Efficient Representatives of Some Transcendental Functions

Surattana Sungnul, Kanokwan Pananu and Vimolyut Varnasavang

Abstract—**Generating representatives of some transcendental functions in finite power series is one of several methods applied in many fields of engineering and applied mathematics. Many problems are in differential equations both linear and non-linear or integral equations which should consider the convergence and rate of convergence of the approximation solution.**

In this work, we study a method to generate approximation of transcendental functions in a finite power series form such as Taylor's, Chebyshev and Legendre series expansions. We then developed Maple software to generate approximation of functions and compared efficiency and accuracy of all proposed methods under the same condition. In addition, we analyze the convergence of all methods both analytically and numerically.

Index Terms—**Taylor's series expansion, Chebyshev series expansion, Legendre series expansion, Convergence, Transcendental Functions.**

I. Introduction

ALMOST all problems in many fields of engineering, applied mathematics and sciences are in the models of differential equations both linear and non-linear form or integral equation. Sometimes these problems are not simple to solve for an exact solution. Numerical solutions or analytical solutions are one way to gain an approximation solution which should consider the convergence and rate of convergence of the approximation solution. Generating approximation function in a power series form is one of several methods applied to solve this problem.

Approximation of functions in a power series form have been widely used in many fields of engineering, applied mathematics and sciences. The methods of generating approximation function in power series form as Taylor's, Chebyshev and Legendre series expansions can be read in text books [1]-[3]. From a general survey it was found that there are many researches which applied approximation function in power series forms, for example, Abdul-Majid Wazwaz [4] and Shih-Hsiang Chang [5] applied Taylor series to solve linear and non-linear ordinary differential equations. In 2000, L.P.Streltsov [6] developed the theory of function interpolation by polynomials from functional space on a discrete point set. Comparative investigation shows that Chebyshev polynomials of the first kind are more effective than other interpolating functions and their derivatives. However, Legendre polynomials are more preferable for solving Fredholm equations. In 2000, Robert Piessens [7] conducted a survey on the use of Chebyshev polynomials in the numerical computation of integral transforms and the solutions of integral equations, focusing especially on problems showing singularity. In 2006, M.M. Hosseini [8] presented the Adomian decomposition method with Chebyshev polynomials and applied it to linear and non-linear ordinary differential equations. In 2007, K. Maleknejad, K. Nouri and M. Yousefi [9] proposed a method to solve Fredholm integral equations of the second kind by using Legendre polynomials and collocation methods. They investigated convergence and rate of convergence. In addition they gave examples to show the accuracy of this method. In 2008, Lan Chen and He-Ping Ma [10] presented the Legendre-Galerkin-Chebyshev collocation method (LGCC) to approximate the eigenvalues of regular SturmLiouville problems with three kinds of boundary conditions. Also, they showed that this method preserves the symmetry of the problem and numerical results with high accuracy. In 2009, Yucheng Liu [11] proposed the modification of Adomian decomposition method for orthogonal polynomials such as Chebyshev and Legendre polynomials to approximate function on the interval (-1,1). In 2009, Wei-Chung Tien and Chao-Kuang Chen [12] presented an efficient modification of the Adomian decomposition method by using Legendre polynomials to solve linear and non-linear models. In 2009, Yucheng Liu [13] proposed a method to solve the Fredholm integral equations of the second kind, where Chebyshev polynomials are applied to approximate a solution for an unknown function in the Fredholm integral equations. Also, this work showed convergence and rate of convergence and gave a few numerical examples and compared the results with previous researches. In 2009, M. P. Ramirez T. and et.al. [14] have studied the general solution of the two dimensional Electrical Impedance Equation in terms of Taylor series in formal powers, for the case when the electrical conductivity is a separable-variables function. In 2013, A.H. Bhrawy, M.M. Tharwat and A. Yildirim [15] derived a new explicit formula for the integrals of shifted Chebyshev polynomials of any degree for any fractional-order in terms of shifted Chebyshev polynomials themselves. They presented efficient direct solvers for the general fractional-order differential equation by using the Chebyshev approximation. In 2013, Nichaphat Patanarapeelert and Vimolyut Varnasavang [16] studied series approximation and Convergence between Chebyshev and Legendre Series. They found that the rate of convergence of infinite series generated from the same function in terms of Chebyshev polynomial are more rapid than Legendre polynomial, while Chebyshev and Legendre Series gave the close rate of convergence. In 2014, Piotr Ruta and Jozef Szybinski [17] presented a method to solve the free vibration of Non-Prismatic Sandwich Beams problem by using the

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Chebyshev Series to approximate the solution. In 2016, H.M. Jaradat and et.al. [18] have developed an iterative technique based on the generalized Taylor series residual power series (RPS) to explore nonlinear fractional differential equations.

In this work, we study a method to generate approximation of transcendental functions in a power series form such as Taylor's, Chebyshev and Legendre series expansions. We then develop Maple software to generate an approximation of function and compare efficiency and accuracy of all proposed methods under the same condition. In addition, we demonstrate the error analysis in generating the representatives of some transcendental function both analytically and numerically.

