The Boundary Value Problems of Higher Order Mixed Type of Delay Differential Equations

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Abstract—**In this paper, by using a fixed-point theorem in cones to study the boundary value problem for a class of higher order mixed type of delay differential equations with singularity. The sufficient condition of existence of their solutions is derived. Some examples are included to illustrate the results.**

Index Terms—**Boundary value problem, higher order mixed type of delay differential equations, positive solution, fixed point.**

I. Introduction

IN this paper, we consider the existence of positive solutions to the following higher order mixed-type of delay differential equations

$$\begin{cases} -u^{(n)}(t) = \lambda p(t) f[t, u(t-\tau), \int_0^t k(t,s)u(s)ds], \\ \qquad 0 < t < 1, \tau > 0, \\ u(t) = u'(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \\ \qquad -\tau \le t \le 0, \\ u^{(n-2)}(1) = (n-1)! a u(\eta). \end{cases} \quad (1.1)$$

where λ is a positive real parameter; $n \ge 2$ is an integer.

In equation (1.1), we assume that the following conditions $(H_1)-(H_5)$ hold

$(H_1) f \in C(J \times R \times R, R), J = [0,1], 0 < a \le 1,$
$0 < \eta < 1, 0 < \tau < \frac{1}{2}.$

$(H_2) p(s) \in C(J_1, R^+), J_1 = (0,1).$

$(H_3) v(t) = \int_0^t k(t,s)u(s)ds, D = \{(t,s) \in J \times J : t > s\}, k(t,s) \in C(D, R^+).$
$k_0 = \min\{k(s,t) : (s,t) \in D\},$
$k_1 = \max\{k(s,t) : (s,t) \in D\}$

$(H_4) u \in C[-\tau, 1] \bigcap C^{(n)}[0,1]; u(t) \ge 0, t \in [-\tau, 1].$

$(H_5) \int_0^1 s(1-s)p(s)ds < \infty, \exists \theta[\frac{\tau}{2}, \frac{1-\tau}{2})$
such that $\int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds > 0.$

Boundary value problems(BVPS) for higher-order delay differential equations arise naturally in various applications to physical, biological, and chemical processes. Frequently, these occur in the form of a multipoint boundary value problem for an n^{th}-order ordinary differential equations, such as an n-degrees of freedom in which n states are observed at n times[2,3]. In recent years, many researchers have done a great deal of research works upon lower order differential equations with delay, and some good results were produced, see, for example [4-22]. But higher order cases haven not been focused. BVPS of higher-order differential equations have received a few of attention (see [23,24,25,26]).

For $n = 2, v = 0$, in the case of ordinary differential equations, BVPs analogous to (1.1) with singularity has been widely studies by many authors, see,for example, [12,15-17]. For the case $\tau = 0, v = 0$, and $a = 0$, Graef and Yang [23] obtain respectfully the existence of positive solutions to two-point and multi-point BVPs (1.1) when $u^{(n)} = \lambda p(t) f(u(t))$. In the case $v = 0, a = 0$, Shen and Dong [24]have applied a fixed-point theorem to derive the sufficient conditions which assure that the equation $(-u^{(n)} = \lambda p(t) f(t, u(t-\tau)), 0 < t < 1)$ with the boundary conditions $(u(0) = u'(0) = \cdots = u^{(n-3)} = u^{(n-2)} = 0, -\tau \le t \le 0; u^{(n-2)}(1) = 0)$ have the positive solutions, and $p(t)$ has some suitable singularity at the ends of (0,1).

In the paper, we prove that the existence of positive solutions of the more general BVPs for n-order $(n \ge 2)$ differential equations (1.1).

For the existence of positive solutions of the boundary value problem of two or higher order differential equations, we mainly adopt the scheme which transforms it into integral equations. During the process of transformation, several kinds of Green functions play important roles. Here Green functions are defined as follows

$$G_2(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1, \\ t(1-s), 0 \le t \le s \le 1. \end{cases}$$

For $n \ge 3$, we define

$$G_n(t,s) = \int_0^t G_{n-1}(v,s)dv.$$

Lemma 1.1. $G_n(t,s)$ satisfies:

(i) $G_n(t,s) \le G_2(s,s), \quad (t,s) \in [0.1] \times [0,1],$
$n \ge 2, n \in N.$

(ii) Let $0 < \theta \le 1-\theta \le 1-\tau, \quad J_\theta = [\theta, 1-\theta],$
for $t \in J_\theta, s \in [0,1]$, one has

$$G_2(t,s) \ge \min\{t, 1-t\} G_2(s,s) \ge \theta G_2(s,s), \quad (1.2)$$

$$G_n(t,s) \ge \theta^{n-1} G_2(s,s), t \in J_\theta, \quad n \ge 2, n \in N. \quad (1.3)$$

Proof At first, we prove the conclusion (i) of Lemma 1.1 by induction.

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Clearly, $G_2(t,s) \leq G_2(s,s)$.

Assuming that when $n=k$, $G_k(t,s) \leq G_2(s,s)$.

Then $n=k+1$, for $(t,s) \in [0.1] \times [0,1]$

$$\begin{aligned} G_{k+1}(t,s) &= \int_0^t G_k(v,s)dv, \\ &\leq \int_0^1 G_2(s,s)dv, \\ &= G_2(s,s). \end{aligned}$$

Therefore the conclusion (i) of Lemma 1.1 holds.

For the conclusion (ii) of Lemma 1.1, the relation formula (1.2) is clear. We prove the relation formula (1.3) in the following.

For $t \in J_\theta$, by (1.2) we have $G_2(t,s) \geq \theta G_2(s,s)$.

Assuming when $n=k$, $G_k(t,s) \geq \theta^{k-1} G_2(s,s)$.

