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Solvability of Quasilinear Euler-Lagrange Equations

Quincy Stevene Nkombo *

Abstract—**In this paper we deal with the solvability of quasilinear Euler-Lagrange equation**

$$-\mathbf{div}((a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{p-2}\nabla u)+\frac{\gamma}{p}\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{p}$$

$$=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad\textbf{in}\quad\Omega$$

with zero Dirichlet boundary condition, under the assumption $1<\theta<p<q<\frac{p*}{p}(\gamma+p)$ and $\gamma>1$. We concern with the existence of multiplicity solutions for the above equation in employing the critical point methods. Moreover, we obtain the trivial solution of such equation when Ω is a smooth star-shaped domain in R^N.

Keywords: Euler-Lagrange equation; Weak solution; Truncated function; Nonsmooth critical point theory

1 Introduction

In this paper we study the following equation.

$$-\mathrm{div}((a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{p-2}\nabla u)+\frac{\gamma}{p}\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{p}$$

$$=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad\text{in}\quad\Omega \tag{1.1}$$

with zero Dirichlet boundary condition.

$$u=0\quad\text{on}\quad\partial\Omega. \tag{1.2}$$

In this case, the corresponding functional to the quasilinear Euler-Lagrange equation J is

$$J(u)=\frac{1}{p}\int_{\Omega}(a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{p}-\frac{\lambda}{\theta}\int_{\Omega}\mid u\mid^{\theta}-\frac{1}{q}\int_{\Omega}\mid u\mid^{q} \tag{1.3}$$

where $\gamma>1$, Ω is a bounded, open subset of R^N with $N>2$, $1<p<N$ and a(x) is a measurable function such that for two constants α and β

$$0<\alpha\leq a(x)\leq\beta\quad\text{a.e}\quad x\in\Omega. \tag{1.4}$$

We notice that the functional J is not Gâteau differentiable in $W_0^1(\Omega)$ but is only differentiable through the direction of $W_0^{1,p}(\Omega)\cap Ł^{\infty}(\Omega)$.

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The main difficulty of this work is due to the term $\mid u\mid^{\gamma}$ in which the functional J is well defined in $W_0^{1,p}(\Omega)\cap Ł^{\infty}(\Omega)$, if we impose an additional condition on γ, namely, $\gamma+p<p^*$. We point out that our approach has been studied in [1], including $L^{\infty}(\Omega)$ a priori estimates. We apply the Theorem 2.8 in [3] to establish the existence of multiplicity critical points under hypotheses $0<\lambda<\widetilde{\lambda}_0$ and $1<\theta<p<q<\frac{p*}{p}(\gamma+p)$ for $\gamma>1$ in which such critical points $u_{\overline{m},\overline{n}}$ of $J_{\overline{m},\overline{n}}$ for $\overline{m},\overline{n}$ large enough are solutions of (1.1)-(1.2) without passing to the limit on m and n. Moreover, no such solution when $\lambda>\widetilde{\lambda}_0$ and Ω is star-shaped for $1<\theta<p<q<\frac{p*}{p}(\gamma+p)$ and $\gamma>1$. We notice that the multiplicity results for p-Laplacian with critical growth of concave-convex functions has been intensively studied. Recently, the existence of multiplicity of bounded weak solutions for the quasilinear singular Euler-Lagrange equation with natural growth with $p=N$ has been investigated by Quincy Stevene Nkombo (see [10]). Finally, the novelty of this paper is that we study the existence of multiplicity bounded weak solutions for quasilinear Euler-Lagrange equation with $1<p<N$.

Notation: in the rest of this work we make use of the following notation. $L^p(\Omega)$, $1\leq p\leq\infty$, denote lebesgue spaces. The usual norm in $L^p(\Omega)$ is denoted by $|\ \ |_p$. $W_0^{k,p}(\Omega)$ denote sobolev spaces ; the norm in $W_0^{1,p}(\Omega)$ is denoted by $\|\ \ \|_p$.
$C_0,C_1,C_2,C_3,....$ denote (possibly different) positive constants.

