On an Almost Periodic Gilpin-Ayala Competition Model of Phytoplankton Allelopathy

Lijun Xu and Yongzhi Liao

***Abstract*—By using some analytical techniques, modified inequalities and Mawhin's continuous theorem of coincidence degree theory, some sufficient conditions for the existence of positive almost periodic solutions to a class of Gilpin-Ayala competition model of phytoplankton allelopathy are established. Further, by constructing a suitable Lyapunov functional, the global asymptotical stability of almost periodic solution for the model is also studied. The work of this paper extends and improves some results in recent years. Finally, some examples and simulations are given to illustrate the main results in this paper.**

***Index Terms*—Almost periodic oscillation; Coincidence degree; Gilpin-Ayala; Phytoplankton allelopathy.**

I. Introduction

ONe of the most interesting questions considering ecological populations is how species which grow in the same environment influence each other since they usually compete for food, territory and other requirements. A specific type of ecological interaction corresponding to this issue is called facultative mutualism and it means that each species enhances the average growth rate of the other although each one of them can survive in the absence of other species.

The first model that regards ecological population systems was suggested independently by Lotka [1] and Volterra [2] in the 1920s and was described by the following differential equation

$$\dot{x}_i = r_i x_i\left[1-\frac{x_i}{K_i}-a_{ij}\frac{x_j}{K_j}\right], \quad i,j=1,2,\ i\neq j,$$

where x_i and r_i are the population size and exponential rate of the growth of the ith species, respectively, K_i is the carrying capacity of the ith species in the absence of its competitor-the jth species, and a_{ij} is the linear reduction (in terms of K_i) of the ith species' rate of growth by its competitor-the jth species, $i,j=1,2$, $i\neq j$.

During the last decades, ecological population systems have been intensively studied and there exist a lot of excellent papers in this field. Most of them are mainly grounded on the classical Lotka-Volterra competition system but they have many different forms (see, for example, [3,4]). However, regardless of this fact, the Lotka-Volterra competition models have often been severely criticized. One of the remarks refers to the fact that this model is linear, i.e. the rate of change in the size of each species is a linear function of sizes of the interacting species. Particularly, in 1973, Gilpin and Ayala [5] pointed out that the Lotka-Volterra systems are the linearization of the per capita growth rates $\dot{x}_i/x_i$ about the point of equilibrium. They claimed that a little more complicated model was needed in order to yield more realistic solutions. Hence, they proposed the following competition model:

$$\begin{cases}\dot{x}_1 = r_1x_1\left[1-\left(\frac{x_1}{K_1}\right)^{\theta_1}-a_{12}\frac{x_2}{K_2}\right],\\ \dot{x}_2 = r_2x_2\left[1-\left(\frac{x_2}{K_2}\right)^{\theta_2}-a_{21}\frac{x_1}{K_1}\right],\end{cases} \tag{1.1}$$

where x_i is the population density of the ith species, r_i is the intrinsic exponential growth rate of the ith species, K_i is the environmental carrying capacity of species i in the absence of competition, θ_i provides a nonlinear measure of interspecific interference, $i=1,2$, a_{12} and a_{21} provides a measure of interspecific interference. In recent years, many generalizations and modifications of system (1.1) have been proposed and studied (see [6-11] for more detail).

The aim of this paper is to consider the following almost periodic Gilpin-Ayala competition model of phytoplankton allelopathy with time-varying delays:

$$\begin{cases}\dot{x}(t) = x(t)\Big[r_1(t)-a_1(t)x^{\alpha_1}(t)-b_1(t)y^{\beta_1}(\mu_1(t))\\ \qquad -c_1(t)x^{\gamma_1}(\nu_1(t))y^{\delta_1}(\sigma_1(t))\Big],\\ \dot{y}(t) = y(t)\Big[r_2(t)-a_2(t)y^{\alpha_2}(t)-b_2(t)x^{\beta_2}(\mu_2(t))\\ \qquad -c_2(t)x^{\gamma_2}(\nu_2(t))y^{\delta_2}(\sigma_2(t))\Big],\end{cases} \tag{1.2}$$

where $\alpha_i,\beta_i,\gamma_i,\delta_i$ are nonnegative constants, $r_i,a_i,b_i,c_i,t-\mu_i(t),t-\nu_i(t)$ and $t-\sigma_i(t)$ are nonnegative almost periodic functions, $i=1,2$. Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity.