Then when $n=k+1$, $t \in J_\theta$

$$\begin{aligned} G_{k+1}(t,s) &= \int_0^t G_k(v,s)dv, \\ &\geq \int_0^t \theta^{k-1} G_2(s,s)dv, \\ &= t\theta^{k-1} G_2(s,s), \\ &\geq \theta^k G_2(s,s). \end{aligned}$$

Therefor (1.3) holds.

Let

$$\begin{aligned} E = \{u \in C[-\tau,1] : u(t) \geq 0, \quad for \quad t \in J; \quad u(t) = \\ u^{'}(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \\ for \ t \in [-\tau,0]; \quad u^{(n-2)}(1) = (n-1)!au(\eta)\}. \end{aligned}$$

With the norm $\|\cdot\|$ given by $||u|| = \sup\{|u(t)| : -\tau \leq t \leq 1\}$, then $(E, \quad \|\cdot\|)$ is a Banach space. It is obvious that$\|\cdot\| = \|\cdot\|_{[0,1]}$ for $u \in E$.

Define a cone $K \in E$ by

$$K = \{u \in E : u(t) \geq 0, fort \in [0,1]; \quad \min_{t \in J_\theta} u(t) \geq \gamma ||u||\},$$

where $\gamma = \frac{\theta^{n-1}(1-a\eta^{n-1})}{1+a-a\eta^{n-1}}$.

II. Some preliminaries

For convenience of the reader, in the section, we also present some definitions and some lemmas.

Definition 2.1. Let X be a real Banach space and $K \in X$ be a closed, convex set. K is a cone if only if the following conditions are satisfied

(i) $\lambda u \in K$,if $\lambda > 0$ and $u \in K$.

(ii) if $u \in K$ and $-u \in K$,then $u = 0$.

Definition 2.2. $u(t)$ is the positive solution of BVP(1.2) if and only if it satisfies the following conditions:

(i) $u \subset C[-\tau,1] \cap C^n; u(t) > 0, t \in (0,1)$.

(ii) When $t \in [-\tau,0]$, $u(t) = u^{'}(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0$, and $u(1) = (n-1)!au(\eta)(0 < \eta < 1)$.

(iii) $u^{(n)}(t) = -\lambda p(s)f(t, u(t-\tau), v(t)), \forall t \in (0,1)$.

If $u(t)$ is the solution of BVP (1.1), then $u(t)$ can be represented as

$$u(t) = \begin{cases} 0, -\tau \leq t \leq 0. \\ \lambda \int_0^1 G_n(t,s)p(s)f(s,u,v)ds + \\ \frac{a\lambda t^{n-1}}{1-a\eta^{n-1}} \int_0^1 G_n(\eta,s)p(s)f(s,u,v)ds, 0 < t < 1. \end{cases}$$

We define the operator $\Phi : C[-\tau,1] \to C[-\tau,1]$ by

$$\Phi u(t) = \begin{cases} 0, -\tau \leq t \leq 0. \\ \lambda \int_0^1 G_n(t,s)p(s)f(s,u,v)ds + \\ \frac{a\lambda t^{n-1}}{1-a\eta^{n-1}} \int_0^1 G_n(\eta,s)p(s)f(s,u,v)ds, 0 < t < 1. \end{cases}$$

Based on the above, we derive the following lemmas.

Lemma 2.1. The fixed-point of the map Φ is the solution of equation (1.1).

Proof It's easy to get

$\Phi u(t) = \Phi^{'} u(t) = \cdots = \Phi^{(n-3)} u(t) = \Phi^{(n-2)} u(t) = 0, -\tau \leq t \leq 0,$

$\Phi^{(n-2)} u(1) = (n-1)!a\Phi u(\eta).$

We prove by reduction that $\Phi^{(n)} u(t) = -\lambda p(t) f(t,u,v)$ also hold.

For $n=2$, calculate easily $\Phi'' u(t) = -\lambda p(t) f(t,u,v)$.

Assuming that when $n=k$, $\Phi^{(k)} u(t) = -\lambda p(t) f(t,u,v)$.

Then $n=k+1$, from

$$\begin{aligned} \Phi u(t) &= \lambda \int_0^1 G_{k+1}(t,s)p(s)f(s,u,v)ds \\ &+ \frac{a\lambda t^k}{1-a\eta^k} \int_0^1 G_{k+1}(\eta,s)p(s)f(s,u,v)ds, \\ &= \lambda \int_0^1 \Big(\int_0^t G_k(x,s)dx \Big) p(s)f(s,u,v)ds \\ &+ \frac{a\lambda t^k}{1-a\eta^k} \int_0^1 G_{k+1}(\eta,s)p(s)f(s,u,v)ds. \end{aligned}$$

one has that

$$\begin{aligned} \Phi^{'} u(t) = \lambda \int_0^1 G_k(t,s)p(s)f(s,u,v)ds \\ + \frac{ak\lambda t^{k-1}}{1-a\eta^k} \int_0^1 G_{k+1}(\eta,s)p(s)f(s,u,v)ds, \end{aligned}$$

$$\Phi^{(k+1)} u(t) = (\Phi^{'} u(t))^{(k)} = -\lambda p(t) f(t,u,v).$$

Therefore the fixed point of Φ is the solution of the equation (1.1). The proof is complete.

Lemma 2.2. $\Phi : K \to K$ is a completely continuous operator.