2. The case $0<\lambda<\widetilde{\lambda}_0$
Definition 2.1 A measurable function u is called a weak solution to the equation (1.1)-(1.2), if $u\in W_0^{1,p}(\Omega)$ such that $\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{p}\in L^1(\Omega)$ and

$$\int_{\Omega}(a(x)+\mid u\mid^{\gamma})\mid\nabla u\mid^{p-2}\nabla u\nabla v$$

$$+\frac{\gamma}{p}\int_{\Omega}\mid u\mid^{\gamma-2}u\mid\nabla u\mid^{p}v$$

$$=\lambda\int_{\Omega}\mid u\mid^{\theta-2}uv+\int_{\Omega}\mid u\mid^{q-2}uv$$

holds for every $v\in W_0^{1,p}(\Omega)\cap Ł^{\infty}(\Omega)$.
The main result of this paper is focus on the existence of multiplicity bounded weak solutions to the equation (1.1)-(1.2). For that the result is given by the following

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theorem.

Theorem 2.1 Suppose that γ satisfies the condition $\gamma + p < p^*$. Moreover, there exists $\widetilde{\lambda}_0$ such that

$$1 < \theta < p < q < \frac{p*}{p}(\gamma + p); \quad 0 < \lambda < \widetilde{\lambda}_0. \qquad (2.2)$$

Then, there exist infinitely many weak solutions for the problem (1.1)-(1.2).

Proof. We use the theorem 2.8 in [3] in order to prove the existence of multiplicity weak solutions to the problem (1.1)-(1.2). So that we divide this proof into several steps.

• **Step 1**: A truncated function

If m is positive integer, we consider the truncated function at level m, $T_m(t)$ is given

$$T_m(t) = \begin{cases} -m - \frac{1}{2} \;\; if \;\; t \leq -m-1 \\ (m+1)t + \frac{t^2+m^2}{2} \;\; if \;\; -m-1 \leq t \leq -m \\ t \;\; if \;\; -m \leq t \leq m \\ (m+1)t - \frac{t^2+m^2}{2} \;\; if \;\; m \leq t \leq m+1 \\ m + \frac{1}{2} \;\; if \;\; t \geq m+1 \end{cases} \qquad (2.3)$$

which is introduced in [1].

Assuming that q_0 and q_1 are two numbers such that $1 < q_0 < \theta < p < q_1 < q$ and the truncated function $f_{n,\lambda}(t)$ is defined by

$$f_{n,\lambda}(t) = \lambda h_n(t) + g_n(t),$$

where

$$h_n(t) = \begin{cases} \frac{|t|^\theta}{\theta} \;\; if \;\; |\, t \,| < n, \\ n^\theta\left(\frac{1}{\theta} - \frac{1}{q_0}\right) + n^{\theta - q_0}\frac{|t|^{q_0}}{q_0} \;\; if \;\; |\, t \,| \geq n. \end{cases} \qquad (2.4)$$

$$g_n(t) = \begin{cases} \frac{|t|^q}{q} \;\; if \;\; |\, t \,| < n \\ n^q\left(\frac{1}{q} - \frac{1}{q_1}\right) + n^{q - q_1}\frac{|t|^{q_1}}{q_1} \;\; if \;\; |\, t \,| \geq n. \end{cases} \qquad (2.5)$$

By observing the definition of $h_n(t)$ and $g_n(t)$, we deduce the following inequalities

$$0 \leq h_n(t) \leq \frac{n^{\theta - q_0}}{q_0} |\, t \,|^{q_0} \quad \text{and} \quad 0 \leq h_n(t) \leq \frac{|\, t \,|^\theta}{\theta}. \qquad (2.6)$$

$$0 \leq g_n(t) \leq \frac{n^{q - q_1}}{q_1} |\, t \,|^{q_1} \quad \text{and} \quad 0 \leq g_n(t) \leq \frac{|\, t \,|^q}{q}. \qquad (2.7)$$

Consequently, we infer that the estimate of the function $f_{n,\lambda}(t)$ as follows

$$0 \leq f_{n,\lambda}(t) \leq \frac{\lambda n^{\theta - q_0}}{q_0} \,|\, t \,|^{q_0} + \frac{n^{q - q_1}}{q_1} \,|\, t \,|^{q_1}. \qquad (2.8)$$

Let us consider the truncated functional,

$$J_{m,n}(u) = \frac{1}{p}\int_\Omega (a(x) + |\, T_m(u) \,|^\gamma) \,|\, \nabla u \,|^p - \int_\Omega f_{n,\lambda}(u) \qquad (2.9)$$

for $\;\; u \in W_0^{1,p}(\Omega)$.