II. BACKGROUND

In this section, we first give a review of the definitions related to the representatives of functions obtained by Taylor's, Chebyshev and Legendre series expansions.

A. Taylor's Series Expansion

Definition A [3] : Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x), \tag{1}$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \end{aligned} \tag{2}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}. \tag{3}$$

Here $P_n(x)$ is called the ***n*th Taylor polynomial** for f about x_0, and $R_n(x)$ is called the **remainder term** (or **truncation error**) associated with $P_n(x)$. Since the number $\xi(x)$ in the truncation error $R_n(x)$ depends on the value of x at which the polynomial $P_n(x)$ is being evaluated, it is a function of the variable x. However, we are unable to explicitly determine the function $\xi(x)$. Taylor's Theorem simply ensures that such a function exists, and that its value lies between x and x_0. In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi(x))$ when x is in some specified interval.

B. Chebyshev Series Expansion

Definition B [1] : If function $f(x)$ is a continuous function in the interval $[-1,1]$, it can be expressed as an infinite series of $T_n(x)$ as

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(x), \tag{4}$$

when

$$T_n(x) = \cos(n \ \cos^{-1} x), \quad (|x| \le 1) \tag{5}$$

where

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx$$

and

$$c_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx. \qquad ; n = 1,2,3,...$$

C. Legendre Series Expansion

Definition C [1] : If function $f(x)$ is a continuous function in the interval $[-1,1]$, it can be expressed as an infinite series as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad (-1 \le x \le 1) \tag{6}$$

where $P_n(x)$ is Legendre polynomial of the first kind of degree n can be write in the form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \tag{7}$$

which is often referred to as Rodrigur's formula and the coefficients a_n is given by

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx. \qquad ; n = 0,1,2,3,...$$

III. Representatives of Transcendental Functions and Error Analysis

In this section, we show how to generate representatives of transcendental functions by Taylor's, Chebyshev and Legendre series expansions. We then develop Maple software to generate the approximation function and compare efficiency and accuracy of all proposed methods under the same condition. In addition, we demonstrate the error analysis of all methods both analytically and numerically.

A. Trigonometrical Functions

Let us consider
$f(x) = \sin(x)$ and $f(x) = \cos(x)$; $x \in [0,\pi]$

(i) By Definition A, Taylor's series expansion of $\sin(x)$ is in the form

$$\sin(x) = P_s(x) + R_s(x), \tag{8}$$

where

$$P_s(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11},$$

and

$$R_s(x) = \frac{1}{13!}x^{13}\sin(\xi(x)), \quad 0 < \xi < 1$$

Thus,

$$|R_s(x)| \le 4.66 \times 10^{-4}. \tag{9}$$

Similarly,

$$\cos(x) = P_c(x) + R_c(x), \tag{10}$$

where

$$P_c(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10},$$

and

$$|R_c(x)| \leq 1.93 \times 10^{-3}. \quad (11)$$

By substituting a new variable, for example,

$$w = \frac{\frac{\pi}{2} - x}{2\pi}, \quad (12)$$

we found that in the case of $x \in [0, \pi]$ it will transform to $w \in [-\frac{1}{4}, \frac{1}{4}]$.
Since

$$\sin(x) = \cos(\frac{\pi}{2} - x),$$

then we have

$$\sin(x) = \cos(2\pi w). \quad (13)$$

And by *Taylor's series expansion of* $\cos(x)$ in form (10), we have

$$\sin(x) = 1 - 2\pi^2 w^2 + \frac{2}{3}\pi^4 w^4 - \frac{4}{45}\pi^6 w^6 + \frac{2}{315}\pi^8 w^8 - \frac{4}{14175}\pi^{10} w^{10} + R_s(w), \quad (14)$$

where

$$R_s(w) = \frac{1}{12!}(2\pi w)^{12} - \frac{1}{14!}(2\pi w)^{14} + \frac{1}{16!}(2\pi w)^{16} + \cdots$$

Thus,

$$|R_s(w)| \leq \frac{1}{12!}(2\pi w)^{12}\left[\frac{1}{1 - w^2}\right] \leq 5.02 \times 10^{-7}. \quad (15)$$

Similarly, since

$$\cos(x) = \sin(2\pi w). \quad (16)$$

$$\cos(x) = 2\pi w - \frac{4}{3}\pi^3 w^3 + \frac{4}{15}\pi^5 w^5 - \frac{8}{315}\pi^7 w^7 + \frac{4}{2835}\pi^9 w^9 - \frac{8}{155925}\pi^{11} w^{11} + R_c(w), \quad (17)$$

where

$$|R_c(w)| \leq 6.07 \times 10^{-8}. \quad (18)$$

(ii) By Definition B, Chebyshev series expansion of $\sin(x); x \in [0, \pi]$, after changing to new variable w from (12)

$$f(w) = \cos(2\pi w) = \sum_{m=0}^{\infty} c_{2m} T_{2m}(w) \quad ; \; w \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

where

$$c_0 = \frac{1}{\pi}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{\cos(2\pi w)}{\sqrt{1 - w^2}} dw$$

$$c_{2m} = \frac{2}{\pi}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{\cos(2\pi w) \cdot \cos\{(2m)\cos^{-1}(w)\}}{\sqrt{1 - w^2}} dw \; ; (m = 1, 2, 3...)$$