Proof Clearly, we have $||\Phi u|| = ||\Phi u||_{[0,1]}, \Phi u(t) \geq 0, \forall u(t) \in K$, and

$$\begin{aligned} ||\Phi u|| &\leq \lambda \int_0^1 G_2(s,s)p(s)f(s,u,v)ds \\ &+ \frac{a\lambda}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)f(s,u,v)ds, \\ &= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)f(s,u,v)ds. \end{aligned}$$

for $t \in J_\theta$, we have

$$\begin{aligned} \Phi u(t) &\geq \lambda \int_0^1 G_n(t,s)p(s)f(s,u,v)ds \\ &\geq \lambda\theta^{n-1} \int_0^1 G_2(s,s)p(s)f(s,u,v)ds, \\ &= \frac{\theta^{n-1}(1-a\eta^{n-1})}{1+a-a\eta^{n-1}} ||\Phi u||_{[0,1]} \geq \gamma ||\Phi u||. \end{aligned}$$

Then $\Phi : K \to K$. Because Φ is a sequential compact set, we can conclude that Φ is a completely continuous operator by Arzela-Ascoli Theorem.

Lemma 2.3 [1] . Let X be a Banach space, K a conic in X, Ω_1, Ω_2 two open subsets in X, and $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $\Phi : K \cap (\Omega_2 \backslash \Omega_1) \to K$ is a completely continuous operator and satisfies

(i) $||\Phi u|| \leq ||u||, u \in K \cap \partial\Omega_1$ and $||\Phi u|| \geq ||u||, u \in K \cap \partial\Omega_2$, or

(ii) $||\Phi u|| \leq ||u||, u \in K \cap \partial\Omega_2$ and $||\Phi u|| \geq ||u||, u \in K \cap \partial\Omega_1$,

then Ω has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Let α, β, φ and ϕ be non-negative continuous concave functional on K. Then for positive real numbers b, c, d and m, we define the following convex sets:

$P(\beta, m) = \{x \in K | \beta(x) < m\}$,

$P(\beta, \alpha, b, m) = \{x \in K | b \leq \alpha(x), \beta(x) \leq m\}$,

$P(\beta, \varphi, \alpha, b, d, m) = \{x \in K | b \leq \alpha(x), \varphi(x) \leq d, \beta(x) \leq m\}$, and a closed set

$R(\beta, \phi, c, m) = \{x \in K | c \leq \phi(x), \beta(x) \leq m\}$.

Lemma 2.4 [4]. Let K be a cone in a real Banach space X. Let β and φ be non-negative continuous convex functionals on K, α be a non-negative continuous concave functional on K, and ϕ be a non-negative continuous functional on K satisfying $\phi(\lambda u) \leq \lambda\phi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and m,

$$\alpha(u) \leq \phi(u), \ \|u\| \leq M\beta(u), \ for\ all\ u \in \overline{P(\beta, m)}.$$

Suppose $\Phi : \overline{P(\beta, m)} \longrightarrow \overline{P(\beta, m)}$ is completely continuous and there exist positive numbers b, c and m with $b > c$ such that

$(S_1)\{u \in P(\beta, \varphi, \alpha, b, d, m) | \alpha(u) > b\} \neq 0$, and $\alpha(\Phi u) > b$ for $u \in P(\beta, \varphi, \alpha, b, d, m)$.

$(S_2)\alpha(\Phi(u)) > b$ for $P(\beta, \alpha, b, m)$ with $\varphi(\Phi(u)) > d$.

$(S_3)0 \notin R(\beta, \phi, c, m)$ and $\phi(\Phi(u)) < c$ for $u \in R(\beta, \phi, b, m)$ with $\phi(\Phi(u)) = c$.

then Φ has at least three fixed points $u_{1,} u_2, u_3 \in \overline{P(\beta, m)}$, such that

$\beta(u_i) \leq m$, for $i = 1, 2, 3$;

$b < \alpha(u_1)$;

$c < \phi(u_2)$, with $\alpha(u_2) < b$;

$\phi(u_3) < c$.

III. The case of no less than one solution

Let

$$M_0 = \lim_{\substack{u \to 0 \\ v \to 0}} \inf \min_{t \in [-\tau, 1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}.$$

$$M_\infty = \lim_{\substack{u \to \infty \\ v \to \infty}} \inf \min_{t \in [-\tau, 1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}.$$

$$M^0 = \lim_{\substack{u \to 0 \\ v \to 0}} \sup \max_{t \in [-\tau, 1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}.$$

$$M^\infty = \lim_{\substack{u \to \infty \\ v \to \infty}} \sup \max_{t \in [-\tau, 1]} \frac{f(t,u,v)}{\sqrt{u^2+v^2}}.$$

$$L_1 = \lambda\gamma\theta^{n-1} \int_{\theta+\tau}^{1-\theta} G_2(s,s)p(s)(1+k_0 s)ds.$$

$$L_2 = \frac{1+a-a\eta^{n-1}}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)(1+k_1 s)ds.$$

In the following, we discuss the existence of at least the positive solutions for all kinds of values and compositions of M_0, M_∞, M^0 and M^∞. In the theorem 3.1 and theorem 3.2, we take $\varepsilon > 0$, such that satisfy $(M_0 - \varepsilon) > 0, (M_\infty - \varepsilon) > 0$.

Theorem 3.1 If the conditions$(H_1) - (H_5)$and the following conditions

$$0 < M_\infty < +\infty, \qquad (3.1)$$

$$0 < M^0 < +\infty, \qquad (3.2)$$

$$\frac{1}{L_1(M_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{L_2(M^0 + \varepsilon)}, \qquad (3.3)$$

hold, then the equation (1.2) has at least one solution.