Which is clearly well defined for $q_0 < q_1 < p^*$.

• **Step 2**: Geometry of truncated function

Let r a positive real constant such that

$$B_r = \{u \in W_0^{1,p}(\Omega) / \parallel u \parallel_p \leq r\}.$$

The fact that, $a(x) + |\, T_m(u) \,|^\gamma \geq \alpha$ and integrating inequality (2.8) on Ω, we get

$$\int_\Omega f_{n,\lambda}(u) \leq \lambda C_0 n^{\theta - q_0} \parallel u \parallel_p^{q_0} + C_1 n^{q - q_1} \parallel u \parallel_p^{q_1} \quad (2.10)$$

where C_0 and C_1 are nonnegative constants.

Combining (2.10) with the hypothesis, $a(x) + |\, T_m(u) \,|^\gamma \geq \alpha$, we obtain the following result

$$J_{m,n}(u) \geq \frac{\alpha}{p} \parallel u \parallel_p^p - \lambda C_0 n^{\theta - q_0} \parallel u \parallel_p^{q_0} - C_1 n^{q - q_1} \parallel u \parallel_p^{q_1}.$$

Thereby, there exist nonnegative constants $r_{n,\lambda}$, $\overline{r}_{n,\lambda}$ and $\widetilde{\lambda}_0$ such that

$$J_{m,n}(u) > 0 \quad \text{in} \quad B_{r_{n,\lambda}} \quad \text{and} \quad J_{m,n}(u) \geq \overline{r}_{n,\lambda} \quad \text{in} \quad \partial B_{r_{n,\lambda}}$$

for all $0 < \lambda < \widetilde{\lambda}_0$.

• **Step 3**: Compactness of the truncated function.

Let $\{w_k\}$ be a sequence in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying, for every $n \in N$ the following conditions

$$J_{m,n}(w_k) \leq C_1.$$

$$|\, w_k \,|_\infty \leq 2b_k.$$

$$\langle J'_{m,n}(w_k), w \rangle \leq \varepsilon_k\left(\frac{|\, w \,|_\infty}{b_k} + \parallel w \parallel_p\right).$$

$\forall w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Where C_1 is a nonnegative constant, $\{b_k\} \subset R^+ - \{0\}$ is a nonnegative sequence and $\{\varepsilon_k\} \subset R^+ - \{0\}$ is a sequence converging to zero.

Suppose that

$$\widetilde{g}(\lambda, t) = \frac{\frac{1}{p+\gamma} + f_{n,\lambda}(t)}{t f'_{n,\lambda}(t)}.$$

And

$$g_0(\lambda) = \max_{t \in R} \widetilde{g}(\lambda, t).$$

Where $\lambda > 0$ and $t > 0$.

Let $\varepsilon > 0$, be given and choose $t_0 > 0$ such that

$$\max_{t \in R} \widetilde{g}(\lambda, t) \leq \widetilde{g}(\lambda, t_0) + \varepsilon.$$

Clearly $\widetilde{g}(\lambda, t_0)$ is an increasing and continuous function with respect to λ and there exists $\widetilde{\lambda}_0$ a nonnegative number $0 < \widetilde{\lambda}_0 < \infty$ and such that

$$\widetilde{g}(\widetilde{\lambda}_0, t_0) \leq \frac{1}{p+\gamma}, \quad \text{for all} \quad 0 < \lambda < \widetilde{\lambda}_0.$$

Therefore

$$\max_{t\in R}\widetilde{g}(\lambda,t)\leq\widetilde{g}(\lambda,t_0)+\varepsilon\leq\widetilde{g}(\widetilde{\lambda}_0,t_0)+\varepsilon.$$