Thus,

$$\sin(x) = 0.89 - 14.4220w^2 + 73.1563w^4 - 157.9471w^6 + 152.2193w^8 - 53.9414w^{10} + R_s(w). \quad (19)$$

where

$$|R_s(w)| \leq 4.08 \times 10^{-4}.$$

Similarly,

$$\cos(x) = 5.5490w - 68.9338w^3 + 293.9390w^5 - 561.8103w^7 + 494.7377w^9 - 163.5550w^{11} + R_c(w). \quad (20)$$

where

$$|R_c(w)| \leq 3.64 \times 10^{-4}.$$

(iii) By Definition C, Legendre series expansion of $\sin(x); x \in [0, \pi]$, after changing to new variable w from (12)

$$f(w) = cos(2\pi w) = \sum_{m=0}^{\infty} a_{2m} P_{2m}(w) \quad ; \; w \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

where

$$a_{2m} = \frac{4m+1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}} \cos(2\pi w) P_{2m}(w) dw \; ; (m = 0, 1, 2, 3, ...)$$

Thus,

$$\sin(x) = 0.9100 - 15.5798w^2 + 83.5877w^4 - 190.2764w^6 + 192.5844w^8 - 71.4019w^{10} + R_s(w). \quad (21)$$

where

$$|R_s(w)| \leq 4.45 \times 10^{-4}.$$

Similarly,

$$\cos(x) = 5.9447w - 77.8503w^3 + 349.2304w^5 - 699.9298w^7 + 644.1507w^9 - 221.8501w^{11} + R_c(w). \quad (22)$$

where

$$|R_c(w)| \leq 3.60 \times 10^{-4}.$$

We have generated the representatives of $f(x) = \sin(x)$ and $f(x) = \cos(x)$ for $x \in [0, \pi]$ in Taylor's series expansions in variable x, Taylor's series expansions in new variable w, Chebyshev and Legendre series expansions respectively for the first six non-zero terms. We have compared the efficiency of all proposed methods by considering the bounded truncation error and found that the representatives of $\sin(x)$ and $\cos(x)$ derived from the modified Taylor's series expansions (in the new variable w) provide the least truncation errors in comparison with the respective ones derived from Chebyshev and Legendre series expansions, namely, $|R_s(w)| \leq 10^{-7}$ and $|R_c(w)| \leq 10^{-8}$.

Table I, II, III and IV illustrate the absolute errors between actual value of $\sin(x)$ and the approximate values denoted by $E(T_s(x))$, $E(T_s(w))$, $E(C_s(w))$ and $E(L_s(w))$ of Taylor's series expansion in 4, 5, 6 and 7 in terms of variables x and new variable w, Chebyshev and Legendre series expansions respectively.

TABLE I
COMPARISON OF THE ABSOLUTE ERROR BETWEEN ACTUAL AND APPROXIMATE VALUES OF $\sin(x)$ BY TAYLOR'S SERIES EXPANSION IN TERMS OF VARIABLE x AND NEW VARIABLE w, CHEBYSHEV SERIES EXPANSION AND LEGENDRE SERIES EXPANSION IN 4,5,6 AND 7 TERMS.

x	w	Absolute error (Approximate value in 4 terms)			
		$E(T_x)$	$E(T_w)$	$E(C_w)$	$E(L_w)$
0	$\frac{1}{4}$	0	9×10^{-4}	3.8×10^{-1}	3.8×10^{-1}
$\frac{\pi}{3}$	$\frac{1}{12}$	4.1×10^{-6}	1.4×10^{-7}	2.5×10^{-1}	2.2×10^{-1}
$\frac{\pi}{6}$	$\frac{1}{6}$	7.9×10^{-9}	3.5×10^{-5}	2.2×10^{-2}	3.8×10^{-2}
$\frac{3\pi}{4}$	$-\frac{1}{8}$	5.9×10^{-3}	3.6×10^{-6}	1.3×10^{-1}	1.1×10^{-1}
π	$-\frac{1}{4}$	7.5×10^{-2}	9×10^{-4}	3.8×10^{-1}	3.8×10^{-1}