It is well known that Mawhin's continuation theorem of coincidence degree theory is an important method to investigate the existence of positive periodic solutions of some kinds of non-linear ecosystems (see [12-22]). However, it is difficult to be used to investigate the existence of positive almost periodic solutions of non-linear ecosystems. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of

Manuscript received May 9, 2016; revised January 1, 2017.

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positive almost periodic solutions of system (1.2) by using Mawhin's continuation theorem [23-24]. Motivated by the above reason, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of system (1.2) by using Mawhin's continuous theorem of coincidence degree theory.

Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbb{R}} f(s), \quad f^M = \sup_{s\in\mathbb{R}} f(s),$$

$$|f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Throughout this paper, we always make the following assumption for system (1.2):

(H_1) $\bar{r}_i > 0$ and $\bar{a}_i > 0$, $i = 1, 2$.

The paper is organized as follows. In Section 2, some basic definitions and necessary lemmas are given. In Section 3, by using Mawhin's continuous theorem of coincidence degree theory, some new sufficient conditions for the existence of at least one positive almost periodic solution of system (1.2) are established. In Section 4, some simple applications are stated. Finally, some illustrative examples are given in Section 5.

II. Preliminaries

Definition 1. ([25, 26]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R}, \mathbb{R})$.

Next, we present and prove several useful lemmas which will be used in later section.

Lemma 1. ([24]) *Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$. For arbitrary interval $[a, b]$ with $b - a = \omega > 0$, let $\xi \in [a, b]$ and*

$$I = \{s \in [\xi, b] : \dot{x}(s) \geq 0\},$$

then ones have

$$x(t) \leq x(\xi) + \int_I \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b].$$

Lemma 2. ([24]) *If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a, b]$ with $I = b - a = \omega > 0$, there exist $\eta \in [a, b]$, $\underline{\eta} \in (-\infty, a]$ and $\bar{\eta} \in [b, +\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \textit{and} \quad x(\eta) \geq x(s), \ \forall s \in [\underline{\eta}, \bar{\eta}].$$

Lemma 3. ([24]) *If $x \in AP(\mathbb{R})$, then for $\forall n \in \mathbb{N}^+$, there exists $\alpha_n \in \mathbb{R}$ such that $x(\alpha_n) \in [x^* - \frac{1}{n}, x^*]$, where $x^* = \sup_{s\in\mathbb{R}} x(s)$.*

For $x \in AP(\mathbb{R})$, we denote by

$$\bar{x} = m(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s,$$

$$a(x, \varpi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\,\mathrm{d}s,$$

$$\Lambda(x) = \left\{\varpi \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\}$$

the mean value and the set of Fourier exponents of x, respectively.

Lemma 4. ([25]) *Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$, there exists a positive constant T_0 independent of t_0 such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in \left[\frac{\bar{x}}{2}, \frac{3\bar{x}}{2}\right], \quad \forall T \geq T_0.$$

III. Main results

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [27].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 5. ([27]) *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic solution for system (1.2).

From (H_1), for $\forall a \in \mathbb{R}$, there exists a constant $\omega \in (0, +\infty)$ independent of a such that

$$\frac{1}{T}\int_a^{a+T} r_i(s)\,\mathrm{d}s \in \left[\frac{\bar{r}_i}{2}, \frac{3\bar{r}_i}{2}\right],$$

$$\frac{1}{T}\int_a^{a+T} a_i(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_i}{2}, \frac{3\bar{a}_i}{2}\right], \quad \forall T \geq \omega, \ i = 1, 2. \quad (\Box)$$

In addition, we introduce two numbers, two functions and a assumption as follows:

$$k_i := \frac{1}{\alpha_i}\ln\frac{3\bar{r}_i}{\bar{a}_i} + \frac{3\bar{r}_i\omega}{2}, \ i = 1, 2,$$

$$\Delta_1(s) = r_1(s) - e^{\beta_1 k_2} b_1(s),$$

$$\Delta_2(s) = r_2(s) - e^{\beta_2 k_1} b_2(s), \quad \forall s \in \mathbb{R},$$

where ω is defined as that in (□).
(H_2) $\bar{\Delta}_i > 0$, $i = 1, 2$.