Which leads to

$$\max_{t\in R}\widetilde{g}(\lambda,t)\leq\frac{1}{p+\gamma}+\varepsilon,\quad\text{for all}\quad 0<\lambda<\widetilde{\lambda}_0.$$

Then it is easily verified by induction that

$$\frac{f_{n,\lambda}(t)}{tf'_{n,\lambda}(t)}<g_0(\lambda)<\frac{1}{p+\gamma},\quad\text{for all}\quad 0<\lambda<\widetilde{\lambda}_0.$$

After straightforward calculation of the term

$$J_{m,n}(w_k)-g_0(\lambda)\langle J'_{m,n}(w_k),w_k\rangle$$

yields

$$\left(\frac{1}{p}-g_0(\lambda)\right)\int_\Omega a(x)\mid\nabla w_k\mid^p$$
$$+\int_\Omega\left(\frac{1}{p}-g_0(\lambda)-\frac{\gamma}{p}g_0(\lambda)w_k\frac{T'_m(w_k)}{T_m(w_k)}\right)$$
$$\times\mid T_{m)}(w_k)\mid^\gamma\mid\nabla w_k\mid^p$$
$$+\int_\Omega\left(g_0(\lambda)w_kf'_{n,\lambda}(w_k)-f_{n,\lambda}(w_k)\right)$$
$$\leq C_1+\varepsilon_k\left(\frac{\mid w_k\mid_\infty}{b_k}+\parallel w_k\parallel_p\right).$$

Notice that all the left-hand side terms are positives. Indeed, the first one is nonnegative due to consequence of the definition of $g_0(\lambda)$, namely, $\frac{f_{n,\lambda}(t)}{tf'_{n,\lambda}(t)}<g_0(\lambda)<\frac{1}{p+\gamma}$. For the second term, it is enough to use the assumption, $0\leq\frac{tT'_m(t)}{T_m(t)}\leq 1$, and that $g_0(\lambda)<\frac{1}{p+\gamma}$. The positiveness of the third term is easily verified in using the definition of $g_0(\lambda)$. Therefore, we can conclude that the sequence $\{w_k\}$ is bounded in $W_0^{1,p}(\Omega)$ for every p such that $1<p<N$. So that the sequence $\{w_k\}$ admit a subsequence that we still denote $\{w_k\}$, which converges to a function w □

• **Step 4**: Existence of critical points of the truncated function.

We point out that the main idea of this proof is in [4], for that we adapt the arguments of Theorem 2.8 in [3] in order to prove the existence of multiplicity critical points of $J_{m,n}$.

Let H_k be a k-dimensional subspace of $W_0^{1,p}(\Omega)$ as we take $w_k\in H_k$, the norm of w_k, $\parallel w_k\parallel_p$ is finite.

We set

$$\Sigma=\{C\subset W_0^{1,p}(\Omega)/\ \ 0\overline{\in}C\ \ ,C=-C\}.$$

For $C\in\Sigma$ the Z_2-genus of C is denoted by $\gamma(C)$. According to the step 2 and step 3, the assumptions (I_1) and (I_3) of Theorem 2.3 hold true (see [3]).

Moreover, letting

$$A_{m,n}=B_{r_{n,\lambda}}\cup\{J_{m,n}\geq 0\}.$$

We can clearly assert that $H_k\cap A_{m,n}$ is bounded for all $n\in N$, the assumption (I_5) Theorem 2.3 is complete.

Next we set

$$\Gamma^*=\{h\in C(W_0^{1,p}(\Omega),W_0^{1,p}(\Omega)):$$

h is an odd homeomorphism $h(0)=0$ and $h(B_1)\subset A_{m,n}\}$.