TABLE II
CONTINUED

x	w	Absolute error (Approximate value in 5 terms)			
		$E(T_x)$	$E(T_w)$	$E(C_w)$	$E(L_w)$
0	$\frac{1}{4}$	0	2.5×10^{-5}	3.2×10^{-1}	3.1×10^{-1}
$\frac{\pi}{3}$	$\frac{1}{12}$	4.1×10^{-8}	1.2×10^{-10}	1.4×10^{-1}	1.2×10^{-1}
$\frac{\pi}{6}$	$\frac{1}{6}$	2.2×10^{-10}	4.3×10^{-7}	5.3×10^{-2}	5.6×10^{-2}
$\frac{3\pi}{4}$	$-\frac{1}{8}$	3×10^{-4}	2.5×10^{-8}	5.8×10^{-2}	4.5×10^{-2}
π	$-\frac{1}{4}$	6.9×10^{-3}	2.5×10^{-5}	3.2×10^{-1}	3.1×10^{-1}

TABLE III
CONTINUED

x	w	Absolute error (Approximate value in 6 terms)			
		$E(T_x)$	$E(T_w)$	$E(C_w)$	$E(L_w)$
0	$\frac{1}{4}$	0	4.6×10^{-7}	2.4×10^{-1}	2.2×10^{-1}
$\frac{\pi}{3}$	$\frac{1}{12}$	7.6×10^{-10}	3.1×10^{-10}	7.3×10^{-2}	6×10^{-2}
$\frac{\pi}{6}$	$\frac{1}{6}$	2×10^{-10}	3.8×10^{-9}	4.3×10^{-2}	3.8×10^{-2}
$\frac{3\pi}{4}$	$-\frac{1}{8}$	1.1×10^{-5}	1.2×10^{-11}	2.5×10^{-2}	2.1×10^{-2}
π	$-\frac{1}{4}$	4.5×10^{-4}	4.6×10^{-7}	2.4×10^{-1}	2.2×10^{-1}

TABLE IV
CONTINUED

x	w	Absolute error (Approximate value in 7 terms)			
		$E(T_x)$	$E(T_w)$	$E(C_w)$	$E(L_w)$
0	$\frac{1}{4}$	0	6.6×10^{-9}	1.6×10^{-1}	1.5×10^{-1}
$\frac{\pi}{3}$	$\frac{1}{12}$	4.7×10^{-10}	3.1×10^{-10}	3.2×10^{-2}	2.6×10^{-2}
$\frac{\pi}{6}$	$\frac{1}{6}$	2×10^{-10}	1.4×10^{-10}	1.1×10^{-2}	3.8×10^{-3}
$\frac{3\pi}{4}$	$-\frac{1}{8}$	2.9×10^{-7}	1.3×10^{-10}	2×10^{-2}	2.1×10^{-2}
π	$-\frac{1}{4}$	2.1×10^{-5}	6.6×10^{-9}	1.6×10^{-1}	1.5×10^{-1}

We can see that approximation of $f(x) = \sin(x)$ derived from Taylor's series expansion in the new variable w in 6 terms provide values more efficient than the respective values derived from Chebyshev and Legendre series expansions for 4 and 5 terms. Graphs of approximation of $\sin(x)$ by Taylor's series expansion in variable x and new variable w, Chebyshev and Legendre series expansions are shown in Fig 1.

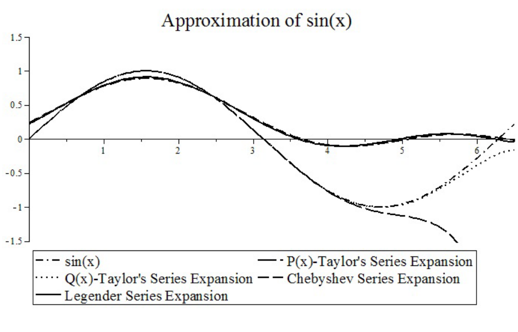

Fig. 1. Graphs of approximation of $\sin(x)$ by Taylor's series expansion in variable x and new variable w, Chebyshev series expansion and Legendre series expansion in 6 terms.

Similarly, absolute error between actual value of $\cos(x)$ and approximate values derived from Taylor's series expansion in the new variable w in 6 terms provides the value which is better than the ones derived from Taylor's series expansion in variable x, Chebyshev and Legendre series expansions in 4, 5 and 7 terms. Graphs of approximation of $\cos(x)$ by Taylor's series expansion in variable x and new variable w, Chebyshev and Legendre series expansions are shown in Fig 2.

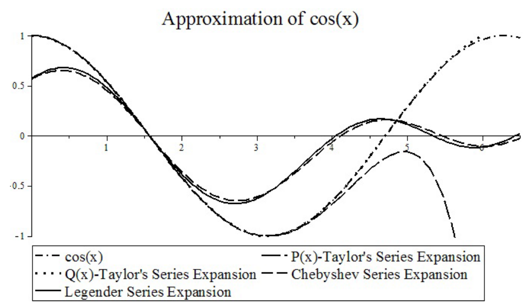

Fig. 2. Graphs of approximation of $\cos(x)$ by Taylor's series expansion in variable x and new variable w, Chebyshev series expansion and Legendre series expansion in 6 terms.

