{q-p} \psi_k^{p-q} \phi_k^q \Big[i^q y \left(1+(-1)^q\right) \\
& + i^{q+1}(k+x)\left(1-(-1)^q\right) \Big] \\
= & y \sum_{j=0}^{[\frac{p}{2}]} (-1)^j C_p^{2j} 2^{2j+1-p} \psi_k^{p-2j} \phi_k^{2j} + (k+x) \\
& \times \sum_{j=0}^{[\frac{p-1}{2}]} (-1)^{j+1} C_p^{2j+1} 2^{2j+2-p} \psi_k^{p-2j-1} \phi_k^{2j+1}.
\end{aligned} \tag{27}$$

For $q = 1, 2, \ldots, p$, we have

$$\frac{d}{dz}\left(\frac{\ln^q z}{z}\right) = z^{-2} \ln^{q-1} z (q - \ln z) < 0, \quad z > e^q. \tag{28}$$

When $n > e^{p/2}$, one has

$$\begin{aligned}
& \left((n+x)^2+y^2\right)^{-1} \psi_n^q \\
= & \left((n+x)^2+y^2\right)^{-1} \ln^q \left((n+x)^2+y^2\right) \\
\leq & n^{-2} \ln^q n^2.
\end{aligned} \tag{29}$$

Using this monotonicity, we yield

$$\begin{aligned}
& \int_0^{\infty} \frac{1}{e^{2\pi y}-1} \left((n+x)^2+y^2\right)^{-1} A_n(x,y) dy \\
\leq & \sum_{j=0}^{[\frac{p}{2}]} C_p^{2j} 2^{2j+1-p} (\pi/2)^{2j} \int_0^{\infty} \frac{y}{e^{2\pi y}-1} dy \\
& \times \frac{\ln^{p-2j} n^2}{n^2} + \sum_{j=0}^{[\frac{p-1}{2}]} C_p^{2j+1} 2^{2j+2-p} (\pi/2)^{2j} \\
& \times \int_0^{\infty} \frac{(n+x) \arctan \frac{y}{n+x}}{e^{2\pi y}-1} dy \times \frac{\ln^{p-2j-1} n^2}{n^2} \\
\leq & \sum_{j=0}^{[\frac{p}{2}]} C_p^{2j} 2^{2j+1-p} (\pi/2)^{2j} \int_0^{\infty} \frac{y}{e^{2\pi y}-1} dy \\
& \times \frac{\ln^{p-2j} n^2}{n^2} + \sum_{j=0}^{[\frac{p-1}{2}]} C_p^{2j+1} 2^{2j+2-p} (\pi/2)^{2j} \\
& \times \int_0^{\infty} \frac{y}{e^{2\pi y}-1} dy \times \frac{\ln^{p-2j-1} n^2}{n^2} \\
= & O\left(\frac{\ln^p n}{n^2}\right).
\end{aligned} \tag{30}$$

Combining (30) and (25), we obtain

$$\begin{aligned}
& N - \lim_{n\to\infty} \sum_{j=0}^{n} \frac{\ln^p(j+x)}{j+x} \\
= & \frac{\ln^p x}{2x} - \frac{\ln^{p+1} x}{p+1} \\
& + \int_0^{\infty} \frac{1}{e^{2\pi y}-1} \left(x^2+y^2\right)^{-1} A_0(x,y) dy.
\end{aligned} \tag{31}$$

With the help of (30) and (31), (25) implies that (23) holds.

2) Using (7), we get

$$\begin{aligned}
& N - \lim_{n\to\infty} \sum_{j=0}^{n} (j+x)^{\alpha} \ln^p (j+x) \\
= & N - \lim_{n\to\infty} \frac{\partial^p}{\partial \alpha^p} \sum_{j=0}^{n} (j+x)^{\alpha} \\
= & \frac{\partial^p}{\partial \alpha^p} \left(N - \lim_{n\to\infty} \sum_{j=0}^{n} (j+x)^{\alpha} \right) \\
= & \frac{\partial^p}{\partial \alpha^p} \left(\frac{x^{\alpha}}{2} - \frac{x^{\alpha+1}}{\alpha+1} \right. \\
& \left. -2 \int_0^{\infty} \frac{\sin\left(\alpha \arctan \frac{y}{x}\right)}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)^{-\frac{\alpha}{2}}} dy \right).
\end{aligned} \tag{32}$$

So, it is easy to obtain (24) by using the Leibniz derivation rule. ■

Remark 2.5 By comparing Theorems 2.3 and 2.4 with (2), we find that the definition (3) is equivalent to the Hermite's integral of $\zeta(\alpha, x)$ for $\alpha \neq 1, 0, -1, \ldots$. Moreover, $\zeta(1, x)$ and $\zeta^{(p)}(1, x) (p \in \mathbb{N})$ can be given by (3), (6) and