And

$$\Gamma_k=\{K\in\Sigma:\gamma(K\cap h(\partial B_1))\geq k\ \ \forall h\in\Gamma^*\}.$$

And then

$$S_k=\inf_{K\in\Gamma_k}\max_{w\in K}J_{m,n}(w).$$

So that we can state that lemma 2.7 in [3] holds. We then choose

$$h(w)=r_{n,\lambda}w$$

where $r_{n,\lambda}$ a nonnegative real which has been defined in the step 2 and h belongs to Γ^*. Consequently, we infer that $K\cap B_{r_{n,\lambda}}\neq\emptyset$ for all $K\in\Gamma_k$. Since $J_{m,n}$ is bounded from below on $\partial B_{r_{n,\lambda}}$, then

$$S_k=\inf_{K\in\Gamma_k}\max_{w\in K}J_{m,n}(w)\geq a_{n,\lambda}>0.$$

Since all assumptions of Theorem 2.8 in [3] are satisfied, thus there exist infinitely many critical points of $J_{m,n}$. Hence, the Dirichlet problem (2.11)-(1.2) possesses infinitely many nontrivial weak solutions □

• **Step 5**: Uniformly L^∞ - estimates

Consider the following equation

$$-\text{div}\{(a(x)+\mid T_m(w_{m,n})\mid^\gamma)\mid\nabla w_{m,n}\mid^{p-2}\nabla w_{m,n}\}$$
$$+\frac{\gamma}{p}\frac{T'_m(w_{m,n})}{T_m(w_{m,n})}\mid T_m(w_{m,n})\mid^\gamma\mid\nabla w_{m,n}\mid^p$$
$$=f'_{n,\lambda}(w_{m,n}).\tag{2.11}$$

Assuming that either $w_{m,n}=u_{m,n}$ or $w_{m,n}=u^0_{m,n}$ or …… or $w_{m,n}=u^k_{m,n}$ or …… solution of (2.11)-(1.2). Setting that $T_m(w_{m,n})=w_{m,n}$ and $v=\mid w_{m,n}\mid^b w_{m,n}$ as a test function , then we have

$$(b+1)\int_\Omega(a(x)+\mid w_{m,n}\mid^\gamma)\mid\nabla w_{m,n}\mid^p\mid w_{m,n}\mid^b$$
$$+\frac{\gamma}{p}\int_\Omega\mid w_{m,n}\mid^{b+\gamma}\mid\nabla w_{m,n}\mid^p$$
$$\leq(\lambda+1)n^{\theta-q_0+q-q_1}\int_\Omega\mid w_{m,n}\mid^{b+q}.\tag{2.12}$$

Dropping the positive terms on the left hand side of (2.12), we get

$$(b+1)\int_\Omega a(x)\mid w_{m,n}\mid^b\mid\nabla w_{m,n}\mid^p$$
$$\leq(\lambda+1)n^{\theta-q_0+q-q_1}\int_\Omega\mid w_{m,n}\mid^{b+q}.\tag{2.13}$$

On the other hand, we obtain the following result after using the sobolev inequality

$$C_2^p \left(\int_\Omega | w_{m,n} |^{\frac{b+p}{p} p^*} \right)^{\frac{p}{p^*}} \leq \frac{(b+p)^p}{p^p}$$

$$\times \int_\Omega | w_{m,n} |^b | \nabla w_{m,n} |^p.$$

Therefore

$$C_2^p \left(\int_\Omega | w_{m,n} |^{\frac{b+p}{p} p^*} \right)^{\frac{p}{p^*}}$$

$$\leq \frac{(b+p)^p}{p^p} \frac{1}{\alpha} \int_\Omega a(x) | w_{m,n} |^b | \nabla w_{m,n} |^p. \tag{2.14}$$

Combining (2.13) with (2.14), we have

$$\left(\int_\Omega | w_{m,n} |^{\frac{b+p}{p} p^*} \right)^{\frac{p}{p^*}} \leq \left(\frac{(b+p)^p(\lambda+1)}{\alpha(pC_2)^p(b+1)} \right)$$

$$\times n^{\theta - q_0 + q - q_1} \int_\Omega | w_{m,n} |^{b+q}.$$

It follows that

$$| w_{m,n} |^{b+q}_{\frac{b+p}{p} p^*} \leq \left(\frac{(b+p)^p(\lambda+1)}{\alpha(pC_2)^p(b+1)} \right) n^{\theta - q_0 + q - q_1} | w_{m,n} |^{b+q}_{b+q}.$$