B. Inverse Trigonometrical Functions

Next, we present how to generate the representative of $f(x) = \sin^{-1}(x)$ in a power series form derived from Taylor's, Chebrshev and Legendre series expansions. Approximation of $\sin^{-1}(x)$ for some values by all proposed methods are shown in Table V.

(i) By Definition A, Taylor's series expansion of $\sin^{-1}(x)$ is in the form

$$\sin^{-1}(x) = P_{as}(x) + R_{as}(x), \tag{23}$$

where

$$P_{as}(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{x^9}{9} \; ; \; |x| < 1,$$

$$R_{as}(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\frac{x^{11}}{11} + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\frac{x^{13}}{13} + \cdots$$

and

$$|R_{as}(x)| \leq 2.24 \times 10^{-2}\left|\frac{x^{11}}{1-x^2}\right| \leq 3.7 \times 10^{-2}. \tag{24}$$

(ii) By Definition B, Chebyshev series expansion of $\sin^{-1}(x)$ is in the form

$$f(x) = \sum_{m=0}^{\infty} c_{2m+1}T_{2m+1}(x), \tag{25}$$

where

$$c_{2m+1} = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)\cos\{(2m+1)\cos^{-1}(x)\}}{\sqrt{1-x^2}}dx \; ; m = 0, 1, ...$$

then substitute the new variable θ such that $x = cos(\theta)$, we have

$$c_{2m+1} = \frac{2}{\pi}\int_{0}^{\pi}\sin^{-1}(\cos\theta)\cdot\cos\{(2m+1)\cos^{-1}(\cos\theta)\}d\theta$$

$$= \frac{4}{\pi(2m+1)^2}.$$

Thus,

$$\sin^{-1}(x) = 1.0631x - 0.8839x^3 + 4.6952x^5 - 7.3911x^7 + 4.0241x^9 + R_{as}(x). \quad (26)$$

where

$$R_{as}(x) = 432.0020x^{11} - 316.8202x^{13} + 92.7145x^{15} + \cdots$$

(iii) By Definition C, Legendre series expansion of $\sin^{-1}(x)$ is in the form

$$f(x) = \sum_{m=0}^{\infty} a_{2m+1} P_{2m+1}(x), \quad (27)$$

where

$$a_{2m+1} = \frac{4m+3}{2} \int_{-1}^{1} \sin^{-1}(x) P_{2m+1}(x) dx \,;\, m = 0, 1, 2, 3, ...$$

Thus,

$$\sin^{-1}(x) = 1.0232x - 0.2587x^3 + 2.1155x^5 - 3.5234x^7 + 2.1190x^9 + R_{as}(x). \quad (28)$$

where

$$R_{as}(x) = 168.2477x^{11} - 129.0731x^{13} + 39.51x^{15} + \cdots$$

We now present the absolute error of $\sin^{-1}(x)$ generated in a power series form for the first non-zero five terms by Taylor's series expansion and compare the absolute error of $\sin^{-1}(x)$ generated by Chebyshev series expansion in Nichaphat P. and Vimolyut V. paper [16].

TABLE V
COMPARISON OF THE ABSOLUTE ERROR BETWEEN ACTUAL AND APPROXIMATE VALUES OF $\sin^{-1}(x)$ IN 5 TERMS.

x	$\sin^{-1}(x)$	*Absolute error (Approximate value in 5 terms)*			
		$E(T_x)$	$E(C_x)$	$E(L_x)$	$E(C_x)$ [16]
$\frac{1}{2}$	0.5236	0.0001	0.0057	0.0016	0.0057
$\frac{\sqrt{2}}{2}$	0.7854	0.0008	0.0084	0.0029	0.0084
$\frac{\sqrt{3}}{2}$	1.0472	0.0118	0.0111	0.0052	0.0111

Table V shows the absolute error between actual value and approximate values of $f(x) = \sin^{-1}(x)$ derived from Taylor's, Chebyshev and Legendre series expansions. In conclusion, we compare the absolute error of $f(x) = \sin^{-1}(x)$ generated by Chebyshev series expansion in paper [16] as shown in last column.

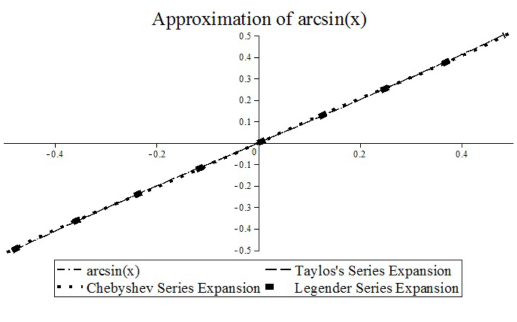

Fig. 3. Graphs of approximation $\sin^{-1}(x)$ by Taylor's series expansion, Chebyshev series expansion and Legendre series expansion in 5 terms.