(23) and expressed as follows,

$$\begin{aligned} &\zeta(1,x)\\ =\;& \frac{1}{2x}-\ln x+2\int_0^\infty \frac{y}{(e^{2\pi y}-1)(x^2+y^2)}dy,\\ &\zeta^{(p)}(1,x)\\ =\;& \frac{1}{2x}\ln^p\frac{1}{x}+\sum_{j=0}^{\left[\frac{p}{2}\right]}\frac{(-1)^{p-j}C_p^{2j}}{2^{p-2j-1}}\\ &\times\int_0^\infty \frac{y\left(\arctan\frac{y}{x}\right)^{2j}\ln^{p-2j}\left(x^2+y^2\right)}{(e^{2\pi y}-1)(x^2+y^2)}dy\\ &+\frac{1}{p+1}\ln^{p+1}\frac{1}{x}-x\sum_{j=0}^{\left[\frac{p-1}{2}\right]}\frac{(-1)^{p-j}C_p^{2j+1}}{2^{p-2j-2}}\\ &\times\int_0^\infty \frac{\left(\arctan\frac{y}{x}\right)^{2j+1}\ln^{p-2j-1}\left(x^2+y^2\right)}{(e^{2\pi y}-1)(x^2+y^2)}dy. \end{aligned} \tag{33}$$

According to the following relation,

$$\frac{d^p}{d\alpha^p}\zeta(\alpha)=\frac{\partial^p}{\partial\alpha^p}\zeta(\alpha,1),\quad \alpha\in\mathbb{C},\quad p\in\mathbb{N}_0, \tag{34}$$

we can redefine $\zeta(\alpha)$ as follows.

Remark 2.6 Let $p\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}\setminus\mathbb{N}_0^-$. Then

$$\zeta^{(p)}(\alpha)=(-1)^p\times N-\lim_{n\to\infty}\sum_{j=1}^{n} j^{-\alpha}\ln^p j. \tag{35}$$

III. Properties of Hurwitz zeta function

Theorem 3.1 Let $\alpha\in\mathbb{C}\setminus\{0\}$ and $x>0$. Then

$$\frac{\partial}{\partial x}\zeta(\alpha,x)=-\alpha\zeta(\alpha+1,x). \tag{36}$$

Proof. It is is well known that $\frac{\partial}{\partial x}\zeta(\alpha,x)=-\alpha\zeta(\alpha+1,x)(\alpha\neq 0,1)$ before we establish the definition of $\zeta(1,x)$. Therefore, we just need to prove (36) holds for $\alpha=1$. With the help of (33), (7) and (3), we obtain

$$\begin{aligned} &\frac{\partial}{\partial x}\zeta(1,x)\\ =\;& -\frac{1}{2x^2}-\frac{1}{x}-2\int_0^\infty \frac{2xy}{(e^{2\pi y}-1)(x^2+y^2)^2}dy\\ =\;& -\frac{1}{2x^2}-\frac{1}{x}-2\int_0^\infty \frac{\sin\left(2\arctan\frac{y}{x}\right)}{(e^{2\pi y}-1)(x^2+y^2)}dy\\ =\;& -N-\lim_{n\to\infty}\sum_{j=0}^{n}(j+x)^{-2}\\ =\;& -\zeta(2,x), \end{aligned} \tag{37}$$

which means that (36) holds for $\alpha=1$. ■

Lemma 3.2 (Lemma 3 in [22]) For $k\in\mathbb{N}$, there is

$$N-\lim_{\varepsilon\to 0}\frac{z^\varepsilon}{\varepsilon^k}=\frac{\ln^k z}{k!},\; z\in\mathbb{C},\; 0<|z|<1, \tag{38}$$

where N is the neutrix having domain $N'=\{\varepsilon:0<\varepsilon<\infty\}$and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$\varepsilon^\lambda\ln^{r-1}\varepsilon,\quad \ln^r\varepsilon\qquad(\lambda<0,\quad r\in\mathbb{N}) \tag{39}$$

and all functions which converge to zero in the usual sense as $\varepsilon\to 0$, see [13], [14], [15], [17], [18], [19], [20], [21], [22].

Theorem 3.3 Let $x>0$. Then

$$\zeta(1,x)=N-\lim_{\varepsilon\to 0}\zeta(1+\varepsilon,x) \tag{40}$$

Proof. Taking $\alpha=1+\varepsilon$ in (2) and using (38), we get

$$\begin{aligned} &N-\lim_{\varepsilon\to 0}\zeta(1+\varepsilon,x)\\ =\;& N-\lim_{\varepsilon\to 0}\left[\frac{1}{2x^{1+\varepsilon}}+\frac{\left(\frac{1}{x}\right)^\varepsilon}{\varepsilon}\right.\\ &\left.+2\int_0^\infty \frac{\sin\left[(1+\varepsilon)\arctan\frac{y}{x}\right]}{(e^{2\pi y}-1)(x^2+y^2)^{\frac{1+\varepsilon}{2}}}dy\right]\\ =\;& \frac{1}{2x}-\ln x+2\int_0^\infty \frac{\sin\left(\arctan\frac{y}{x}\right)}{(e^{2\pi y}-1)(x^2+y^2)^{\frac{1}{2}}}dy\\ =\;& \frac{1}{2x}-\ln x+2\int_0^\infty \frac{y}{(e^{2\pi y}-1)(x^2+y^2)}dy. \end{aligned} \tag{41}$$

Comparing (41) with (33), we conclude that (40) holds. ■

Theorem 3.4 Let $x>0$. Then

$$\zeta(1,x)=-\psi(x), \tag{42}$$

where $\psi(x)$ is the digamma function.

