The General L_p-Dual Mixed Brightness Integrals

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***Abstract*—Based on general L_p-mixed brightness integrals of convex bodies and general L_p-intersection bodies of star bodies, this paper is going to define the general L_p-dual mixed brightness integrals. After studying their extremum values and establishing Aleksandrov-Frenchel inequality, cyclic inequality and the Brunn-Minkowski inequality for the general L_p-dual mixed brightness integrals, we obtain a more general result than the Brunn-Minkowski inequality for the general L_p-dual mixed brightness integrals.**

***Index Terms*—general L_p-mixed brightness integrals, general L_p-intersection body, general L_p-dual mixed brightness integrals.**

I. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$. $\mathcal{S}_o^n$ denotes the set of star bodies (about the origin) in $\mathbf{R}^n$. Let S^{n-1} denote the unite sphere in $\mathbf{R}^n$, and let $V(K)$ denote the n-dimensional volume of body K. For the standard unit ball B in $\mathbf{R}^n$, we write $\omega_n = V(B)$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,.) : \mathbf{R}^n \to (-\infty, \infty)$, is defined by (see [2])

$$h(K,x) = \max\{x \cdot y : y \in K\}, x \in \mathbf{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For a compact set K in $\mathbf{R}^n$, which is star shaped with respect to the origin, the radial function, $\rho_K(u) = \rho(K,u)$, of K is defined by (see [2])

$$\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \qquad u \in S^{n-1}. \tag{1}$$

If ρ_K is positive and continuous, then K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $c > 0$ and $K \in \mathcal{S}_o^n$, then $\rho(cK, \cdot) = c\rho(K, \cdot)$.

Let $GL(n)$ denote the group of general (nonsingular) linear transformations, if $\phi \in GL(n)$, from (1), we easily have

$$\rho(\phi K, x) = \rho(K, \phi^{-1}x), \quad x \in \mathbf{R}^n \setminus \{0\}, \tag{2}$$

where ϕ^{-1} denotes the reverse of transformation ϕ.

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The notion of mixed brightness-integrals of convex bodies was defined by Li(see [8]). After that, Yan and Wang extended mixed brightness-integrals to the general mixed brightness-integrals of convex bodies: For $K_1, \cdots, K_n \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, the general L_p-mixed brightness integrals, $D_p^{(\tau)}(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n$ is defined by (see [22])

$$D_p^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \delta_p^{(\tau)}(K_1,u)\cdots\delta_p^{(\tau)}(K_n,u)dS(u),$$

where $\delta_p^{(\tau)}(K,u) = \frac{1}{2}h(\Pi_p^\tau K, u)$ denotes the half general L_p-brightness of $K \in \mathcal{K}_o^n$ and $\Pi_p^\tau K$ denotes the general L_p-projection body of $K \in \mathcal{K}_o^n$. Further, they established some inequalities for the general L_p-mixed brightness integrals(see [22]).

Recently, Wang and Li used the function $\varphi_\tau : \mathbf{R} \to [0, +\infty)$ which is given by

$$\varphi_\tau(t) = |t| - \tau t, \qquad \tau \in [-1,1] \tag{3}$$

to define the general L_p-intersection body with parameter τ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$, and $\tau \in [-1,1]$, the general L_p-intersection body, $I_p^\tau K \in \mathcal{S}_o^n$, of K is defined by (see [20])

$$\rho(I_p^\tau K, u)^p = i(\tau)\int_K \varphi_\tau(u \cdot x)^{-p}dx, \quad u \in S^{n-1}, \tag{4}$$

where

$$i(\tau) = \frac{(1+\tau)^p(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}.$$

In this paper, based on the general L_p-intersection bodies and the general L_p-mixed brightness integrals, we are going to define the general L_p-dual mixed brightness integrals of star bodies as follows:

For $K_1, \cdots, K_n \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1,1]$, the general L_p-dual mixed brightness integrals, $D_p^\tau(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n$ is defined by

$$D_p^\tau(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K_1,u)\cdots\delta_p^\tau(K_n,u)dS(u), \tag{5}$$

where $\delta_p^\tau(K,u) = \frac{1}{2}\rho(I_p^\tau K, u)$ denotes the half general L_p-dual brightness of $K \in \mathcal{S}_o^n$ in direction $u \in S^{n-1}$.