We can see that approximation of $\sin^{-1}(x)$ by Taylor's series expansion provides value which is better than the ones derived from Chebyshev, Legendre and Chebyshev series expansions in paper [16]. Graphs of approximation of $\sin^{-1}(x)$ by Taylor's, Chebyshev and Legendre series expansions are shown in Fig 3.

C. Logarithm Function

Let us consider

$$f(x) = \ln(x), x > 0$$

In the process of finding the approximate values of $f(x)$ to compare with the actual value, we limit the domain of $f(x)$ in $(-1, 1)$ by using the new variable : w

$$w = \frac{8}{7}\left(\frac{\sqrt[8]{x} - 1}{\sqrt[8]{x} + 1}\right). \quad (29)$$

For $w \in (-1, 1)$; we have $\frac{1}{15^8} < x < 15^8$ and

$$\frac{1}{8}\ln(x) = \ln(8 + 7w) - \ln(8 - 7w)$$

$$\ln(x) = 8\left[\ln\left(1 + \frac{7}{8}w\right) - \ln\left(1 - \frac{7}{8}w\right)\right] \quad (30)$$

$$= 8\left[\ln\left(\frac{8 + 7w}{8}\right) - \ln\left(\frac{8 - 7w}{8}\right)\right] \quad (31)$$

(i) By Definition A, Taylor's series expansion of $\ln(1 + x)$ is in the form

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots \quad ; \; -1 < x \le 1, \quad (32)$$

Therefore,

$$\ln\left(1 + \frac{7}{8}w\right) = \frac{7w}{8} - \frac{1}{2}\left(\frac{7w}{8}\right)^2 + \frac{1}{3}\left(\frac{7w}{8}\right)^3 - \frac{1}{4}\left(\frac{7w}{8}\right)^4 + \frac{1}{5}\left(\frac{7w}{8}\right)^5 \quad (33)$$

and

$$\ln\left(1 - \frac{7}{8}w\right) = -\frac{7w}{8} - \frac{1}{2}\left(\frac{7w}{8}\right)^2 - \frac{1}{3}\left(\frac{7w}{8}\right)^3 - \frac{1}{4}\left(\frac{7w}{8}\right)^4 - \frac{1}{5}\left(\frac{7w}{8}\right)^5. \quad (34)$$

Substituting equation (33) and (34) under five terms into the equation (30), then gives

$$\ln(x) \approx P_{\ln}(w) = 16\left[\frac{7w}{8} + \frac{1}{3}\left(\frac{7w}{8}\right)^3 + \frac{1}{5}\left(\frac{7w}{8}\right)^5 + \frac{1}{7}\left(\frac{7w}{8}\right)^7 + \frac{1}{9}\left(\frac{7w}{8}\right)^9 + \frac{1}{11}\left(\frac{7w}{8}\right)^{11} + \cdots\right]$$

$$= 16\left[\frac{7w}{8} + \frac{1}{3}\left(\frac{7w}{8}\right)^3 + \frac{1}{5}\left(\frac{7w}{8}\right)^5 + \frac{1}{7}\left(\frac{7w}{8}\right)^7 + \frac{1}{9}\left(\frac{7w}{8}\right)^9\right] + R_{\ln}(w). \quad (35)$$

(ii) By Definition B, Chebyshev series expansion of $\ln(x)$ is in the form

$$\ln(x) = 8\ln\left(\frac{8+7w}{8-7w}\right) = \sum_{m=0}^{\infty} c_{2m+1}T_{2m+1}(w) \quad (36)$$

where

$$c_{2m+1} = \frac{16}{\pi}\int_{-1}^{1}\frac{\ln\left(\frac{8+7w}{8-7w}\right)\cdot\cos\{(2m+1)\cos^{-1}(w)\}}{\sqrt{1-w^2}}dw.$$
$$; m = 0, 1, 2, 3, ...$$

Substituting $w = \cos(\theta)$, give

$$c_{2m+1} = \frac{16}{\pi}\int_{0}^{\pi}\ln\left(\frac{8+7\cos(\theta)}{8-7\cos(\theta)}\right)\cdot\cos[(2m+1)\theta]d\theta$$
$$; m = 0, 1, 2, 3, ...$$

Thus,

$$\ln(x) = 14.0710w + 2.2924w^3 + 7.8370w^5 - 10.3825w^7 + 7.8342w^9 + R_{\ln}(w). \quad (37)$$

where

$$R_{\ln}(w) = 45.9461w^{11} - 36.9215w^{13} + 12.6416w^{15} + \cdots$$

(iii) By Definition C, Legendre series expansion of $\ln(x)$ is in the form

$$\ln(x) = 8\ln\left(\frac{8+7w}{8-7w}\right) = \sum_{m=0}^{\infty} a_{2m+1}P_{2m+1}(w). \quad (38)$$

Where,

$$a_{2m+1} = \frac{4m+3}{2}\int_{-1}^{1}8\ln\left(\frac{8+7w}{8-7w}\right)\cdot P_{2m+1}(w)dw$$
$$; m = 0, 1, 2, 3, ...$$