Theorem 1. *Assume that* (H_1)-(H_2) *hold, then system* (1.2) *admits at least one positive almost periodic solution.*

Proof: Under the invariant transformation $(x, y)^T = (e^u, e^v)^T$, system (1.2) reduces to

$$\begin{cases} \dot{u}(t) = a_1(t) - b_1(t)e^{\alpha_1 u(t)} - c_1(t)e^{\beta_1 v(\mu_1(t))} \\ \qquad -d_1(t)e^{\gamma_1 u(\nu_1(t))}e^{\delta_1 v(\sigma_1(t))} := F_1(t), \\ \dot{v}(t) = a_2(t) - b_2(t)e^{\alpha_2 v(t)} - c_2(t)e^{\beta_2 u(\mu_2(t))} \\ \qquad -d_2(t)e^{\gamma_2 u(\nu_2(t))}e^{\delta_2 v(\sigma_2(t))} := F_2(t). \end{cases} \quad (3.0)$$

It is easy to see that if system (3.0) has one almost periodic solution $(u, v)^T$, then $(x, y)^T = (e^u, e^v)^T$ is a positive almost periodic solution of system (1.2). Therefore, to complete the proof it suffices to show that system (3.0) has one almost periodic solution.

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \left\{(u, v)^T \in AP(\mathbb{R}) : \forall \varpi \in \Lambda(u) \cup \Lambda(v), |\varpi| \geq \gamma_0\right\},$$

$$\mathbb{V}_2 = \left\{z = (u, v)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R}\right\},$$

where γ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}} = \max\left\{\sup_{s\in\mathbb{R}} |u(s)|, \sup_{s\in\mathbb{R}} |v(s)|\right\}, \quad \forall z \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|_{\mathbb{X}}$. Set

$$L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}, \quad Lz = L(u, v)^T = (\dot{u}, \dot{v})^T,$$

where $\mathrm{Dom}L = \{z = (u, v)^T \in \mathbb{X} : u, v \in C^1(\mathbb{R}), \dot{u}, \dot{v} \in C(\mathbb{R})\}$ and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Nz = N\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}.$$

With these notations system (3.0) can be written in the form

$$Lz = Nz, \quad \forall z \in \mathbb{X}.$$

It is not difficult to verify that $\mathrm{Ker}L = \mathbb{V}_2$, $\mathrm{Im}L = \mathbb{V}_1$ is closed in $\mathbb{Y}$ and $\dim \mathrm{Ker}L = 2 = \mathrm{codim}\,\mathrm{Im}L$. Therefore, L is a Fredholm mapping of index zero (see Lemma 2.12 in [24]). Now define two projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ as

$$Pz = P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m(u) \\ m(v) \end{bmatrix} = Qz, \quad \forall z = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then P and Q are continuous projectors such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. Furthermore, through an easy computation we find that the inverse $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of L_P has the form

$$K_P z = K_P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \int_0^t u(s)\,\mathrm{d}s - m\left[\int_0^t u(s)\,\mathrm{d}s\right] \\ \int_0^t v(s)\,\mathrm{d}s - m\left[\int_0^t v(s)\,\mathrm{d}s\right] \end{bmatrix}.$$

Then $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I-Q)N : \mathbb{X} \to \mathbb{X}$ read

$$QNz = QN\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m(F_1) \\ m(F_2) \end{bmatrix},$$

$$K_P(I-Q)Nz = \begin{bmatrix} f[u(t)] - Qf[u(t)] \\ f[v(t)] - Qf[v(t)] \end{bmatrix}, \quad \forall z \in \mathrm{Im}L,$$

where $f(x)$ is defined by $f[x(t)] = \int_0^t \left[Nx(s) - QNx(s)\right]\mathrm{d}s$. Then N is L-compact on $\bar{\Omega}$ (see Lemma 2.13 in [24]).