Proof By (3.2),(3.3), for a given $\varepsilon > 0, \exists r_1 > 0$, when $0 < \sqrt{u^2+v^2} \leq r_1, f(t,u,v) \leq (M^0 + \varepsilon)\sqrt{u^2+v^2}$. Let

$$\Omega_1 = \{t \in [-\tau, 1] : ||\sqrt{u^2+v^2}|| < r_1\},$$

for $u, v \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
||\Phi u|| &\leq \lambda \int_0^1 G_n(s,s) f(s, u(s-\tau), v) ds \\
&+ \frac{\lambda a}{1 - a\eta^{n-1}} \int_0^1 G_n(s,s) f(s, u(s-\tau), v) ds, \\
&= \frac{\lambda(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\int_0^1 G_n(s,s) p(s) f(s, u(s-\tau), v) ds, \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\int_0^1 G_2(s,s) p(s) \sqrt{u^2(s-\tau) + v^2} ds, \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\int_0^1 G_2(s,s) p(s) [u(s-\tau) + v] ds, \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\Big[\int_0^1 G_2(s,s) p(s) u(s-\tau) ds \\
&+ \int_0^1 G_2(s,s) p(s) \int_0^s k(t,s) u(t) dt ds\Big], \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\Big[\int_0^{1-\tau} G_2(s+\tau, s+\tau) p(s+\tau) u(s) ds \\
&+ k_1 \int_0^1 G_2(s,s) p(s) \int_0^s u(t) dt ds\Big], \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\Big[\int_0^{1-\tau} G_2(s+\tau, s+\tau) p(s+\tau) ds \\
&+ k_1 \int_0^1 G_2(s,s) p(s) s ds\Big] ||u||, \\
&\leq \frac{\lambda(M^0 + \varepsilon)(1 + a - a\eta^{n-1})}{1 - a\eta^{n-1}} \times \\
&\Big[\int_0^1 G_2(s,s) p(s)(1 + k_1 s) ds\Big] ||u|| \\
&= \lambda(M^0 + \varepsilon) L_2 ||u|| \leq ||u||.
\end{aligned}$$

For the same above $\varepsilon > 0$, from (3.1) and (3.3), $\exists R_1 > r$, when $\sqrt{u^2 + v^2} \geq R_1, f(t,u,v) > (M_\infty - \varepsilon)\sqrt{u^2 + v^2}$. Since $\gamma < \frac{1-\tau}{2}$, then $\gamma + \tau < 1 - \gamma$. Let

$$\Omega_2 = \{t \in [-\tau, 1] : ||\sqrt{u^2 + v^2}|| < R_1\},$$

for $u, v \in K \cap \partial\Omega_2$, we get

$$\begin{aligned}
||\Phi u|| &\geq \lambda \sup_{t \in J} \int_0^1 G_n(t,s) p(s) f(s,u,v) ds, \\
&\geq \lambda(M_\infty - \varepsilon) \sup_{t \in J} \int_0^1 G_n(t,s) P(s) \sqrt{u^2 + v^2} ds, \\
&\geq \lambda(M_\infty - \varepsilon) \sup_{t \in J} \int_0^1 G_n(t,s) P(s) \frac{u+v}{2} ds, \\
&= \frac{\lambda(M_\infty - \varepsilon)}{2} \sup_{t \in J} \Big[\int_0^1 G_n(t,s) P(s) u(s-\tau) ds \\
&+ \int_0^1 G_n(t,s) p(s) \int_0^s K(t,s) u(t) dt ds\Big], \\
&\geq \frac{\lambda(M_\infty - \varepsilon)}{2} \sup_{t \in J} \Big[\int_{-\tau}^{1-\tau} G_n(t, s+\tau) p(s+\tau) \\
&u(s) ds + k_0 \int_0^1 G_n(t,s) p(s \int_0^s k_0) u(t) dt ds\Big], \\
&\geq \frac{\lambda(M_\infty - \varepsilon)}{2} \sup_{t \in J} \Big[\int_\theta^{1-\theta} G_n(t, s+\tau) p(s+\tau)\gamma \\
&||u|| ds + k_0 \int_\theta^{1-\theta} G_n(t,s) p(s) s\gamma ||u|| dt ds\Big], \\
&\geq \frac{\lambda\gamma(M_\infty - \varepsilon)}{2} \sup_{t \in J} \Big[\int_{\theta+\tau}^{1-\theta+\tau} G_n(t,s) p(s) ||u|| ds \\
&+ k_0 ||u|| \int_\theta^{1-\theta} G_n(t,s) p(s) s ds\Big], \\
&\geq \frac{(M_\infty - \varepsilon)\lambda\gamma\theta^{n-1}}{2} \times \\
&\Big[\int_{\theta+\tau}^{1-\theta} G_2(s,s) p(s)(1 + k_0 s) ds\Big] ||u||, \\
&= \lambda(M_\infty - \varepsilon) L_1 \geq ||u||.
\end{aligned}$$

Therefore, by Lemma 2.3, Φ has a fixed point $u \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, and $u(t)$ is a positive solution of equation (1.2), completing the proof of Theorem 3.1

Theorem 3.2 If the conditions $(H_1) - (H_5)$ and the following conditions

$$0 < M_0 < +\infty, \tag{3.4}$$

$$0 < M^\infty < +\infty, \tag{3.5}$$

$$\frac{1}{L_1(M_0 - \varepsilon)} \leq \lambda \leq \frac{1}{L_2(M^\infty + \varepsilon)}, \tag{3.6}$$

hold, then the equation (1.2) has at least one solution.