Thus,

$$\ln(x) = 14.0434w + 2.7213w^3 + 6.0845w^5 - 7.7786x^7 + 7.4231w^{11} + R_{\ln}(w). \quad (39)$$

where

$$R_{\ln}(w) = 34.9284w^{11} - 29.1304w^{13} + 10.4475w^{15} + \cdots$$

Table VI shows the absolute error between actual value and approximate values of $f(x) = \ln(x)$ derived from Taylor's, Chebyshev and Legendre series expansions in 5 terms, respectively. We can see that approximation of $\ln(x)$ derived from Taylor's series expansion provides value which is better than the ones derived from Chebyshev and Legendre series expansions. Graphs of approximation of $\ln(x)$ by Taylor's, Chebyshev and Legendre series expansions are shown in Fig 4.

TABLE VI
COMPARISON OF THE ABSOLUTE ERROR BETWEEN ACTUAL AND APPROXIMATE VALUES OF $\ln(x)$ BY TAYLOR'S, CHEBYSHEV AND LEGENDRE SERIES EXPANSIONS IN 5 TERMS.

x	w	*Absolute error (Approximate value in 5 terms)*		
		$E(T_w)$	$E(C_w)$	$E(L_w)$
0.5	-0.0495	3×10^{-10}	3.36×10^{-3}	2.05×10^{-3}
1	0	0	0	0
1.01	0.0007	1.48×10^{-9}	5.05×10^{-5}	3.08×10^{-5}
1.5	0.0290	3.80×10^{-9}	2.03×10^{-3}	1.24×10^{-3}
4	0.0988	1×10^{-9}	5.84×10^{-3}	3.51×10^{-3}
4.2	0.1022	4×10^{-9}	5.96×10^{-3}	3.58×10^{-3}
10	0.1633	0	6.71×10^{-3}	3.87×10^{-3}
30	0.2393	5×10^{-8}	3.82×10^{-3}	1.83×10^{-3}

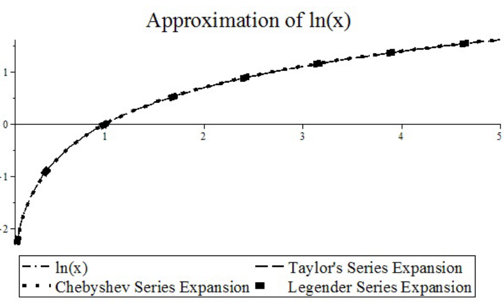

Fig. 4. Graphs of approximated values of $\ln(x)$ derived from Taylor's series expansion, Chebyshev series expansion and Legendre series expansion in 5 terms.

IV. APPLICATIONS

In this section, we illustrate the applications of approximated transcendental function in a power series form.

A. Application I

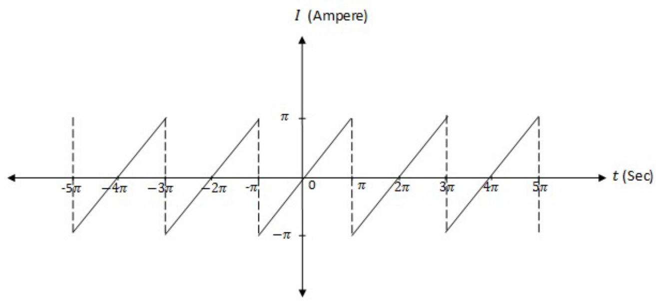

Fig. 5. Graph of the alternating current I.

Fig. 5 shows the profile of the saw-tooth wave arising from the alternating current I passing through conductor which has the form [19]

$$I = 2\left(\sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) + ...\right) \quad (40)$$

If we substitute in each term of current I in equation (40) by the representative of $f(x) = \sin(x)$ derived from Taylor's series in the new variable w (14), we found that the absolute error between actual value and approximate value of current I in 27 terms is 10^{-3} shown in Table VII.

TABLE VII
ABSOLUTE ERROR BETWEEN ACTUAL VALUE AND APPROXIMATE VALUE OF CURRENT I

Number terms of I	Absolute error
10	0.1010
15	0.0326
19	0.0139
25	0.0325
27	0.0079
30	0.0259

B. Application II

Root finding of equation $\sin(x) = \frac{x}{3\pi}$; $x \in (0, \pi]$ and $(2\pi, 3\pi]$.

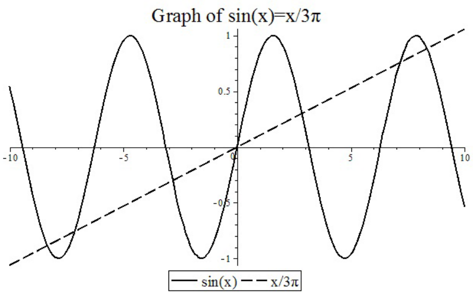

Fig. 6. Graph of $y_1 = \sin(x)$ and $y_2 = \frac{x}{3\pi}$.