In order to apply Lemma 5, we need to search for an appropriate open-bounded subset Ω.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0, 1)$, we have

$$\begin{cases} u'(t) = \lambda\left[r_1(t) - a_1(t)e^{\alpha_1 u(t)} - b_1(t)e^{\beta_1 v(\mu_1(t))}\right. \\ \qquad \left.-c_1(t)e^{\gamma_1 u(\nu_1(t))}e^{\delta_1 v(\sigma_1(t))}\right], \\ v'(t) = \lambda\left[r_2(t) - a_2(t)e^{\alpha_2 v(t)} - b_2(t)e^{\beta_2 u(\mu_2(t))}\right. \\ \qquad \left.-c_2(t)e^{\gamma_2 u(\nu_2(t))}e^{\delta_2 v(\sigma_2(t))}\right]. \end{cases} \quad (3.1)$$

Suppose that $(u, v)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda \in (0, 1)$. By Lemma 3, there exist two sequences $\{T_n : n \in \mathbb{N}^+\}$ and $\{P_n : n \in \mathbb{N}^+\}$ such that

$$u(T_n) \in \left[x^* - \frac{1}{n}, u^*\right], \quad u^* = \sup_{s\in\mathbb{R}} u(s), \quad n \in \mathbb{N}^+, (3.2)$$

$$v(P_n) \in \left[v^* - \frac{1}{n}, v^*\right], \quad v^* = \sup_{s\in\mathbb{R}} v(s), \quad n \in \mathbb{N}^+. (3.3)$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $[T_{n_0} - \omega, T_{n_0}]$ and $[P_{n_0} - \omega, P_{n_0}]$, where ω is defined as that in (□). By Lemma 1, there exist $\xi \in [T_{n_0} - \omega, T_{n_0}]$, $\underline{\xi} \in (-\infty, T_{n_0} - \omega]$ and $\bar{\xi} \in [T_{n_0}, +\infty)$ such that

$$u(\underline{\xi}) = u(\bar{\xi}) \quad \text{and} \quad u(\xi) \leq u(s), \quad \forall s \in [\bar{\xi}, \underline{\xi}]. \quad (3.4)$$

Integrating the first equation of system (3.1) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} \left[r_1(s) - a_1(s)e^{\alpha_1 u(s)} - b_1(s)e^{\beta_1 v(\mu_1(s))}\right. \\ \left.-c_1(s)e^{\gamma_1 u(\nu_1(s))}e^{\delta_1 v(\sigma_1(s))}\right]\mathrm{d}s = 0, \quad (3.5)$$

which yields from (3.4) that

$$\int_{\underline{\xi}}^{\bar{\xi}} a_1(s)e^{\alpha_1 u(\xi_u^{n_0})}\,\mathrm{d}s \leq \int_{\underline{\xi}}^{\bar{\xi}} a_1(s)e^{\alpha_1 u(s)}\,\mathrm{d}s \leq \int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\,\mathrm{d}s,$$

which yields from (□) that

$$u(\xi) \leq \frac{1}{\alpha_1}\ln\left[\frac{\int_{\underline{\xi}}^{\bar{\xi}} r_1(s)\,\mathrm{d}s}{\int_{\underline{\xi}}^{\bar{\xi}} a_1(s)\,\mathrm{d}s}\right] \leq \frac{1}{\alpha_1}\ln\frac{3\bar{r}_1}{\bar{a}_1}. \quad (3.6)$$

Similar to the argument as that in (3.6), there exists $\zeta \in [P_{n_0} - \omega, P_{n_0}]$ so that

$$v(\zeta) \leq \frac{1}{\alpha_2}\ln\frac{3\bar{r}_2}{\bar{a}_2}. \quad (3.7)$$

Let $I_1 = \{s \in [\xi, T_{n_0}] : \dot{u}(s) \geq 0\}$. It follows from system (3.1) that

$$\int_{I_1} \dot{u}(s)\,\mathrm{d}s = \int_{I_1} \lambda\left[r_1(s) - a_1(s)e^{\alpha_1 u(s)} - b_1(s)e^{\beta_1 v(\mu_1(s))}\right. \\ \left.-c_1(s)e^{\gamma_1 u(\nu_1(s))}e^{\delta_1 v(\sigma_1(s))}\right]\mathrm{d}s$$

$$\begin{aligned}&\leq \int_{I_1} \lambda r_1(s)\,\mathrm{d}s \leq \int_{T_{n_0}-\omega_0}^{T_{n_0}} r_1(s)\,\mathrm{d}s\\ &\leq \frac{3\bar{r}_1\omega_0}{2}. \qquad (3.8)\end{aligned}$$

Let $I_2 = \{s \in [\zeta, P_{n_0}] : \dot{v}(s) \geq 0\}$. Similar to the argument as that in (3.8), we can easily obtain that

$$\int_{I_2} \dot{v}(s)\,\mathrm{d}s \leq \int_{P_{n_0}-\omega}^{P_{n_0}} r_2(s)\,\mathrm{d}s \leq \frac{3\bar{r}_2\omega}{2}. \qquad (3.9)$$