If $\tau = 0$, we write $D_p^\tau(K_1, \cdots, K_n) = D_p(K_1, \cdots, K_n)$ and $\delta_p(K,u) = \frac{1}{2}\rho(I_pK, u)$, then

$$D_p(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \delta_p(K_1,u)\cdots\delta_p(K_n,u)dS(u),$$

we call $D_p(K_1, \cdots, K_n)$ the L_p-dual mixed brightness integrals of $K_1, \cdots, K_n \in \mathcal{S}_o^n$.

Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$ $(i = 0, 1, \cdots, n)$, we write

$$D_{p,i}^{\tau}(K, L) = D_p^{\tau}(K, \cdots K, L \cdots, L).$$

If i is any real, $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, then the general L_p-dual mixed brightness integrals, $D_{p,i}^{\tau}(K, L)$, of K and L is defined by

$$D_{p,i}^{\tau}(K, L) = \frac{1}{n}\int_{S^{n-1}} \delta_p^{\tau}(K, u)^{n-i}\delta_p^{\tau}(L, u)^i dS(u). \quad (6)$$

Let $\tau = 0$ in (6), we write $D_{p,i}^{\tau}(K, L) = D_{p,i}(K, L)$.

Let $i = 0$ in (6), we write

$$D_{p,0}^{\tau}(K, K) = D_p^{\tau}(K) = \frac{1}{n}\int_{S^{n-1}} \delta_p^{\tau}(K, u)^n dS(u), \quad (7)$$

which is called the general L_p-dual brightness integrals of K.

Let $\tau = 0$ in (7), we write $D_p^{\tau}(K) = D_p(K)$; for $\tau = \pm 1$, we write $D_p^{\tau}(K) = D_p^{\pm}(K)$.

In this paper, we will establish the following inequalities for the general L_p-dual mixed brightness integrals.

Initinally, we give the extremal values of the general L_p-dual mixed brightness integrals.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, then*

$$D_p(K) \le D_p^{\tau}(K) \le D_p^{\pm}(K), \quad (8)$$

if K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Furthermore, we establish the following Fenchel-Aleksandrov type inequality for the general L_p-dual mixed brightness integrals.

Theorem 1.2. *If $K_1, \cdots, K_n \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$ and $1 < m \le n$, then*

$$D_p^{\tau}(K_1, \cdots, K_n)^m \le \prod_{i=1}^{m} D_p^{\tau}(K_1, \cdots, K_{n-m}, K_{n-i+1}, \cdots, K_{n-i+1}) \quad (9)$$

equality holds if and only if $I_p^{\tau}K_{n-m+1}, \cdots, I_p^{\tau}K_n$ are dilates of each other.

Let $\tau = 0$ in Theorem 1.2, we obtain the following inequality.

Corollary 1.1. *If $K_1, \cdots, K_n \in \mathcal{S}_o^n$, $0 < p < 1$ and $1 < m \le n$, then*

$$D_p(K_1, \cdots, K_n)^m \le \prod_{i=1}^{m} D_p(K_1, \cdots, K_{n-m}, K_{n-i+1}, \cdots, K_{n-i+1}),$$

equality holds if and only if $I_pK_{n-m+1}, \cdots, I_pK_n$ are dilates of each other.

Additionally, we establish the following cyclic inequality for the general L_p-dual mixed brightness integrals.

Theorem 1.3. *Iet $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, if $\frac{k-i}{k-j} > 1$, then*

$$D_{p,i}^{\tau}(K, L)^{k-j} D_{p,k}^{\tau}(K, L)^{j-i} \ge D_{p,j}^{\tau}(K, L)^{k-i}, \quad (10)$$

equality holds if and only if $I_p^{\tau}K$ and $I_p^{\tau}L$ are dilates. If $0 < \frac{k-i}{k-j} < 1$, the inequality (10) is reversed.

Let $\tau = 0$ in Theorem 1.3, we obtain the following inequality.