Fig. 6 shows the solutions of equation $\sin(x) = \frac{x}{3\pi}$ in interval $(0, \pi]$ and $(2\pi, 3\pi]$.
Case I: $x \in (0, \pi]$, we can find a solution by solving

$$f(x) = \sin(x) - \frac{x}{3\pi} = 0, \quad x \in (0, \pi] \tag{41}$$

we use approximation of $\sin(x)$ in (41) by Taylor's series approximation in (14)

$$\sin(x) = 1 - \frac{1}{2!}(2\pi w)^2 + \frac{1}{4!}(2\pi w)^4 - \frac{1}{6!}(2\pi w)^6 + \frac{1}{8!}(2\pi w)^8 \tag{42}$$

where

$$w = \frac{\frac{\pi}{2} - x}{2\pi}, \; 0 < x < 2\pi, \quad -\frac{3}{4} < w < \frac{1}{4}.$$

The computation has terminated when $|\sin(x) - \frac{x}{3\pi}| < \epsilon$.

TABLE VIII
ROOT FINDING $\sin(x) = \frac{x}{3\pi}$ $(0 < x \leq \pi)$

x	w	$\sin(x) - \frac{x}{3\pi}$
$\pi - 0.1 = 3.041593$	-0.234085	-0.222877
$\pi - 0.2 = 2.941593$	-0.218169	-0.113437
$\pi - 0.3 = 2.841593$	-0.202254	-0.005979
$\pi - 0.301 = 2.840593$	-0.202094	-0.004918
$\pi - 0.302 = 2.839593$	-0.201935	-0.003857
$\vdots$	$\vdots$	$\vdots$
$\pi - 0.306 = 2.835593$	-0.201299	0.000384

We can see from Table VIII, that the approximated solution of the equation $\sin(x) = \frac{x}{3\pi}$ for $0 < x \leq \pi$ is $x^* = 2.835593$.
Case II: $2\pi < x \leq 3\pi$
By property ;

$$\sin(x) = \sin(3\pi - x) \; ; \quad x \in (2\pi, 3\pi] \tag{43}$$

That is, we can find solution by solving

$$f(x) = \sin(x) - \frac{x}{3\pi} = 0, \quad x \in (2\pi, 3\pi]$$

or

$$\sin(3\pi - x) - \frac{x}{3\pi} = 0. \tag{44}$$

which we can calculate by formula

$$x_{i+1} = \sin(3\pi - (x_0 - \delta_i)) \tag{45}$$

where initial data $x_0 = 2\pi + 0.2 = 6.483185$
and $\delta_i = ih, \; i = 0, 1, 2, ...N$;
we can guess the value of h by considering the value of $f(x)$.

We use property (43) and apply approximation of $\sin(x)$ to generate the function by Taylor's series expansion (42) to gain the results. The computation has terminated when $|\sin(x) - \frac{x}{3\pi}| < \epsilon$.

TABLE IX
ROOT FINDING $\sin(x) = \frac{x}{3\pi}$ $(2\pi < x \leq 3\pi)$

δ_i	x_0	$x_0 + \delta_i$	x	w	$\sin(x) - \frac{x}{3\pi}$
0.2	2π	$2\pi + 0.2$	2.94	-0.22	-0.48921
		$2\pi + 0.4$	2.74	-0.19	-0.31969
		$2\pi + 0.6$	2.54	-0.15	-0.16569
		$2\pi + 0.8$	2.34	-0.12	-0.03419
0.01	$2\pi + 0.8$	$2\pi + 0.81$	2.33	-0.12	-0.02832
		$2\pi + 0.82$	2.32	-0.12	-0.02253
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.001	$2\pi + 0.86$	$2\pi + 0.861$	2.28	-0.11	0.00047

We can see from Table IX, that the approximated solution of the equation $\sin(x) = \frac{x}{3\pi}$ for $2\pi < x \leq 3\pi$ is $x_1^* = 2\pi + 0.861 = 7.144185$.

From Figure 6, we can see that equation $\sin(x) = \frac{x}{3\pi}$ has two solutions for $x \in (2\pi, 3\pi]$ and we found the first solution from the previous computation. Similarly, we can find a second solution of equation $\sin(x) = \frac{x}{3\pi}$ for $x \in (2\pi, 3\pi]$ which is $x_2^* = 8.337778$.

V. CONCLUSION

In this work, the methods to generate approximation functions in a power series form have been presented focusing on Taylor's, Chebyshev and Legendre series expansions. We have developed Maple software to analyze the rate of convergence in an attempt to select the one with the least truncation error to represent the efficient representative of a transcendental function.

We have found that the Taylor's series expansion in the modified form with the substitution of an appropriate variable is most efficient.

The simplicity of the calculation by using the "substitution method" in the Taylor's series expansion enable the process to be carried out on a personal computer of medium capacity, thus yielding benefits to the theoretical studies of some electricity problems and other practical works in applied science.

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