Proof. Using (40) and the Laurent expansion [9]

$$\zeta(1+\varepsilon,x)=\frac{1}{\varepsilon}-\psi(x)+O(\varepsilon), \tag{43}$$

we have

$$\begin{aligned} \zeta(1,x)&=N-\lim_{\varepsilon\to 0}\zeta(1+\varepsilon,x)\\ &=N-\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}-\psi(x)+O(\varepsilon)\right]\\ &=-\psi(x), \end{aligned} \tag{44}$$

so (42) holds. ■

Theorem 3.5 Let $p\in\mathbb{N}$ and $x>0$. Then

$$\zeta^{(p)}(1,x)=N-\lim_{\varepsilon\to 0}\zeta^{(p)}(1+\varepsilon,x). \tag{45}$$

Proof. Combining (3) with (24), we come to

$$\begin{aligned} &N-\lim_{\varepsilon\to 0}\zeta^{(p)}(1+\varepsilon,x)\\ =\;& (-1)^pN-\lim_{\varepsilon\to 0}\left[N-\lim_{n\to\infty}\sum_{k=0}^{n}(k+x)^{-(1+\varepsilon)}\right.\\ &\left.\times\ln^p(k+x)\right]\\ =\;& (-1)^pN-\lim_{\varepsilon\to 0}\left[p!\sum_{k=0}^{p}\frac{\ln^k x}{k!}\times\frac{\left(\frac{1}{x}\right)^\varepsilon}{\varepsilon^{p-k+1}}\right.\\ &+\frac{\ln^p x}{2x^{1+\varepsilon}}-\sum_{k=0}^{p}2^{1+k-p}C_p^k\\ &\times\int_0^\infty \frac{\ln^{p-k}\left(x^2+y^2\right)\left(\arctan\frac{y}{x}\right)^k}{(e^{2\pi y}-1)(x^2+y^2)^{\frac{1+\varepsilon}{2}}}\\ &\left.\times\sin\left(-(1+\varepsilon)\arctan\frac{y}{x}+\frac{k\pi}{2}\right)dy\right]. \end{aligned} \tag{46}$$

With the aid of (38), we deduce that

$$\begin{aligned} & N-\lim_{\varepsilon\to 0}\zeta^{(p)}(1+\varepsilon,x) \\ = & (-1)^p\Bigg[p!\sum_{k=0}^{p}\frac{\ln^k x}{k!}\times\frac{\ln^{p-k+1}\frac{1}{x}}{(p-k+1)!}+\frac{\ln^p x}{2x} \\ & -\sum_{k=0}^{p}2^{1+k-p}C_p^k\int_0^\infty\frac{\ln^{p-k}\left(x^2+y^2\right)}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)^{\frac{1}{2}}} \\ & \times\left(\arctan\frac{y}{x}\right)^k\sin\left(-\arctan\frac{y}{x}+\frac{k\pi}{2}\right)dy\Bigg] \\ = & \frac{1}{p+1}\ln^{p+1}\frac{1}{x}+\frac{1}{2x}\ln^p\frac{1}{x}-(-1)^p\sum_{k=0}^{p}C_p^k \\ & \times 2^{1+k-p}\int_0^\infty\frac{\ln^{p-k}\left(x^2+y^2\right)\left(\arctan\frac{y}{x}\right)^k}{\left(e^{2\pi y}-1\right)\left(x^2+y^2\right)^{\frac{1}{2}}} \\ & \times\sin\left(-\arctan\frac{y}{x}+\frac{k\pi}{2}\right)dy, \end{aligned} \tag{47}$$

where

$$\begin{aligned} & \sin\left(-\arctan\frac{y}{x}+\frac{k\pi}{2}\right)=\frac{(-1)^j}{\sqrt{x^2+y^2}} \\ & \times\begin{cases} -y, & k=2j\left(j=0,1,\ldots,\left[\frac{p}{2}\right]\right), \\ x, & k=2j+1\left(j=0,1,\ldots,\left[\frac{p-1}{2}\right]\right). \end{cases} \end{aligned} \tag{48}$$