Proof By (3.4) and (3.6), for a given $\varepsilon > 0, \exists r_2 > 0$, when $\sqrt{u^2 + v^2} \leq r_2, f(t,u,v) \geq (M_0 - \varepsilon)\sqrt{u^2 + v^2}$. Let

$$\Omega_1 = \{t \in [-\tau, 1] : ||\sqrt{u^2 + v^2}|| < r_2\},$$

for $u, v \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
||\Phi u|| &\geq \lambda \sup_{t \in J} \int_0^1 G_n(t,s) p(s) f(s,u,v) ds, \\
&\geq \lambda(M_0 - \varepsilon) \sup_{t \in J} \int_0^1 G_n(t,s) p(s) \sqrt{u^2 + v^2} ds, \\
&\geq \lambda(M_0 - \varepsilon) \sup_{t \in J} \int_0^1 G_n(t,s) p(s) \frac{u+v}{2} ds, \\
&= \frac{\lambda(M_0 - \varepsilon)}{2} \sup_{t \in J} \Big[\int_0^1 G_n(t,s) p(s) u(s-\tau) ds \\
&+ \int_0^1 G_n(t,s) p(s) \int_0^s K(t,s) u(t) dt ds\Big], \\
&\geq \frac{\lambda(M_0 - \varepsilon)}{2} \sup_{t \in J} \Big[\int_{-\tau}^{1-\tau} G_n(t, s+\tau) p(s+\tau)
\end{aligned}$$

$$
\begin{aligned}
& u(s)ds + k_0\int_0^1 G_n(t,s)p(s\int_0^s k_0)u(t)dtds\Big],\\
\geq\ & \frac{\lambda(M_0-\varepsilon)}{2}\sup_{t\in J}\Big[\int_\theta^{1-\theta} G_n(t,s+\tau)p(s+\tau)\\
& \gamma||u||ds + k_0\int_\theta^{1-\theta} G_n(t,s)p(s)s\gamma||u||dtds\Big],\\
\geq\ & \frac{\lambda\gamma(M_0-\varepsilon)}{2}\sup_{t\in J}\Big[\int_{\theta+\tau}^{1-\theta+\tau} G_n(t,s)p(s)||u||ds\\
& +k_0||u||\int_\theta^{1-\theta} G_n(t,s)p(s)sds\Big],\\
\geq\ & \frac{(M_0-\varepsilon)\lambda\gamma\theta^{n-1}}{2}\times\\
& \Big[\int_{\theta+\tau}^{1-\theta} G_2(s,s)p(s)(1+k_0 s)ds\Big]||u||,\\
=\ & \lambda(M_0-\varepsilon)L_1 \geq ||u||.
\end{aligned}
$$

For the same above $\varepsilon > 0$, from (3.5) and (3.6), $\exists R_2 > r_2$, when $\sqrt{u^2+v^2} \geq R_2, f(t,u,v) > (M^\infty+\varepsilon)\sqrt{u^2+v^2}$. Let

$$\Omega_2 = \{t \in [-\tau, 1] : ||\sqrt{u^2+v^2}|| < R_2\},$$

for $u, v \in K \cap \partial\Omega_2$, we get

$$
\begin{aligned}
||\Phi u|| \leq\ & \lambda\int_0^1 G_n(s,s)f(s,u(s-\tau),v)ds\\
& +\frac{\lambda a}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)f(s,u(s-\tau),v)ds,\\
=\ & \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau),v)ds\},\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \int_0^1 G_2(s,s)p(s)\sqrt{u^2(s-\tau)+v^2}ds,\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \int_0^1 G_2(s,s)p(s)[u(s-\tau)+v]ds,\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \Big[\int_0^1 G_2(s,s)p(s)u(s-\tau)ds\\
+\ & \int_0^1 G_2(s,s)p(s)\int_0^s k(t,s)u(t)dtds\Big],\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \Big[\int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)u(s)ds\\
+\ & k_1\int_0^1 G_2(s,s)p(s)\int_0^s u(t)dtds\Big],\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \Big[\int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)ds\\
+\ & k_1\int_0^1 G_2(s,s)p(s)sds\Big]||u||,\\
\leq\ & \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times\\
& \Big[\int_0^1 G_2(s,s)p(s)(1+k_1 s)ds\Big]||u||\\
=\ & \lambda(M^\infty+\varepsilon)L_2||u|| \leq ||u||..
\end{aligned}
$$

Therefore, by Lemma 2.3, Φ has a fixed point $u \in K \cap (\overline{\Omega_2}\backslash\Omega_1)$, and $u(t)$ is a positive solution of equation (1.2), completing the proof of Theorem 3.2.

Theorem 3.3 If the conditions$(H_1)-(H_5)$ satisfy and $M_\infty = \infty, M^0 = 0$. Then there exists two positive numbers λ_1, λ_2, when $\lambda_1 \leq \lambda \leq \lambda_2$, BVP (1.2) has at least a positive solution.

Proof Since $M_\infty = \infty$, we can choose a positive constant $M > 0$ such that $f(t,u,v) \geq M = \alpha R_3 (\alpha > 0)$ for any $\sqrt{u^2+v^2} \geq R_3, t \in J$. Let

$$\lambda_1 = \Big[\alpha\theta^{n-1}\int_0^1 G_2(s,s)p(s)ds\Big]^{-1}.$$

$$\Omega_1 = \{t \in [-\tau, 1] : ||\sqrt{u^2+v^2}|| \geq R_3\},$$

for $u, v \in K \cap \partial\Omega_1, \lambda \geq \lambda_1$, we have

$$
\begin{aligned}
||\Phi u|| \geq\ & \lambda\sup_{t\in J}\int_0^1 G(t,s)f(s,u,v)ds\\
& \geq \lambda M\sup_{t\in J}\int_0^1 G_n(t,s)p(s)ds,\\
\geq\ & \lambda M\sup_{t\in J_\theta}\int_0^1 G_n(t,s)p(s)ds\\
& \geq \lambda M\theta^{n-1}\int_0^1 G_2(s,s)p(s)ds,\\
\geq\ & \lambda\alpha R_3\theta^{n-1}\int_0^1 G_2(s,s)p(s)ds = \frac{\lambda}{\lambda_1}R_3,\\
\geq\ & \|\sqrt{u^2+v^2}\| \geq \|u\|.
\end{aligned}
$$

Because $M^0 = 0$, we choose a value small enough for $\varepsilon > 0$, so that

$$\lambda_2 = \Big[\tfrac{\lambda\theta^{n-1}\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\int_0^1 G_2(s,s)(1+k_1 s)p(s)ds\Big]^{-1}$$