By Lemma 1, it follows from (3.6)-(3.9) that

$$\begin{aligned}u(t) \;&\leq\; u(\xi) + \int_{I_1} \dot{u}(s)\,\mathrm{d}s \leq \frac{1}{\alpha_1}\ln\frac{3\bar{r}_1}{\bar{a}_1} + \frac{3\bar{r}_1\omega}{2}\\ &:= k_1, \quad \forall t \in [\xi, T_{n_0}],\end{aligned}$$

$$\begin{aligned}v(t) \;&\leq\; v(\zeta) + \int_{I_2} \dot{v}(s)\,\mathrm{d}s \leq \frac{1}{\alpha_2}\ln\frac{3\bar{r}_2}{\bar{a}_2} + \frac{3\bar{r}_2\omega}{2}\\ &:= k_2, \quad \forall t \in [\zeta, P_{n_0}],\end{aligned}$$

which imply that

$$u(T_{n_0}) \leq k_1 \quad \text{and} \quad v(P_{n_0}) \leq k_2.$$

In view of (3.2)-(3.3), letting $n_0 \to +\infty$ in the above inequalities leads to

$$u^* = \lim_{n_0\to+\infty} u(T_{n_0}) \leq k_1, \qquad (3.10)$$

$$v^* = \lim_{n_0\to+\infty} v(P_{n_0}) \leq k_2. \qquad (3.11)$$

In view of (3.5), by the integral mean value theorem, there exists $s_0 \in [\underline{\xi}, \bar{\xi}]$ such that

$$\begin{aligned}&r_1(s_0) - a_1(s_0)e^{\alpha_1 u(s_0)} - b_1(s_0)e^{\beta_1 v(\mu_1(s_0))}\\ &\quad -c_1(s_0)e^{\gamma_1 u(\nu_1(s_0))}e^{\delta_1 v(\sigma_1(s_0))} = 0,\end{aligned}$$

which implies that

$$\begin{aligned}p_0 \;&:=\; \inf_{s\in\mathbb{R}}\Big\{r_1(s) - a_1(s)e^{\alpha_1 k_1} - b_1(s)e^{\beta_1 k_2}\\ &\qquad -c_1(s)e^{\gamma_1 k_1}e^{\delta_1 k_2}\Big\}\\ &\leq\; \inf_{s\in\mathbb{R}}\Big\{r_1(s) - a_1(s)e^{\alpha_1 u(s)} - b_1(s)e^{\beta_1 v(\mu_1(s))}\\ &\qquad -c_1(s)e^{\gamma_1 u(\nu_1(s))}e^{\delta_1 v(\sigma_1(s))}\Big\}\\ &\leq\; r_1(s_0) - a_1(s_0)e^{\alpha_1 u(s_0)} - b_1(s_0)e^{\beta_1 v(\mu_1(s_0))}\\ &\qquad -c_1(s_0)e^{\gamma_1 u(\nu_1(s_0))}e^{\delta_1 v(\sigma_1(s_0))}\\ &=\; 0.\end{aligned}$$

Similarly, we can easily obtain that

$$\begin{aligned}q_0 := \inf_{s\in\mathbb{R}}\Big\{&r_2(s) - a_2(s)e^{\alpha_2 k_2}\\ &-b_2(s)e^{\beta_2 k_1} - c_2(s)e^{\gamma_2 k_1}e^{\delta_2 k_2}\Big\} \leq 0.\end{aligned}$$

Substituting (3.10)-(3.11) into system (3.1), we obtain

$$\begin{cases}\dot{u}(t) \;\geq\; \lambda\Big[r_1(t) - a_1(t)e^{\alpha_1 k_1} - b_1(t)e^{\beta_1 k_2}\\ \qquad -c_1(t)e^{\gamma_1 k_1}e^{\delta_1 k_2}\Big] \geq \lambda p_0 \geq p_0,\\ \dot{v}(t) \;\geq\; \lambda\Big[r_2(t) - a_2(t)e^{\alpha_2 k_2} - b_2(t)e^{\beta_2 k_1}\\ \qquad -c_2(t)e^{\gamma_2 k_1}e^{\delta_2 k_2}\Big] \geq \lambda q_0 \geq q_0.\end{cases}$$