Corollary 1.2. *Iet $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, if $\frac{k-i}{k-j} > 1$, then*

$$D_{p,i}(K, L)^{k-j} D_{p,k}(K, L)^{j-i} \ge D_{p,j}(K, L)^{k-i}, \quad (11)$$

equality holds if and only if I_pK and I_pL are dilates. If $0 < \frac{k-i}{k-j} < 1$, the inequality (11) is reversed.

Finally, we obtain the Brunn-Minkowski type inequality for the general L_p-dual mixed brightness integrals as follows:

Theorem 1.4. *Iet $K, K', L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, if $i \le n - p$, then*

$$D_{p,i}^{\tau}(K \tilde{+}_{n-p} K', L)^{\frac{p}{n-i}} \le D_{p,i}^{\tau}(K, L)^{\frac{p}{n-i}} + D_{p,i}^{\tau}(K', L)^{\frac{p}{n-i}}, \quad (12)$$

equality holds if and only if $I_p^{\tau}K$ and $I_p^{\tau}K'$ are dilates. If $i \ge n - p$, the inequality (12) is reversed.

Actually, we prove a more general result than Theorem 1.4 in Section III.

Our work belongs to a new and rapidly evolving asymmetric L_p dual Brunn-Minkowski theory that has its own origin in the work of Ludwig, Haberl and Schuster (see [3], [4], [5], [6], [10], [11]). For the further researches of asymmetric L_p Brunn-Minkowski theory, we can refer to papers [1], [7], [14], [15], [16], [17], [18], [19], [20], [21].

II. Preliminaries

A. Dual mixed volumes

In 1975, Lutwak (see [9]) gave the notion of dual mixed volumes as follows: For $K_1, K_2, \cdots, K_n \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}(K_1, K_2, \cdots, K_n)$, of $K_1, K_2, \cdots, K_n$ is defined by

$$\widetilde{V}(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \quad (13)$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$ in (13), we write $\widetilde{V}_i(K, L) = \widetilde{V}(K, n-i; L, i)$, where K appears $n - i$ times and L appears i times. Let i be any real, we have

$$\widetilde{V}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i}\rho(L, u)^i dS(u). \quad (14)$$

Let $i = 0$ in (14), then

$$\widetilde{V}_0(K, L) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^n dS(u). \quad (15)$$

B. Some L_p-combinations

For $K, L \in \mathcal{S}_o^n, p > 0$ and $\lambda, \mu \ge 0$(not both zero), the L_p-radial linear combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by(see [12])

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p. \quad (16)$$

For $\phi \in GL(n)$, $K, L \in \mathcal{S}_o^n, p > 0$ and $\lambda, \mu \ge 0$ (not both zero), from (2) and (16), we have

$$\phi(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot) = \lambda \circ \phi K \tilde{+}_p \mu \circ \phi L. \quad (17)$$

For $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, from (4),(16) and a transformation to polar coordinate, we obtain

$$\rho(I_p^{\tau}(K \tilde{+}_{n-p} L), \cdot)^p = \rho(I_p^{\tau}K, \cdot)^p + \rho(I_p^{\tau}L, \cdot)^p, \quad (18)$$

i.e.,

$$I_p^\tau(K\tilde{+}_{n-p}L) = I_p^\tau K\tilde{+}_p I_p^\tau L. \tag{19}$$

From (17) and (19), we get

$$I_p^\tau(\phi(K\tilde{+}_{n-p}L)) = I_p^\tau \phi K\tilde{+}_p I_p^\tau \phi L. \tag{20}$$

III. Proofs of Theorems

In this section, firstly we shall prove Theorems 1.1-1.3 , then we prove a more general result than Theorem 1.4, i.e., a quotient form of the Brunn-Minkowski type inequality for the general L_p-dual mixed brightness integrals.

In order to prove Theorem 1.1, we need the following inequality (see [20]).