$> \lambda$, and $\exists 0 < r_3 < R_3$, such that$f(t,u,v) \leq \varepsilon\sqrt{u^2+v^2}$ for any $\sqrt{u^2+v^2} \leq r_3$. Let

$$\Omega_1 = \{t \in [-\tau, 1] : ||\sqrt{u^2+v^2}|| < r_3\},$$

for $u, v \in K \cap \partial\Omega$, we have

$$
\begin{aligned}
||\Phi u|| \leq\ & \lambda\int_0^1 G_n(s,s)p(s)f(s,u,v)ds\\
+\ & \frac{\lambda a}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)p(s)f(s,u,v)ds,\\
\leq\ & \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)p(s)\sqrt{u^2+v^2)}ds,\\
\leq\ & \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)p(s)(u+v)ds,\\
=\ & \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\Big[\int_0^1 G_n(s,s)p(s)u(s-\tau)ds
\end{aligned}
$$

$$+ \quad \int_0^1 G_n(s,s)p(s)\int_0^s K(t,s)u(t)dtds\Big],$$
$$\leq \quad \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times$$
$$\Big[\int_{-\tau}^{1-\tau} G_n(s+\tau,s+\tau)p(s+\tau)u(s)ds$$
$$+k_1\int_0^1 G_n(s,s)p(s)\int_0^s u(t)dtds\Big],$$
$$\leq \quad \frac{\lambda\theta^{n-1}\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times$$
$$\int_0^1 G_2(s,s)(1+k_1 s)p(s)ds||u||$$
$$\leq \frac{\lambda}{\lambda_2}||u|| \leq ||u||.$$

Therefore, by Lemma 2.3, Φ has at least one fixed point $u \in K\cap(\overline{\Omega_2}\backslash\Omega_1)$, and $u(t)$ has at least one positive solution of equation (1.2), completing the proof of Theorem 3.3.

Remark 3.1 If $M_0 = \infty, M^\infty = 0$, similarly we can verify that BVP (1.2) has at least one positive solution.

IV. The case of no less than three solutions

In this section, by Lemma (2.4), we prove that BVP (1.1) has at least three solutions when $f(t,u,v)$ satisfies some certain conditions. First, we define four the non-negative continuous concave functions in K α,β,φ and ϕ

$\alpha(u) = \min_{\theta\leq t\leq 1-\theta}|u(t)|, \beta(u) = \max_{0\leq t\leq 1}|u(t)|,$
$$\varphi(u) = \phi(u) = \max_{\theta-\tau\leq t\leq 1}|u(t)|. \qquad (4.1)$$

Set

(1) $\omega_1(t)$, $\omega_2(t)$ are two non-negative characteristic functions and $\omega_1(t) \in C[0,1], \omega_2(t) \in C[\theta+\tau, 1-\theta]$.

(2) $L_3 = \frac{1+a-a\eta^{n-1}}{1-a\eta^{n-1}}[\int_0^1 G_2(s,s)p(s)\omega_1(s)(1+k_1 s)ds]$.

(3) $N_1 = \frac{c}{\lambda L_3(m+c)}$,
$N_2 = \frac{1}{\lambda\gamma\theta^{n-1}\sqrt{k_0}\int_{\gamma+\tau}^{1-\gamma}\sqrt{s}G_2(s,s)p(s)\omega_2(s)ds}$.

As we know, some researchers [27,28] had discussed BVPs with at least three solutions. In these papers, the relative conclusions were based on the assumption that $f(t,u(t))$ be more than or less than a given constant. In fact, it's very difficult to find such functions. In our paper, $f(t,u,v)$ is assumed to be a function which satisfies the conditions $f(t,u,v) \leq N_1\omega_1(u+v)$ or $f(t,u,v) \geq N_2\omega_2\sqrt{uv}$. Meanwhile, we introduce two characteristic ω_1,ω_2. The conditions $f(t,u,v) \leq N_1\omega_1(u+v)$ or $f(t,u,v) \geq N_2\omega_2\sqrt{uv}$ can be easily satisfied for some equations when we choose suitable characteristic functions. The conclusion is described in Theorem 4.1. In section 5, we give example 2 to illustrate our conclusion.

Theorem 4.1 If the conditions $(H_1)-(H_5)$ hold and there exist positive numbers b,c,m with $m > b > c > 0$ such that the following conditions are satisfied

$(i) f(t,u,v) \leq N_1\omega_1(t)(u+v), (t,u) \in [0,1]\times[0,m]$, where $N_1 \leq \frac{c}{\lambda L_2(m+c)}$,

$(ii) f(t,u,v) \geq N_2\omega_2(t)\sqrt{uv}, (t,u) \in [\theta+\tau, 1-\theta]\times[b, \frac{(\theta+1)^2}{\gamma}b]$,

where $N_2 \geq [\lambda\gamma\gamma_1\sqrt{k_0}\int_{\gamma+\tau}^{1-\gamma}G(s,s)\sqrt{s}ds]^{-1}$.

Then BVP(1.2) has at least three solutions $u_1,u_2,u_3 \in \overline{P(\beta,m)}$, such that

$\beta(u_i) \leq m$, for $i = 1,2,3$;
$b < \alpha(u_1)$;
$c < \phi(u_2)$, with $\alpha(u_2) < b$;
$\phi(u_3) < c$.