Let $r = b + q$, then

$$| w_{m,n} |_{\frac{r-q+p}{p} p^*} \leq \left(\frac{(r-q+p)^p(\lambda+1)}{\alpha(pC_2)^p(r-q+1)} \right)^{\frac{1}{r-q+p}}$$

$$\times n^{\frac{\theta - q_0 + q - q_1}{r-q+p}} | w_{m,n} |_r^{\frac{r}{r-q+p}}.$$

Notice that $w_{m,n}$ belongs to $W_0^{1,p}(\Omega)$ and so to $L^{p^*}(\Omega)$, we can choose $r = r_0 = p^* - q$ to deduce that $w_{m,n}$ belongs to $L^{\frac{r_0+p-q}{p} p^*}(\Omega)$, we can then choose $r = r_1 = \frac{r_0-q+p}{p} p^*$ to obtain $w_{m,n}$ belongs to $L^{\frac{r_0+p-q}{p} p^*}(\Omega)$. Iterating this process and defining by induction r_k as

$$\begin{cases} r_0 = p^* - q \\ r_k = r_{k-1} \frac{p^*}{p} + \frac{p^*}{p}(p-q). \end{cases} \tag{2.15}$$

We infer that $w_{m,n}$ belongs to $L^{r_k}(\Omega)$ with

$$| w_{m,n} |_{r_k} \leq \left(\frac{(r_k-q+p)^p(\lambda+1)}{\alpha(pC_2)^p(r_k-q+1)} \right)^{\frac{1}{r_k-q+p}}$$

$$\times n^{\frac{\theta - q_0 + q - q_1}{r_k-q+p}} | w_{m,n} |^{\frac{p^*}{p} \frac{r_{k-1}}{r_k}}.$$

Therefore

$$| w_{m,n} |_{r_k} \leq \ldots\ldots \leq C_3 | w_{m,n} |_{p^*}^{(\frac{p^*}{p})^k \frac{p^*-q}{r_k^k}} \leq C_4.$$

Because of
$\int_\Omega a(x) | \nabla w_{m,n} |^p$ is bounded with respect to m and n.
Since $\frac{p^*}{p} > 1$, it is enough to show that r_k is increasing sequence which diverges to infinity, thus, if it is such that $\frac{r_k+p-q}{p} p^* \geq \frac{N}{2}$ an adaptation to the quasilinear case of the proof of a result of Stampacchia (see [5]) implies that there exists $M_n > 0$ such that

$$| w_{m,n} |_\infty \leq M_n.$$

Let m_n be an integer such that $m_n \geq max(M_n + p, \bar{t})$, if we define $w_n \stackrel{def}{=} w_{\overline{m},n}$, namely, either $w_n = u_n^0 \stackrel{def}{=} u^0_{\overline{m},n}$ or $w_n = u_n^1 \stackrel{def}{=} u^1_{\overline{m},n}$ or ...... or $w_n = u_n^k \stackrel{def}{=} u^k_{\overline{m},n}$ or $w_n = \ldots \stackrel{def}{=} \ldots$. Then $T_{m_n}(w_n) = w_n$ and $T'_{m_n}(w_n) = 1$, consequently, the equation which is satisfied by w_n is

$$-\text{div}\big((a(x) + | w_n |^\gamma) | \nabla w_n |^{p-2} \nabla w_n\big)$$

$$+\frac{\gamma}{p} | w_n |^{\gamma-2} w_n | \nabla w_n |^p = f'_{n,\lambda}(w_n) \tag{2.16}$$

with zero Dirichlet boundary condition.
Notice that by the assumption $q < \frac{p^*}{p}(p+\gamma)$, then w_n is bounded in $L^q(\Omega)$ using this fact, we are going to show that w_n is uniformly bounded in $L^\infty(\Omega)$.
Let $b > 0$ as before, and choose $v = | w_n |^b w_n$ as a test function in the equation (2.16)-(1.2) satisfied w_n.
The fact that $f'_{n,\lambda}(t) \leq (\lambda+1) | t |^{q+b-1}$ and we drop two nonnegative terms, and then we obtain

$$(b+1) \int_\Omega a(x) | w_n |^{b+\gamma} | \nabla w_n |^p \leq (\lambda+1) \int_\Omega | w_n |^{q+b}.$$