Lemma 3.1. *If $K \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, then*

$$V(I_pK) \le V(I_p^\tau K) \le V(I_p^\pm K). \tag{21}$$

If K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Proof of Theorem 1.1. If $K, L \in \mathcal{S}_o^n$, from (6), then

$$D_{p,i}^\tau(K,L) = \frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K,u)^{n-i}\delta_p^\tau(L,u)^i dS(u)$$

$$= \frac{1}{2^n}\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau K,u)^{n-i}\rho(I_p^\tau L,u)^i dS(u)$$

$$= \frac{1}{2^n}\widetilde{V}_i(I_p^\tau K, I_p^\tau L). \tag{22}$$

Let $i = 0$ in (22), and from (15), we have

$$D_{p,0}^\tau(K,L) = D_p^\tau(K) = \frac{1}{2^n}V(I_p^\tau K).$$

According to (21), we get

$$\frac{1}{2^n}V(I_pK) \le \frac{1}{2^n}V(I_p^\tau K) \le \frac{1}{2^n}V(I_p^\pm K).$$

i.e.,

$$D_p(K) \le D_p^\tau(K) \le D_p^\pm(K). \tag{23}$$

According to (21), we know that if K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$ in (23). And (23) is just the inequality (8).

The proof of Theorem 1.2 requires the following extension of the Hölder inequality (see [8] [13]).

Lemma 3.2. *If $f_0, f_1, \cdots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \cdots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u)f_1(u)\cdots f_m(u)dS(u)$$

$$\le \prod_{i=1}^m \left(\int_{S^{n-1}} f_0(u)f_i^{\lambda_i}(u)dS(u)\right)^{\frac{1}{\lambda_i}}, \tag{24}$$

with equality if and only if there exist positive constants $\alpha_1, \cdots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.2. If $K_1, \cdots, K_n \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, $1 < m \le n$, and let $\lambda_i = m (1 \le i \le m)$, and

$$f_0(u) = \delta_p^\tau(K_1,u)\cdots\delta_p^\tau(K_{n-m},u),\ (f_0 = 1\ if\ m = n),$$

$$f_i(u) = \delta_p^\tau(K_{n-i+1},u),\ (1 \le i \le m).$$

According to (24), we have

$$\int_{S^{n-1}} \delta_p^\tau(K_1,u)\cdots\delta_p^\tau(K_{n-m},u)\cdots\delta_p^\tau(K_n,u)dS(u)$$

$$\le \prod_{i=1}^m \Bigg(\int_{S^{n-1}} \delta_p^\tau(K_1,u)\times\cdots$$

$$\times\delta_p^\tau(K_{n-m},u)\delta_p^\tau(K_{n-i+1},u)^m dS(u)\Bigg)^{\frac{1}{m}}.$$

So

$$\left(\int_{S^{n-1}} \delta_p^\tau(K_1,u)\cdots\delta_p^\tau(K_{n-m},u)\cdots\delta_p^\tau(K_n,u)dS(u)\right)^m$$

$$\le \prod_{i=1}^m \int_{S^{n-1}} \delta_p^\tau(K_1,u)\cdots\delta_p^\tau(K_{n-m},u)\delta_p^\tau(K_{n-i+1},u)^m dS(u).$$

i.e.,

$$D_p^\tau(K_1,\cdots,K_n)^m$$

$$\le \prod_{i=1}^m D_p^\tau(K_1,\cdots,K_{n-m},K_{n-i+1},\cdots,K_{n-i+1}). \tag{25}$$

The equality condition in (25) can be got from the equality condition in inequality (24) if and only if there exist positive constants $\alpha_1, \cdots, \alpha_m$ such that

$$\alpha_1\delta_p^\tau(K_{n-m+1},u)^m = \alpha_2\delta_p^\tau(K_{n-m+2},u)^m$$

$$= \cdots = \alpha_m\delta_p^\tau(K_n,u)^m$$

for all $u \in S^{n-1}$. So equality holds in (25) if and only if $I_p^\tau K_{n-m+1}, \cdots, I_p^\tau K_n$ are dilates of each other. And (25) is just the inequality (9).