Under the invariant transformation $(u, v)^T = (\ln x, \ln y)^T$ for $x > 0$ and $y > 0$, the above system changes to

$$\begin{cases}\dot{x}(t) \geq p_0 x(t),\\ \dot{y}(t) \geq q_0 y(t),\end{cases} \Longleftrightarrow \begin{cases}\big[e^{-p_0 t}x(t)\big]' \geq 0,\\ \big[e^{-q_0 t}y(t)\big]' \geq 0.\end{cases} \qquad (3.12)$$

For any $s \leq t$, integrating (3.12) from s to t leads to

$$\begin{cases}e^{u(s)} \leq e^{-p_0(t-s)}e^{u(t)},\\ e^{v(s)} \leq e^{-q_0(t-s)}e^{v(t)}.\end{cases} \qquad (3.13)$$

Substituting (3.13) into system (3.1) leads to

$$\begin{cases}\dot{u}(t) \;\geq\; \lambda\Big[r_1(t) - a_1(t)e^{\alpha_1 u(t)} - b_1(t)e^{\beta_1 v(\mu_1(t))}\\ \qquad -c_1(t)e^{-p_0(t-\nu_1(t))\gamma_1}e^{\delta_1 v(\sigma_1(t))}e^{\gamma_1 u(t)}\Big],\\ \dot{v}(t) \;\geq\; \lambda\Big[r_2(t) - a_2(t)e^{\alpha_2 v(t)} - b_2(t)e^{\beta_2 u(\mu_2(t))}\\ \qquad -c_2(t)e^{\gamma_2 u(\nu_2(t))}e^{-q_0(t-\sigma_2(t))\delta_2}e^{\delta_2 v(t)}\Big].\end{cases}$$

From (H_2) and Lemma 4, for $\forall a \in \mathbb{R}$, there exists a constant $\hat{\omega} \in [\omega, +\infty)$ independent of a such that

$$\frac{1}{T}\int_a^{a+T} \Delta_i(s)\,\mathrm{d}s \in \left[\frac{\bar{\Delta}_i}{2}, \frac{3\bar{\Delta}_i}{2}\right], \qquad (3.14)$$

where $T \geq \hat{\omega},\ i = 1, 2$. For $\forall n_0 \in \mathbb{Z}$, by Lemma 2, we can conclude that there exist $\eta \in [n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}]$, $\underline{\eta} \in (-\infty, n_0\hat{\omega}]$ and $\bar{\eta} \in [n_0\hat{\omega} + \hat{\omega}, +\infty)$ such that

$$u(\underline{\eta}) = u(\bar{\eta}) \quad \text{and} \quad u(\eta) \geq u(s),\ \forall s \in [\underline{\eta}, \bar{\eta}]. \quad (3.15)$$

Integrating the first inequality of system (3.14) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\begin{aligned}&\int_{\underline{\eta}}^{\bar{\eta}} \Big[r_1(s) - b_1(s)e^{\beta_1 v(\mu_1(s))}\Big]\mathrm{d}s \leq \int_{\underline{\eta}}^{\bar{\eta}} \Big[a_1(s)e^{\alpha_1 u(s)}\\ &+c_1(s)e^{-p_0(s-\nu_1(s))\gamma_1}e^{\delta_1 v(\sigma_1(s))}e^{\gamma_1 u(s)}\Big]\mathrm{d}s, \qquad (3.16)\end{aligned}$$

which implies from (3.16) that

$$\begin{aligned}&\int_{\underline{\eta}}^{\bar{\eta}} \Delta_1(s)\,\mathrm{d}s\\ &= \int_{\underline{\eta}}^{\bar{\eta}} \Big(r_1(s) - b_1(s)e^{\beta_1 k_2}\Big)\mathrm{d}s\\ &\leq \int_{\underline{\eta}}^{\bar{\eta}} \Big(r_1(s) - b_1(s)e^{\beta_1 v(\mu_1(s))}\Big)\mathrm{d}s\\ &\leq \int_{\underline{\eta}}^{\bar{\eta}} \Big(a_1(s)e^{\alpha_1 u(s)}\\ &\qquad +c_1(s)e^{-\gamma_1 p_0(s-\nu_1(s))}e^{\delta_1 v(\sigma_1(s))}e^{\gamma_1 u(s)}\Big)\mathrm{d}s\end{aligned}$$

$$\leq \int_{\underline{\eta}}^{\bar{\eta}} \bigg(a_1(s)e^{\alpha_1 u(\eta)} + c_1(s)e^{-\gamma_1 p_0(s-\nu_1(s))}e^{\delta_1 v(\sigma_1(s))}e^{\gamma_1 u(\eta)} \bigg) \, ds. \quad (3.17)$$