However, we get another inequality when we apply the sobolev inequality to $w_n^{b+p+\gamma}$, we then have

$$C_5^p \left(\int_\Omega | w_n |^{\frac{\gamma+b+p}{p} p^*} \right)^{\frac{p}{p^*}} \leq \left(\frac{\gamma+b+p}{p} \right)^p \int_\Omega | w_n |^{b+\gamma} | \nabla w_n |^p.$$

Thus

$$\left(\int_\Omega | w_n |^{\frac{\gamma+b+p}{p} p^*} \right)^{\frac{p}{p^*}} \leq \frac{(\gamma+b+p)^p(\lambda+1)}{(pC_5)^p(b+1)} \int_\Omega | w_n |^{b+q}.$$

Where w_n belongs to $L^{\frac{\gamma+r-q+p}{p} p^*}(\Omega)$ provided that w_n belongs to $L^r(\Omega)$ with $b = r - q$, yields

$$| w_n |_{\frac{\gamma+b+p}{p} p^*} \leq \left(\frac{(\gamma+b+p)^p(\lambda+1)}{(pC_5)^p(b+1)} \right)^{\frac{p^*}{p}} | w_n |_r^{\frac{r}{\gamma+r-q+p}}.$$

Because of $\frac{p}{b+\gamma+p} \leq p^*$.
Arguing as before, if we consider the sequence r_k as follows

$$\begin{cases} r_0 = \frac{p^*}{p}(\gamma+p) \\ r_k = r_{k-1} \frac{p^*}{p} + \frac{p^*}{p}(\gamma+p-q). \end{cases} \tag{2.17}$$

Thus $w_{m,n}$ belongs to $L^{r_k}(\Omega)$ for every k and so

$$| w_n |_{r_k} \leq \left(\frac{(\gamma+r_{k-1}-q+p)^p(\lambda+1)}{(pC_5)^p(r_{k-1}-q+1)} \right)^{\frac{p^*}{p}} | w_n |_{r_k}^{\frac{p^*}{p} \frac{r_{k-1}}{r_k}}.$$

It follows that

$$| w_n |_{r_k} \leq \ldots\ldots \leq C_6 | w_n |_{\frac{p^*}{p}(\gamma+p)}^{(\frac{p^*}{p})^{k+1} \frac{\gamma+p}{r_k^k}} \leq C_7.$$

The fact that

$$\int_\Omega | w_n |^\gamma | \nabla w_n |^p \text{ is bounded with respect to n.}$$

And $w_n \in L^{\frac{p^*}{p}(\gamma+p)}(\Omega)$, clearly the sequence $\{r_k\}$ is increasing and unbounded for $\frac{p^*}{p} > 1$. So that in a finite number of steps we conclude that

$$\lambda | w_n |^{\theta-2} w_n + | w_n |^{q-2} w_n$$

is bounded in L^r with $r > \frac{N}{2}$. Using again an adaptation of the proof theorem 2.1 in [5] yields that there exists a nonnegative constant $C_0' > 0$, such that

$$| w_n |_\infty \leq C_0', \qquad \forall n \geq max(\bar{t}, \bar{n}).$$

In other words, we obtain

$$| u_n^0 |_\infty \leq C_1', \qquad | u_n^1 |_\infty \leq C_2', \ldots\ldots, | u_n^k |_\infty \leq C_k' \ldots\ldots$$

$$\forall n \geq max(\bar{t}, \bar{n}). \quad \square$$

• **Step 6**: Conclusion
Finally, if $\forall n \geq max(C_0', \bar{t}, \bar{n})$ then

$$f'_{n,\lambda}(w_n) = \lambda | w_n |^{\theta-2} w_n + | w_n |^{q-2} w_n$$

and so $w \overset{def}{=} w_{\overline{n}}$. In other words, either

$$w \overset{def}{=} u^0 \overset{def}{=} u^0_{\overline{n}} \quad \text{or} \quad w \overset{def}{=} u^1 \overset{def}{=} u^1_{\overline{n}} \quad \text{or} \ldots \text{or}$$

$$w \overset{def}{=} u^k \overset{def}{=} u^k_{\overline{n}} \quad \text{or} \ldots\ldots$$