Inserting (48) into (47), we obtain (45). ■

IV. APPLICATIONS

In this Section, we consider the closed forms of the certain integrals involving the Hurwitz zeta function, which can be expressed as a linear combination of the Riemann zeta functions and their derivatives. Specially, $\zeta^{(p)}(-u)$ and $\zeta^{(p)}(1)(p,u\in\mathbb{N}_0)$ will be used. According to the relation (34) and (2), we have

$$\zeta(\alpha)=\frac{1}{2}-\frac{1}{1-\alpha}+2\int_0^\infty\frac{\sin\left(\alpha\arctan y\right)}{\left(e^{2\pi y}-1\right)\left(1+y^2\right)^{\frac{\alpha}{2}}}dy, \tag{49}$$

where $\alpha\in\mathbb{C}\setminus\{1\}$. Therefore, $\zeta^{(p)}(-u)(p,u\in\mathbb{N}_0)$ can be given by (49) while $\zeta^{(p)}(1)(p\in\mathbb{N}_0)$ can be given by (33) after setting $x=1$. In particular, $\zeta(-u)(u\in\mathbb{N}_0)$ can be expressed as follows [6],

$$\zeta(-u)=\zeta(-u,1)=-\frac{B_{u+1}(1)}{u+1}, \tag{50}$$

where $B_m(q)$ are the Bernoulli polynomials expressed by

$$B_m(q)=\sum_{k=0}^{m}C_m^k B_k q^{m-k} \tag{51}$$

and B_k are the Bernoulli numbers.

A. The closed form of the integral $\int_0^1 x^m\zeta(\alpha,x+1)dx$

Theorem 4.1 Let $\alpha\in\mathbb{C}$. Then

$$\int_0^1\zeta(\alpha,x+1)dx=\begin{cases}\dfrac{1}{\alpha-1}, & \alpha\neq 1, \\ 0, & \alpha=1.\end{cases} \tag{52}$$

Proof. For $\alpha\neq 1,0,-1,\ldots$, we have

$$\begin{aligned} & \int_0^1\zeta(\alpha,x+1)dx \\ = & \int_0^1 N-\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(x+1+k)^\alpha}dx \\ = & \frac{1}{\alpha-1}N-\lim_{n\to\infty}\sum_{k=0}^{n}\Bigg[\frac{1}{(k+1)^{\alpha-1}} \\ & -\frac{1}{(k+2)^{\alpha-1}}\Bigg] \\ = & \frac{1}{\alpha-1}\left[\zeta(\alpha-1)-(\zeta(\alpha-1)-1)\right] \\ = & \frac{1}{\alpha-1}. \end{aligned} \tag{53}$$

Due to the continuity of $\zeta(\alpha,x)$ at $\alpha\neq 1$, we yield

$$\begin{aligned} \int_0^1\zeta(\alpha,x+1)dx & =\lim_{\varepsilon\to 0}\int_0^1\zeta(\alpha+\varepsilon,x+1)dx \\ & =\lim_{\varepsilon\to 0}\frac{1}{\alpha+\varepsilon-1} \\ & =\frac{1}{\alpha-1} \end{aligned} \tag{54}$$

for $\alpha=0,-1,-2,\ldots$.

Moreover,

$$\begin{aligned} & \int_0^1\zeta(1,x+1)dx \\ = & \int_0^1 N-\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{x+k+1}dx \\ = & N-\lim_{n\to\infty}\sum_{k=0}^{n}\left[\ln(k+2)-\ln(k+1)\right] \\ = & N-\lim_{n\to\infty}\left[\ln n+\ln(1+2/n)\right] \\ = & 0. \end{aligned} \tag{55}$$

Combining (53), (54) and (55), we obtain (52). ■

Theorem 4.2 Let $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$.

1) If $\alpha\neq 1,2,\ldots,m+1$, then

$$\begin{aligned} & \int_0^1 x^m\zeta(\alpha,x+1)dx \\ = & m!\sum_{u=1}^{m}\frac{(-1)^{u-1}\zeta(\alpha-u)}{(m-u+1)!\,(1-\alpha)_u}-\frac{1}{m-\alpha+1}. \end{aligned} \tag{56}$$

2)

$$\begin{aligned} & \int_0^1 x^m\zeta(m+1,x+1)dx \\ = & \sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\sum_{u=1}^{i}(-1)^u C_i^u\zeta(u)+H_m, \end{aligned} \tag{57}$$

where $H_m=\sum_{u=1}^{m}\frac{1}{u}$.

Proof. 1) (I). When $m = 1$, (56) reduces to

$$\begin{aligned}&\int_0^1 x\zeta(\alpha, x+1)dx\\ =\ & \frac{1}{1-\alpha}\int_0^1 xd\zeta(\alpha-1, x+1)\\ =\ & \frac{1}{1-\alpha}\left[\zeta(\alpha-1,2) - \int_0^1 \zeta(\alpha-1,x+1)dx\right]\\ =\ & \frac{1}{1-\alpha}\left[\zeta(\alpha-1) - 1 - \frac{1}{\alpha-2}\right]\\ =\ & \frac{\zeta(\alpha-1)}{1-\alpha} - \frac{1}{2-\alpha},\end{aligned} \tag{58}$$

which means that (56) holds for $m = 1$.