If $e^{u(\eta)} < 1$, then $\max\{e^{\alpha_1 u(\eta)}, e^{\gamma_1 u(\eta)}\} \leq e^{\kappa^l u(\eta)}$, where $\kappa^l = \min\{\alpha_1, \gamma_1\}$. By (3.17), we obtain from (□) and (3.15) that

$$\begin{aligned} &u(\eta) \\ &\geq \frac{1}{\kappa^l} \ln \frac{\int_{\underline{\eta}}^{\bar{\eta}} \Delta_1(s)\, ds}{\int_{\underline{\eta}}^{\bar{\eta}} \Big[a_1(s) + c_1(s)e^{-\gamma_1 p_0(s-\nu_1(s))}e^{\delta_1 v(\sigma_1(s))} \Big] \, ds} \\ &\geq \frac{1}{\kappa^l} \ln \frac{\bar{\Delta}_1}{2a_1^M + 2c_1^M[e^{-p_0(s-\nu_1(s))^M}+1]^{\gamma_1}[e^{k_2}+1]^{\delta_1}} \\ &= \frac{1}{\kappa^l}\Gamma_1, \end{aligned} \quad (3.18)$$

where

$$\Gamma_1 = \ln \left\{ \frac{\bar{\Delta}_1}{2a_1^M + 2c_1^M[e^{-p_0(s-\nu_1(s))^M}+1]^{\gamma_1}[e^{k_2}+1]^{\delta_1}} \right\}.$$

If $e^{u(\eta)} \geq 1$, then $\max\{e^{\alpha_1 u(\eta)}, e^{\gamma_1 u(\eta)}\} \leq e^{\kappa^M u(\eta)}$, where $\kappa^M = \max\{\alpha_1, \gamma_1\}$. Similar to (3.18), we obtain from (□) and (3.15) that

$$u(\eta) \geq \frac{1}{\kappa^M}\Gamma_1. \quad (3.19)$$

From (3.18)-(3.19), we obtain

$$u(\eta) \geq \min\left\{ \frac{1}{\kappa^l}\Gamma_1, \frac{1}{\kappa^M}\Gamma_1 \right\} := \Pi_1. \quad (3.20)$$

Similar to the argument as that in (3.20), it is not difficult to obtain that there exists $\varsigma \in [n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}]$ so that

$$v(\varsigma) \geq \min\left\{ \frac{1}{\upsilon^l}\Gamma_2, \frac{1}{\upsilon^M}\Gamma_2 \right\} := \Pi_2, \quad (3.21)$$

where $\upsilon^l = \min\{\alpha_2, \gamma_2\}$, $\upsilon^M = \max\{\alpha_2, \gamma_2\}$ and

$$\Gamma_2 = \ln \left\{ \frac{\bar{\Delta}_2}{2a_2^M + 2c_2^M[e^{-q_0(s-\sigma_2(s))^M}+1]^{\delta_2}[e^{k_1}+1]^{\gamma_2}} \right\}.$$

Further, it follows from system (3.1) that

$$\begin{aligned} &\int_{n_0\hat{\omega}}^{n_0\hat{\omega}+\hat{\omega}} |\dot{u}(s)|\, ds \\ &= \int_{n_0\hat{\omega}}^{n_0\hat{\omega}+\hat{\omega}} \lambda \bigg| r_1(s) - a_1(s)e^{\alpha_1 u(s)} \\ &\quad -b_1(s)e^{\beta_1 v(\mu_1(s))} - c_1(s)e^{\gamma_1 u(\nu_1(s))}e^{\delta_1 v(\sigma_1(s))} \bigg| \, ds \\ &\leq \Big\{ r_1^M + a_1^M[e^{k_1}+1]^{\alpha_1} + b_1^M[e^{k_2}+1]^{\beta_1} \\ &\quad +c_1^M[e^{k_1}+1]^{\gamma_1}[e^{k_2}+1]^{\delta_1} \Big\} \hat{\omega} := \Theta_1, \end{aligned} \quad (3.22)$$