Hence, we can conclude that the problem (1.1)-(1.2) has an infinitely many positive bounded weak solutions. $\square$

3. The case $\lambda > \widetilde{\lambda}_0$

We complete the study of the equation (1.1)-(1.2) by showing that such equation does not have nontrivial solution. In order to prove this fact, we assume that Ω is star-shaped, ie, $x.\nu > 0$ on $\partial\Omega$. Where ν is outward normal to $\partial\Omega$. For that we use the idea of [9] in the next proposition.

Proposition 3.1 If Ω is a smooth star-shaped in R^N containing 0, then $u \equiv 0$ is the unique $H^2(\Omega) \cap H_0^1(\Omega)$ nonnegative solution of (1.1)-(1.2).
Proof. Let u belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ be a nonnegative solution of (1.1). The divergence of the vector field $(a(x)+ | u |^\gamma) \mid \nabla u \mid^{p-2} \nabla u(x\nabla u)$ can be written as follows

$$\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u(x\nabla u)\right\}$$

$$= (x\nabla u)\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u\right\}$$

$$+ (a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u.\nabla(x\nabla u).$$

Since

$$\nabla u.\nabla(x\nabla u) =| \nabla u |^2 + \frac{1}{2}\left(x\nabla(| \nabla u |^2)\right).$$

And

$$\frac{1}{2}(a(x)+ | u |^\gamma) | \nabla u |^{p-2}\left(x\nabla(| \nabla u |^2)\right)$$

$$= \frac{1}{p}(a(x)+ | u |^\gamma)(x\nabla(| \nabla u |^p)).$$

Consequently,

$$\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u(x\nabla u)\right\}$$

$$= (x\nabla u)\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u\right\}$$

$$+ (a(x)+ | u |^\gamma) | \nabla u |^p$$

$$+ \frac{1}{p}(a(x)+ | u |^\gamma)(x\nabla(| \nabla u |^p)). \tag{3.2}$$

Multiplying the equation (1.1) by $x\nabla u$, yields

$$(x\nabla u)\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u\right\}$$

$$= \frac{\gamma}{p} | u |^{\gamma-2} u | \nabla u |^p (x\nabla u)$$

$$-\lambda | u |^{\theta-2} u(x\nabla u) - | u |^{q-2} u(x\nabla u). \tag{3.3}$$

Replacing (3.3)into (3.2), we have

$$\text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u(x\nabla u)\right\}$$

$$= \frac{\gamma}{p} | u |^{\gamma-2} u | \nabla u |^p (x\nabla u)$$

$$-\lambda | u |^{\theta-2} u(x\nabla u) - | u |^{q-2} u(x\nabla u)$$

$$+ (a(x)+ | u |^\gamma) | \nabla u |^p$$

$$+ \frac{1}{p}(a(x)+ | u |^\gamma)(x\nabla(| \nabla u |^p)). \tag{3.4}$$

On the other hand, applying Gauss formula to the vector field $(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u(x\nabla u)$, we obtain

$$\int_\Omega \text{div}\left\{(a(x)+ | u |^\gamma) | \nabla u |^{p-2} \nabla u(x\nabla u)\right\}$$

$$= \int_{\partial\Omega}(a(x)+ | u |^\gamma) | \nabla u |^p (x.\nu)d\sigma. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$-\frac{\gamma N}{p}\int_\Omega | u |^\gamma | \nabla u |^p - \lambda\int_{\partial\Omega} | u |^\theta (x.\nu)d\sigma$$

$$-\int_{\partial\Omega} | u |^q (x.\nu)d\sigma + \left(1 - \frac{N}{p}\right)\int_\Omega (a(x)+ | u |^\gamma) | \nabla u |^p$$

$$= \int_{\partial\Omega}(a(x)+ | u |^\gamma) | \nabla u |^p(x.\nu)d\sigma.$$

The fact that $p < N$ and $x.\nu > 0$ on $\partial\Omega$.
Therefore $u \equiv 0$. $\square$

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