(II). Now assume that (56) holds for $m = k$, *i.e.*,

$$\begin{aligned}&\int_0^1 x^k\zeta(\alpha, x+1)dx\\ =\ & k!\sum_{v=1}^{k}\frac{(-1)^{v-1}\zeta(\alpha-v)}{(k-v+1)!\,(1-\alpha)_v} - \frac{1}{k-\alpha+1}.\end{aligned} \tag{59}$$

Integrating by parts and using (59), we obtain

$$\begin{aligned}&\int_0^1 x^{k+1}\zeta(\alpha, x+1)dx\\ =\ & \frac{1}{1-\alpha}\int_0^1 x^{k+1}d\zeta(\alpha-1, x+1)\\ =\ & \frac{\zeta(\alpha-1,2)}{1-\alpha} - \frac{k+1}{1-\alpha}\int_0^1 x^k\zeta(\alpha-1,x+1)dx\\ =\ & \frac{\zeta(\alpha-1)-1}{1-\alpha} - \frac{k+1}{1-\alpha}\times\\ & \left[k!\sum_{v=1}^{k}\frac{(-1)^{v-1}\zeta(\alpha-1-v)}{(k-v+1)!\,(2-\alpha)_v} - \frac{1}{k-\alpha+2}\right]\\ =\ & \frac{\zeta(\alpha-1)}{1-\alpha} + (k+1)!\sum_{v=1}^{k}\frac{(-1)^v}{(1-\alpha)_{v+1}}\\ & \times\frac{\zeta(\alpha-1-v)}{(1-\alpha)_{v+1}} - \frac{1}{1-\alpha}\left[1 - \frac{k+1}{k-\alpha+2}\right]\\ =\ & (k+1)!\sum_{u=1}^{k+1}\frac{(-1)^{u-1}\zeta(\alpha-u)}{(k+1-u+1)!\,(1-\alpha)_u}\\ & -\frac{1}{k-\alpha+2},\end{aligned} \tag{60}$$

which means that (56) holds for $m = k+1$. According to the mathematical induction, we conclude that (56) holds.

2) Using (3), we obtain

$$\begin{aligned}&\int_0^1 x^m\zeta(m+1, x+1)dx\\ =\ & N - \lim_{n\to\infty}\sum_{k=1}^{n+1}\int_0^1\frac{x^m}{(x+k)^{m+1}}dx\\ =\ & N - \lim_{n\to\infty}\sum_{k=1}^{n+1}\sum_{i=0}^{m}C_m^i(-1)^i k^i\\ & \times\int_0^1 (x+k)^{-i-1}dx\\ =\ & N - \lim_{n\to\infty}\sum_{k=1}^{n+1}\left(\sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\right.\\ & \left.\times\left[k^i(1+k)^{-i} - 1\right] + \ln(k+1) - \ln k\right),\end{aligned} \tag{61}$$

With the help of the Binomial theorem, we get

$$\begin{aligned}&\int_0^1 x^m\zeta(m+1, x+1)dx\\ =\ & N - \lim_{n\to\infty}\sum_{k=1}^{n+1}\left(\sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\right.\\ & \left.\times\sum_{u=1}^{i}C_i^u(-1)^u\frac{1}{(k+1)^u} + \ln(k+1) - \ln k\right)\\ =\ & N - \lim_{n\to\infty}\left(\sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\sum_{u=1}^{i}C_i^u(-1)^u\right.\\ & \left.\times\sum_{k=1}^{n+1}\frac{1}{(k+1)^u} + \ln(n+1)\right).\end{aligned} \tag{62}$$

Using (35), we have

$$\begin{aligned}&\int_0^1 x^m\zeta(m+1, x+1)dx\\ =\ & \sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\sum_{u=1}^{i}C_i^u(-1)^u\left[\zeta(u) - 1\right]\\ =\ & \sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\sum_{u=1}^{i}C_i^u(-1)^u\zeta(u)\\ & +\sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\\ =\ & \sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i}\sum_{u=1}^{i}(-1)^u C_i^u\zeta(u) + H_m,\end{aligned} \tag{63}$$

where

$$\begin{aligned}\sum_{i=1}^{m}C_m^i\frac{(-1)^{i-1}}{i} &= \sum_{i=1}^{m}C_m^i(-1)^{i-1}\int_0^1 x^{i-1}dx\\ &= \int_0^1\frac{1-(1-x)^m}{x}dx\\ &= \int_0^1\frac{1-t^m}{1-t}dt\\ &= \int_0^1\sum_{j=0}^{m-1}t^j dt\\ &= H_m,\end{aligned} \tag{64}$$

so (57) holds. ■

Theorem 4.3 Let $m \in \mathbb{N}$. Then

$$\begin{aligned}&\int_0^1 x^m\zeta(1, x+1)dx\\ =\ & -\sum_{v=1}^{m-1}(-1)^v C_m^v H_v\zeta(-v)\\ & -\sum_{v=0}^{m-1}(-1)^v C_m^v\zeta'(-v) - \frac{1}{m},\end{aligned} \tag{65}$$

where an empty sum for $m = 1$ is understood to be nil.