Proof of Theorem 1.3. Let $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1, 1]$, if $\frac{k-i}{k-j} > 1$, according to (6), and the Hölder inequality, we have

$$D_{p,i}^\tau(K,L)^{\frac{k-j}{k-i}}D_{p,k}^\tau(K,L)^{\frac{j-i}{k-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K,u)^{n-i}\delta_p^\tau(L,u)^i dS(u)\right)^{\frac{k-j}{k-i}}$$

$$\times\left(\frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K,u)^{n-k}\delta_p^\tau(L,u)^k dS(u)\right)^{\frac{j-i}{k-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} [\delta_p^\tau(K,u)^{(n-i)\frac{k-j}{k-i}}\delta_p^\tau(L,u)^{i\frac{k-j}{k-i}}]^{\frac{k-i}{k-j}} dS(u)\right)^{\frac{k-j}{k-i}}$$

$$\times\left(\frac{1}{n}\int_{S^{n-1}} [\delta_p^\tau(K,u)^{(n-k)\frac{j-i}{k-i}}\delta_p^\tau(L,u)^{k\frac{j-i}{k-i}}]^{\frac{k-i}{j-i}} dS(u)\right)^{\frac{j-i}{k-i}}$$

$$\ge \frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K,u)^{(n-i)\frac{k-j}{k-i}}\delta_p^\tau(K,u)^{(n-k)\frac{j-i}{k-i}}$$

$$\delta_p^\tau(L,u)^{k\frac{j-i}{k-i}}\delta_p^\tau(L,u)^{i\frac{k-j}{k-i}} dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(K,u)^{n-j}\delta_p^\tau(L,u)^j dS(u) = D_{p,j}^\tau(K,L).$$

i.e.,

$$D_{p,i}^\tau(K,L)^{\frac{k-j}{k-i}}D_{p,k}^\tau(K,L)^{\frac{j-i}{k-i}} \ge D_{p,j}^\tau(K,L). \tag{26}$$

The equality condition in (26) can be got from the equality condition in the Hölder inequalitiy if and only if $I_p^\tau K$ and

$I_p^\tau L$ are dilates. Similarly, if $0 < \frac{k-i}{k-j} < 1$, we can obtain the reverse form of (26). And (26) is just the inequality (10).

Now, we give a more general result than Theorem 1.4 as follows:

Theorem 3.1. *For $K, L, K^{'} \in \mathcal{S}_o^n$, $\phi \in GL(n)$, $0 < p < 1$, $\tau \in [-1, 1]$, if $0 \le n - j \le p \le n - i$, then*

$$\left(\frac{D_{p,i}^\tau(\phi(K\tilde{+}_{n-p}K^{'}), L)}{D_{p,j}^\tau(\phi(K\tilde{+}_{n-p}K^{'}), L)}\right)^{\frac{p}{j-i}} \le \left(\frac{D_{p,i}^\tau(\phi K, L)}{D_{p,j}^\tau(\phi K, L)}\right)^{\frac{p}{j-i}} + \left(\frac{D_{p,i}^\tau(\phi K^{'}, L)}{D_{p,j}^\tau(\phi K^{'}, L)}\right)^{\frac{p}{j-i}}, \quad (27)$$

with equality holds in (27) if and only if $I_p^\tau \phi K$ and $I_p^\tau \phi K^{'}$ are dilates. If $n - j \le 0 < n - i \le p$, the inequality (27) is reversed.

The proof of Theorem 3.1 requires the following Dresher's inequality(see [23]).

Lemma 3.3. *Let functions $f_1, f_2, g_1, g_2 \ge 0$, E is a bounded measurable subset in $\mathbf{R}^n$. If $p \ge 1 \ge r \ge 0$, then*

$$\left(\frac{\int_E (f_1 + f_2)^p dx}{\int_E (g_1 + g_2)^r dx}\right)^{\frac{1}{p-r}} \le \left(\frac{\int_E f_1^p dx}{\int_E g_1^r dx}\right)^{\frac{1}{p-r}} + \left(\frac{\int_E f_2^p dx}{\int_E g_2^r dx}\right)^{\frac{1}{p-r}}, \quad (28)$$

equality holds if and only if $\frac{f_1}{f_2} = \frac{g_1}{g_2}$. If $1 \ge p > 0 > r$, the inequality (28) is reversed.