Proof By Lemma 2.1, we can derive that $\Phi : K \to K$ is completely continuous. It is easy to verify that $\phi(\lambda u) = \lambda\phi(u)$ for $0 \leq \lambda \leq 1$ and $\alpha(u) \leq \phi(u)$. According to the definition of norm and (4.1), if taking $M \geq 1$, then we have that $||u|| = \beta(u)$ and $||u|| \leq M\beta(u)$ for all $u \in \overline{P(\beta,m)}$. Therefore if $u \in \overline{P(\beta,m)}$, then $\beta(u) = ||u|| \leq m$, in addition

$$\max_{t\in[0,1]}|u(t-\tau)| = \max_{t\in[-\tau,1-\tau]}|u(t)| = \max_{t\in[0,1-\tau]}|u(t)| \leq \max_{t\in[0,1]}|u(t)| \leq m.$$

From the conditions (i) of theorem 4.1, we have

$$\beta(\Phi u) \quad = \quad \max_{t\in[0,1]}|(\Phi u)(t)| \leq \lambda\int_0^1 G_n(s,s)p(s)f(s,u,v)ds$$
$$+\frac{a\lambda}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)p(s)f(s,u,v)ds,$$
$$\leq \quad \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times$$
$$\int_0^1 G_n(s,s)p(s)\omega_1(s)(u+v)ds,$$
$$= \quad \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times$$
$$\Big[\int_0^1 G_n(s,s)p(s)\omega_1(s)u(s-\tau)ds$$
$$+\int_0^1 G_n(s,s)p(s)\omega_1(s)\int_0^s K(t,s)u(t)dtds\Big],$$
$$\leq \quad \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\times$$
$$\int_0^1 G_2(s,s)p(s)\omega_1(s)(m+k_1 m\int_0^s dt)ds,$$
$$= \quad \frac{\lambda N_1(1+a-a\eta)m}{1-a\eta}\times$$
$$\int_0^1 (1+k_1 s)G_2(s,s)p(s)\omega_1(s)ds$$
$$= \quad \lambda N_1 m L_2 \leq \frac{mc}{(m+c)} < m.$$

So $\Phi : \overline{P(\beta,m)} \to \overline{P(\beta,m)}$.

If one choosing

$$u_0(t) = -\tfrac{4b}{\gamma}(t-\tfrac{1+\theta}{2})^2 + \tfrac{(1+\theta)^2 b}{\gamma}, t \in [0,1],$$

we have that $\varphi(u_0) = \frac{(1+\theta)^2 b}{\gamma}$ and $\alpha(u_0) = \min_{t\in[\theta,1-\theta]}|u_0(t)| = u_0(\theta) = \frac{4b\theta}{\gamma} > b$, then $u_0 \in P(\beta,\varphi,\alpha,b,\frac{(1+\theta)^2 b}{\gamma},m)$.

Therefore $\{u \in P(\beta, \varphi, \alpha, b, \frac{(1+\theta)^2 b}{\gamma}, m) | \alpha(u) > b\} \neq 0$.
On the other hand, if $u \in P(\beta, \varphi, \alpha, b, \frac{(1+\theta)^2 b}{\gamma}, m)$, then

$$\min_{t\in[\theta+\tau,1-\theta]} |u(t-\tau)| = \min_{t\in[\theta,1-\theta-\tau]} |u(t)| \geq \min_{t\in[\theta,1-\theta]} |u(t)| \geq b$$

,

$$\min_{t\in[\theta+\tau,1-\theta]} |u(t)| \geq \min_{t\in[\theta,1-\theta]} |u(t)| \geq b.$$

From the conditions (ii) of theorem 4.1 and Lemma 1.1, we have

$$\begin{aligned}
\alpha(\Phi u) &= \min_{t\in[\theta,1-\theta]} |(\Phi u)(t)| \geq \gamma ||(\Phi u)(t)||, \\
&\geq \lambda\gamma \sup_{t\in J_\theta} \int_0^1 G_n(t,s) f(s,u,v) ds \\
&\geq \lambda\gamma N_2 \sup_{t\in J_\theta} \int_0^1 G_n(t,s) p(s)\omega_2(s)\sqrt{uv} ds, \\
&= \lambda\gamma N_2 \sup_{t\in J_\theta} \int_0^1 G_n(t,s)) p(s)\omega_2(s) \\
&\quad \sqrt{u(t-\tau)\int_0^s k(t,s)u(t)dtds}, \\
&\geq \lambda\gamma N_2 \sup_{t\in J_\theta} \int_{\theta+\tau}^{1-\theta} G_n(t,s) p(s)\omega_2(s) \\
&\quad \sqrt{u(t-\tau)\int_0^s k(t,s)u(t)dtds}, \\
&\geq [\lambda\theta^{n-1}\gamma N_2\sqrt{k_0}\int_{\theta+\tau}^{1-\theta} G_2(s,s) \\
&\quad p(s)\omega_2(s)\sqrt{s}ds] b \geq b.
\end{aligned}$$

So the condition (S_1) of Lemma 2.4 is satisfied.

For all $u \in P(\beta, \alpha, b, m)$ with $\varphi(\Phi u) > \frac{b(1+\theta)^2}{\gamma}$, then one has

$$\alpha(\Phi u) \geq \gamma\varphi(\Phi u) > \gamma\frac{b(1+\theta)^2}{\gamma} > b.$$

So the condition (S_2) of Lemma 2.4 is also satisfied.