Proof. Using the Taylor expansion, we get

$$\begin{aligned}
&\frac{1}{(-\varepsilon)_u} \\
=\ & \frac{1}{-\varepsilon(u-1)!}\times\prod_{k=1}^{u-1}\frac{1}{1-\varepsilon/k} \\
=\ & \frac{1}{-\varepsilon(u-1)!}\times\prod_{k=1}^{u-1}\left[1+\frac{\varepsilon}{k}+O(\varepsilon^2)\right] \\
=\ & \frac{1}{-\varepsilon(u-1)!} \\
& \times\left[1+\left(1+\frac{1}{2}+\cdots+\frac{1}{u-1}\right)\varepsilon+O(\varepsilon^2)\right] \\
=\ & -\frac{1}{(u-1)!}\times\left[\frac{1}{\varepsilon}+H_{u-1}+O(\varepsilon)\right],
\end{aligned} \tag{66}$$

where $u=2,3,\ldots$. Hence,

$$\begin{aligned}
&\frac{\zeta(1+\varepsilon-u)}{(-\varepsilon)_1} \\
=\ & -\frac{\zeta(1+\varepsilon-u)-\zeta(1-u)}{\varepsilon}-\frac{\zeta(1-u)}{\varepsilon},
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
&\frac{\zeta(1+\varepsilon-u)}{(-\varepsilon)_u} \\
=\ & -\frac{\zeta(1+\varepsilon-u)-\zeta(1-u)}{\varepsilon}\times\frac{1}{(1-\varepsilon)_{u-1}} \\
& +\frac{\zeta(1-u)}{(-\varepsilon)_u} \\
=\ & -\frac{\zeta(1+\varepsilon-u)-\zeta(1-u)}{\varepsilon}\times\frac{1}{(1-\varepsilon)_{u-1}} \\
& -\frac{\zeta(1-u)}{(u-1)!}\times\left[\frac{1}{\varepsilon}+H_{u-1}+O(\varepsilon)\right],
\end{aligned} \tag{68}$$

where $u=2,3,\ldots$.

Combining (40), (56), (67) and (68), we obtain

$$\begin{aligned}
&\int_0^1 x^m\zeta(1,x+1)dx \\
=\ & N-\lim_{\varepsilon\to 0}\int_0^1 x^m\zeta(1+\varepsilon,x+1)dx \\
=\ & N-\lim_{\varepsilon\to 0}\left[m!\sum_{u=1}^{m}\frac{(-1)^{u-1}}{(m-u+1)!}\right. \\
& \left.\times\frac{\zeta(1+\varepsilon-u)}{(-\varepsilon)_u}-\frac{1}{m-\varepsilon}\right] \\
=\ & m!\sum_{u=1}^{m}\frac{(-1)^u}{(m-u+1)!}\times\frac{\zeta'(1-u)}{(u-1)!} \\
& +m!\sum_{u=2}^{m}\frac{(-1)^u H_{u-1}}{(m-u+1)!}\times\frac{\zeta(1-u)}{(u-1)!}-\frac{1}{m} \\
=\ & \sum_{u=1}^{m}(-1)^u C_m^{u-1}\zeta'(1-u) \\
& +\sum_{u=2}^{m}(-1)^u C_m^{u-1}H_{u-1}\zeta(1-u)-\frac{1}{m} \\
=\ & -\sum_{v=0}^{m-1}(-1)^v C_m^v\zeta'(-v) \\
& -\sum_{v=1}^{m-1}(-1)^v C_m^v H_v\zeta(-v)-\frac{1}{m}
\end{aligned} \tag{69}$$

for $m\in\mathbb{N}$. ■

Theorem 4.4 Let $n,m\in\mathbb{N}$ and $n\le m$. Then

$$\begin{aligned}
&\int_0^1 x^m\zeta(n,x+1)dx \\
=\ & m!\sum_{u=1}^{n-1}\frac{(-1)^{u-1}\left[\zeta(n-u)-1\right]}{(m+1-u)!(1-n)_u} \\
& -C_m^{n-1}\left[\sum_{u=1}^{m-n}(-1)^u C_{m-n+1}^u H_u\zeta(-u)\right. \\
& \left.+\sum_{u=0}^{m-n}(-1)^u C_{m-n+1}^u\zeta'(-u)+\frac{1}{m-n+1}\right],
\end{aligned} \tag{70}$$

where an empty sum for $n=1$ is understood to be nil.

Proof. Using (36) and integrating by parts, we have

$$\begin{aligned}
&\int_0^1 x^m\zeta(n,x+1)dx \\
=\ & -\frac{1}{n-1}\int_0^1 x^m d\zeta(n-1,x+1) \\
=\ & -\frac{1}{n-1}\left[\zeta(n-1,2)\right. \\
& \left.-m\int_0^1 x^{m-1}\zeta(n-1,x+1)dx\right] \\
=\ & -\frac{1}{n-1}\left[\zeta(n-1)-1\right] \\
& +\frac{m}{n-1}\int_0^1 x^{m-1}\zeta(n-1,x+1)dx \\
=\ & \cdots \\
=\ & m!\sum_{u=1}^{n-1}\frac{(-1)^{u-1}}{(m+1-u)!(1-n)_u}\left[\zeta(n-u)-1\right] \\
& +C_m^{n-1}\int_0^1 x^{m-n+1}\zeta(1,x+1)dx.
\end{aligned} \tag{71}$$

Combining (65) and (71), we yield (70). ■

From the identity $\zeta(\alpha,x)=\zeta(\alpha,x+1)+\frac{1}{x^\alpha}$, we have

$$\begin{aligned}
&\int_0^1 x^m\zeta(\alpha,x)dx \\
=\ & \int_0^1 x^m\zeta(\alpha,x+1)dx+\int_0^1 x^{m-\alpha}dx \\
=\ & \int_0^1 x^m\zeta(\alpha,x+1)dx+\frac{1}{m-\alpha+1}.
\end{aligned} \tag{72}$$

Combining (56), (70) with (72), we give the following Corollary.