Proof of Theorem 3.1. For $K, K^{'}, L \in \mathcal{S}_o^n$, $\phi \in GL(n)$, $0 < p < 1$, $\tau \in [-1, 1]$, if $0 \le n - j \le p \le n - i$, according to (16),(20) and (28), we have

$$\begin{aligned}
&\left(\frac{D_{p,i}^\tau(\phi(K\tilde{+}_{n-p}K^{'}), L)}{D_{p,j}^\tau(\phi(K\tilde{+}_{n-p}K^{'}), L)}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{\frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(\phi(K\tilde{+}_{n-p}K^{'}), u)^{n-i} \delta_p^\tau(L, u)^i dS(u)}{\frac{1}{n}\int_{S^{n-1}} \delta_p^\tau(\phi(K\tilde{+}_{n-p}K^{'}), u)^{n-j} \delta_p^\tau(L, u)^j dS(u)}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau(\phi(K\tilde{+}_{n-p}K^{'})), u)^{n-i} \rho(I_p^\tau L, u)^i dS(u)}{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau(\phi(K\tilde{+}_{n-p}K^{'})), u)^{n-j} \rho(I_p^\tau L, u)^j dS(u)}\right)^{\frac{p}{j-i}} \\
&\le \left(\frac{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau \phi K, u)^{n-i} \rho(I_p^\tau L, u)^i dS(u)}{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau \phi K, u)^{n-j} \rho(I_p^\tau L, u)^j dS(u)}\right)^{\frac{p}{j-i}} \\
&+ \left(\frac{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau \phi K^{'}, u)^{n-i} \rho(I_p^\tau L, u)^i dS(u)}{\frac{1}{n}\int_{S^{n-1}} \rho(I_p^\tau \phi K^{'}, u)^{n-j} \rho(I_p^\tau L, u)^j dS(u)}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{D_{p,i}^\tau(\phi K, L)}{D_{p,j}^\tau(\phi K, L)}\right)^{\frac{p}{j-i}} + \left(\frac{D_{p,i}^\tau(\phi K^{'}, L)}{D_{p,j}^\tau(\phi K^{'}, L)}\right)^{\frac{p}{j-i}}.
\end{aligned}$$

The equality condition in (27) can be got from the equality condition in (28) if and only if $I_p^\tau \phi K$ and $I_p^\tau \phi K^{'}$ are dilates.

If $n - j \le 0 < n - i \le p$, similarly, we can prove that the reverse of the inequality (27) is true.

If ϕ is identic, then we get the following inequality.

Theorem 3.2. *For $K, L, K^{'} \in \mathcal{S}_o^n$, $0 < p < 1$, if $0 \le n - j \le p \le n - i$, then*

$$\left(\frac{D_{p,i}^\tau(K\tilde{+}_{n-p}K^{'}, L)}{D_{p,j}^\tau(K\tilde{+}_{n-p}K^{'}, L)}\right)^{\frac{p}{j-i}} \le \left(\frac{D_{p,i}^\tau(K, L)}{D_{p,j}^\tau(K, L)}\right)^{\frac{p}{j-i}} + \left(\frac{D_{p,i}^\tau(K^{'}, L)}{D_{p,j}^\tau(K^{'}, L)}\right)^{\frac{p}{j-i}}, \quad (29)$$

with equality holds in (29) if and only if $I_p^\tau K$ and $I_p^\tau K^{'}$ are dilates. If $n - j \le 0 < n - i \le p$, the inequality (29) is reversed.

Proof of Theorem 1.4. Let $j = n$ in the inequality (29) and notice that $D_{p,n}^\tau(M, L) = D_p^\tau(L)$ by (6), for $i \le n - p$ and any $L \in \mathcal{S}_o^n$, we get

$$D_{p,i}^\tau(K\tilde{+}_{n-p}K^{'}, L)^{\frac{p}{n-i}} \le D_{p,i}^\tau(K, L)^{\frac{p}{n-i}} + D_{p,i}^\tau(K^{'}, L)^{\frac{p}{n-i}}, \quad (30)$$

which is just the inequality (12). From the equality condition of (29), we see that equality holds in (30) if and only if $I_p^\tau K$ and $I_p^\tau K^{'}$ are dilates.

Similarly, let $j = n$ in the reverse of the inequality (29), and for $i \ge n - p$, we can obtain that the reverse of the inequality (30) is true.

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