Finally, we verify the condition (S_3) of Lemma 2.4 holds. Obviously, $0 \notin R(\beta, \phi, c, m)$. As if $0 \in R(\beta, \phi, c, m)$, then it is conflicts wit $\phi(0) = 0 < c$. For all $u \in R(\beta, \phi, c, m)$ with $\phi(u) = c$, then

$$\max_{t\in[0,1]} |u(t)| \leq m,$$

$$\max_{t\in[\theta,1]} |u(t-\tau)| = \max_{t\in[\theta-\tau,1-\tau]} |u(t)| \leq \max_{t\in[\theta-\tau,1]} |u(t)| = c.$$

From the conditions (i) of theorem 4.1, we have

$$\begin{aligned}
\phi(\Phi u) &= \max_{t\in[\theta-\tau,1]} |(\Phi u)(t)| \leq \max_{t\in[0,1]} |(\Phi u)(t)|, \\
&\leq \lambda\int_0^1 G_n(s,s)p(s)f(s,u,v)ds \\
&\quad + \frac{a\lambda}{1-a\eta^{n-1}}\int_0^1 G_n(s,s)p(s)f(s,u,v)ds, \\
&\leq \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \times \\
&\quad \int_0^1 G_n(s,s)p(s)(u+v)ds, \\
&= \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}[\int_0^1 G_n(s,s)p(s)u(s-\tau)ds \\
&\quad + \int_0^1 G_n(s,s)p(s)\int_0^s K(t,s)u(t)dtds], \\
&= \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\Big[\int_0^\theta G_n(s,s)p(s)u(s-\tau)ds \\
&\quad + \int_\theta^1 G_n(s,s)p(s)u(s-\tau)ds \\
&\quad + \int_0^{\theta-\tau} G_n(s,s)p(s)\int_0^s K(t,s)u(t)dtds \\
&\quad + \int_{\theta-\tau}^1 G_n(s,s)p(s)\int_0^s K(t,s)u(t)dtds\Big], \\
&\leq \frac{\lambda N_1(1+a-a\eta^{n-1})}{1-a\eta^{n-1}}\Big[m\int_0^\theta G_n(s,s)p(s)ds \\
&\quad + c\int_\theta^1 G_n(s,s)p(s)ds \\
&\quad + mk_1\int_0^{\theta-\tau} G_n(s,s)p(s)sds \\
&\quad + k_1 c\int_{\theta-\tau}^1 G_n(s,s)p(s)sds\Big], \\
&\leq \frac{\lambda N_1(m+c)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \\
&\quad \int_0^1 G_2(s,s)p(s)(1+k_1 s)ds \\
&= \lambda N_1(m+c)L_2 \leq c.
\end{aligned}$$

Therefore, by Lemma 2.4, Φ has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\beta, m)}$, then u_1, u_2, u_3 are three positive solution of equation (1.2), and u_1, u_2, u_3 satisfy that

$\beta(u_i) \leq m$, for $i = 1, 2, 3$;
$b < \alpha(u_1)$;
$c < \phi(u_2)$, with $\alpha(u_2) < b$;

V. Example

Consider the equation

$$\begin{cases} -u^{(5)}(t) = \frac{1000}{t}\sqrt{t^2+1}\frac{[u^2(t-\frac{1}{6})+v^2(t)][1+u(t-\frac{1}{6})+v(t)]}{2+u(t-\frac{1}{6})+v(t)}, \\ \quad 0 < t < 1, u(t) = 0, -\tau \leq t \leq 0, \\ u(1) = 4!\frac{1}{2}u(\frac{1}{2}). \end{cases} \quad (5.1)$$

Where $f(t,u,v) = \sqrt{t^2+1}\frac{[u^2(t-\frac{1}{6})+v^2(t)][1+u(t-\frac{1}{6})+v(t)]}{2+u(t-\frac{1}{6})+v(t)}$; $v(t) = \int_0^1 (t+s+1)u(s)ds, k(s,t) = t+s+1$, then $k_1 = 3; p(t) = \frac{1}{t}, t = 0$ is its singularity. Here we have $M_\infty = \infty, M^0 = 0$. If we choose $M = 100, \theta = \frac{1}{8}, \lambda = 1000, \eta = \frac{1}{2}, a = \frac{1}{2}, \alpha = \frac{100}{11}$. Then when $\sqrt{u^2+v^2} \geq 11$, $f(t,u,v) \geq M$. We can calculate that $\lambda_1 = \frac{22528}{25}$. If we choose $\varepsilon = 0.01$, Calculations show that $\lambda_2 = \frac{12679600}{47}$.

From Theorem 3.3, we have that if $\frac{22528}{25} = \lambda_1 \leq \lambda \leq \lambda_2 = \frac{12679600}{47}$, the problem (5.1) has at least one positive solution.

Example 2 Consider the boundary value problem

$$\begin{cases} -u^{(4)}(t) = \frac{10^{24min(1-t)}}{3(1-t)(2+t)}\sqrt{u^2(t-\frac{1}{4})+v^2(t)+u(t-\frac{1}{4})v(t)} \\ \quad t \in (0,1), u(t) = 0, \quad -\tau \leq t \leq 0, \\ u(1) = 3!\frac{1}{2}u(\frac{1}{2}). \end{cases} \quad (5.2)$$

where $\lambda = \frac{1}{3}; \tau = \frac{1}{4}; k(t,s) = 2ts + 1, v(t) = \int_0^1 k(t,s)u(s)ds, (t,s) \in [0,1] \times [0,1]; f(t,u,v) =$

$\frac{10^{24min(1-t)}}{2+t}\sqrt{u^2(t-\frac{1}{4})+v^2(t)+u(t-\frac{1}{4})v(t)}; p(t) = \frac{1}{1-t}, t = 1$ is its singularity; $u(t)$ satisfies the definition of (4.1). If setting $m = 2, c = 1, \theta = \frac{1}{4}$, such that $\max\limits_{0\le t\le 1}|u(t)| \le m$, $\phi(u) = \max\limits_{\theta-\tau\le t\le 1}|u(t)| > c$, we can easily derive that

$$L_2 = \frac{11}{6}, N_1 = \frac{6}{11}, N_2 = 60857.88.$$

Obviously, we have that

$$f(t,u,v) \le \frac{1}{2}(u+v) < N_1(u+v), t \in [0,1],$$

$$f(t,u,v) \ge \frac{10^6\sqrt{3uv}}{3} > N_2\sqrt{uv}, t \in [\frac{1}{2}, \frac{3}{4}].$$

From Theorem 4.1, the problem (5.2) has at least three positive solutions.

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