Corollary 4.5 Let $m\in\mathbb{N}$. Then

1) If $\Re(\alpha)<m+1$ and $\alpha\neq 1,2,\ldots,m$, then

$$\int_0^1 x^m\zeta(\alpha,x)dx=m!\sum_{u=1}^{m}\frac{(-1)^{u-1}\zeta(\alpha-u)}{(m+1-u)!\,(1-\alpha)_u}. \tag{73}$$

2) If $n \in \mathbb{N}$ and $n \le m$, then

$$\begin{aligned}&\int_0^1 x^m \zeta(n,x)dx \\ =\ & m! \sum_{u=1}^{n-1} \frac{(-1)^{u-1}\left[\zeta(n-u)-1\right]}{(m+1-u)!(1-n)_u} + \frac{1}{m-n+1} \\ &\times \left[1 - C_m^{n-1}\right] - C_m^{n-1}\left[\sum_{u=1}^{m-n}(-1)^u C_{m-n+1}^u \right. \\ &\left. \times H_u \zeta(-u) + \sum_{u=0}^{m-n}(-1)^u C_{m-n+1}^u \zeta'(-u)\right],\end{aligned} \tag{74}$$

where an empty sum for $n = 1$ is understood to be nil. It's worth nothing that (73) and (74) have been given by Example 12.3 and Theorem 12.4 in [5], respectively.

B. The closed form of the integral $\int_0^1 x^m \zeta^{(p)}(\alpha, x+1)dx$

Lemma 4.6 Let $m \in \mathbb{N}$ and $\alpha \ne 1, 2, \ldots, m+1$. Then

$$D_{\alpha,m} := \sum_{i=0}^{m} C_m^i \frac{(-1)^i}{i-\alpha+1} = \frac{m!}{(1-\alpha)_{m+1}}. \tag{75}$$

Proof. (I). When $m = 1$, (75) reduces to

$$\sum_{i=0}^{1} C_1^i \frac{(-1)^i}{i-\alpha+1} = \frac{1}{1-\alpha} - \frac{1}{2-\alpha} = \frac{1}{(1-\alpha)_2}, \tag{76}$$

which means that (75) holds for $m = 1$.

(II). Now assume that (75) holds for $m = k$, *i.e.*,

$$\sum_{i=0}^{k} C_k^i \frac{(-1)^i}{i-\alpha+1} = \frac{k!}{(1-\alpha)_{k+1}}. \tag{77}$$

Using (77), we have

$$\begin{aligned}&\sum_{i=0}^{k+1} C_{k+1}^i \frac{(-1)^i}{i-\alpha+1} \\ =\ & \sum_{i=1}^{k}\left(C_k^i + C_k^{i-1}\right)\frac{(-1)^i}{i-\alpha+1} \\ &+ \frac{1}{1-\alpha} + \frac{(-1)^{k+1}}{k+2-\alpha} \\ =\ & \sum_{i=1}^{k} C_k^i \frac{(-1)^i}{i-\alpha+1} + \sum_{j=0}^{k-1} C_k^j \frac{(-1)^{j+1}}{j-\alpha+2} \\ &+ \frac{1}{1-\alpha} + \frac{(-1)^{k+1}}{k+2-\alpha} \\ =\ & \sum_{i=0}^{k} C_k^i \frac{(-1)^i}{i-\alpha+1} - \sum_{j=0}^{k} C_k^j \frac{(-1)^j}{j-(\alpha-1)+1} \\ =\ & \frac{k!}{(1-\alpha)_{k+1}} - \frac{k!}{(2-\alpha)_{k+1}} \\ =\ & \frac{(k+1)!}{(1-\alpha)_{k+2}},\end{aligned} \tag{78}$$

which means that (75) holds for $m = k+1$. According to the mathematical induction, we conclude that (75) holds. ■

Theorem 4.7 Let $m, p \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. If $\alpha \ne 1, 2, \ldots, m+1$, then

$$\begin{aligned}&\int_0^1 x^m \zeta^{(p)}(\alpha, x+1)dx \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v D_{\alpha,u}^{(v)} \zeta^{(p-v)}(\alpha-u-1) \\ & - \frac{p!}{(m-\alpha+1)^{p+1}},\end{aligned} \tag{79}$$

where

$$D_{\alpha,u}^{(v)} = v! \sum_{i=0}^{u} C_u^i \frac{(-1)^i}{(i-\alpha+1)^{v+1}}. \tag{80}$$

Proof. Using (56) and (75), we obtain

$$\begin{aligned}&\int_0^1 x^m \zeta(\alpha, x+1)dx \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u \frac{u!}{(1-\alpha)_{u+1}} \zeta(\alpha-u-1) \\ & - \frac{1}{m-\alpha+1} \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u D_{\alpha,u} \zeta(\alpha-u-1) \\ & - \frac{1}{m-\alpha+1}.\end{aligned} \tag{81}$$

Calculating p-order partial derivatives on α for (81) by using the Leibniz's rule, we can obtain

$$\begin{aligned}&\int_0^1 x^m \zeta^{(p)}(\alpha, x+1)dx + \frac{p!}{(m-\alpha+1)^{p+1}} \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u \frac{\partial^p}{\partial \alpha^p}\left[D_{\alpha,u}\zeta(\alpha-u-1)\right] \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v \frac{\partial^v}{\partial \alpha^v} D_{\alpha,u} \\ & \times \zeta^{(p-v)}(\alpha-u-1) \\ =\ & \sum_{u=0}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v D_{\alpha,u}^{(v)} \zeta^{(p-v)}(\alpha-u-1)\end{aligned} \tag{82}$$

where $D_{\alpha,u}^{(v)}$ is given by (80). Thus, (79) is obtained. ■

Theorem 4.8 Let $m, p \in \mathbb{N}$. Then

$$\begin{aligned}&\int_0^1 x^m \zeta^{(p)}(1, x+1)dx \\ =\ & \sum_{u=1}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v \\ & \times \left[v! \sum_{i=1}^{u} C_u^i \frac{(-1)^i}{i^{v+1}} \zeta^{(p-v)}(-u)\right] \\ & - \frac{1}{p+1}\sum_{u=0}^{m-1}(-1)^u C_m^u \zeta^{(p+1)}(-u) - \frac{p!}{m^{p+1}}.\end{aligned} \tag{83}$$

Proof. Using (80) and the Taylor expansion, we obtain

$$\begin{aligned} & D^{(v)}_{1+\varepsilon,u}\zeta^{(p-v)}(\varepsilon-u) \\ = \quad & v!\sum_{i=0}^{u} C_u^i \frac{(-1)^i}{(i-\varepsilon)^{v+1}}\zeta^{(p-v)}(\varepsilon-u) \\ = \quad & v!\sum_{i=1}^{u} C_u^i \frac{(-1)^i}{(i-\varepsilon)^{v+1}}\zeta^{(p-v)}(\varepsilon-u) \\ & +v!\frac{(-1)^{v+1}}{\varepsilon^{v+1}}\zeta^{(p-v)}(\varepsilon-u) \\ = \quad & v!\sum_{i=1}^{u} C_u^i \frac{(-1)^i}{(i-\varepsilon)^{v+1}}\zeta^{(p-v)}(\varepsilon-u) \\ & +v!(-1)^{v+1}\left[\sum_{k=0}^{v}\frac{\zeta^{(p-v+k)}(-u)}{k!}\varepsilon^{k-v-1} \right. \\ & \left. +\frac{\zeta^{(p+1)}(-u)}{(v+1)!}+O(\varepsilon)\right]. \end{aligned} \tag{84}$$

Using (45) and (84), we have

$$\begin{aligned} & \int_0^1 x^m \zeta^{(p)}(1,x+1)dx \\ = \quad & N-\lim_{\varepsilon\to 0}\int_0^1 x^m \zeta^{(p)}(1+\varepsilon,x+1)dx \\ = \quad & N-\lim_{\varepsilon\to 0}\left[\sum_{u=0}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v \right. \\ & \left. \times D^{(v)}_{1+\varepsilon,u}\zeta^{(p-v)}(\varepsilon-u)-\frac{p!}{(m-\varepsilon)^{p+1}}\right], \\ = \quad & \sum_{u=1}^{m-1}(-1)^u C_m^u \sum_{v=0}^{p} C_p^v \left[v!\sum_{i=1}^{u} C_u^i \frac{(-1)^i}{i^{v+1}} \right. \\ & \left. \times\zeta^{(p-v)}(-u)\right]+\sum_{u=0}^{m-1}(-1)^u C_m^u \zeta^{(p+1)}(-u) \\ & \times\left[\sum_{v=0}^{p} C_p^v \frac{(-1)^{v+1}}{v+1}\right]-\frac{p!}{m^{p+1}}. \end{aligned} \tag{85}$$

Combining (75) with (85), we get (83). ■

V. Conclusion

In this work, the neutrix limit and the Abel-Plana formula are used to define $\zeta^{(p)}(1,x)(p\in\mathbb{N}_0)$. This definition and the Hermite's integral of $\zeta(\alpha,x)$ for $\alpha\neq 1$ can extend the Hurwitz zeta function to the whole complex plane. Moreover, we use the Laurent expansion and the neutrix limit to prove that $\zeta(1,x)$ is the inverse number of the digamma function $\psi(x)$.

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