$$\begin{aligned} &\int_{n_0\hat{\omega}}^{n_0\hat{\omega}+\hat{\omega}} |\dot{v}(s)|\, ds \\ &\leq \Big\{ r_2^M + a_2^M[e^{k_2}+1]^{\alpha_2} + b_2^M[e^{k_1}+1]^{\beta_2} \\ &\quad +c_2^M[e^{k_1}+1]^{\gamma_2}[e^{k_2}+1]^{\delta_2} \Big\} \hat{\omega} := \Theta_2. \end{aligned} \quad (3.23)$$

In view of (3.20)-(3.23), it follows that

$$\begin{aligned} u(t) &\geq u(\eta) - \int_{n_0\hat{\omega}}^{n_0\hat{\omega}+\hat{\omega}} |\dot{u}(s)|\, ds \\ &\geq \Pi_1 - \Theta_1 := k_3, \ \forall t \in [n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}], \end{aligned} \quad (3.24)$$

$$\begin{aligned} v(t) &\geq v(\varsigma) - \int_{n_0\hat{\omega}}^{n_0\hat{\omega}+\hat{\omega}} |\dot{v}(s)|\, ds \\ &\geq \Pi_2 - \Theta_2 := k_4, \ \forall t \in [n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}]. \end{aligned} \quad (3.25)$$

Obviously, k_3 and k_4 are constants independent of n_0. So it follows from (3.24)-(3.25) that

$$u_* = \inf_{s\in\mathbb{R}} u(s) = \inf_{n_0\in\mathbb{Z}} \Big\{ \min_{s\in[n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}]} u(s) \Big\} \geq k_3, \quad (3.26)$$

$$v_* = \inf_{s\in\mathbb{R}} v(s) = \inf_{n_0\in\mathbb{Z}} \Big\{ \min_{s\in[n_0\hat{\omega}, n_0\hat{\omega}+\hat{\omega}]} v(s) \Big\} \geq k_4. \quad (3.27)$$

Set $K = |k_1| + |k_2| + |k_3| + |k_4| + 1$, then $\|w\| = \|(u,v)^T\| < K$. Clearly, K is independent of $\lambda \in (0,1)$. Consider the algebraic equations $QNz_0 = 0$ for $z_0 = (u_0, v_0)^T \in \mathbb{R}^2$ as follows:

$$\begin{cases} m(r_1) - m(a_1)e^{\alpha_1 u_0} - m(b_1)e^{\beta_1 v_0} \\ -m(c_1)e^{\gamma_1 u_0}e^{\delta_1 v_0} = 0, \\ m(r_2) - m(a_2)e^{\alpha_2 v_0} - m(b_2)e^{\beta_2 u_0} \\ -m(c_2)e^{\gamma_2 u_0}e^{\delta_2 v_0} = 0. \end{cases} \quad (3.28)$$

From the first equation of system (3.28), we have

$$m(r_1) \geq m(a_1)e^{\alpha_1 u_0},$$

which implies that

$$u_0 \leq \frac{1}{\alpha_1} \ln \frac{\bar{r}_1}{\bar{a}_1} < k_1. \quad (3.29)$$

Similarly, we also obtain

$$v_0 \leq \frac{1}{\alpha_2} \ln \frac{\bar{r}_2}{\bar{a}_2} < k_2. \quad (3.30)$$

Furthermore, by the first equation of system (3.28) and (3.30), we have

$$\begin{aligned} m(\Delta_1) &= m(r_1) - e^{\beta_1 k_2}m(b_1) \\ &\leq m(r_1) - m(b_1)e^{\beta_1 v_0} \\ &= m(a_1)e^{\alpha_1 u_0} + m(c_1)e^{\gamma_1 u_0}e^{\delta_1 v_0} \\ &\leq \big(a_1^M + c_1^M[e^{k_2}+1]^{\delta_1}\big)\max\{e^{\kappa^l u_0}, e^{\kappa^M u_0}\}, \end{aligned}$$

where $\kappa^l = \min\{\alpha_1, \gamma_1\}$ and $\kappa^M = \max\{\alpha_1, \gamma_1\}$. Therefore, we obtain

$$u_0 \geq k_3. \quad (3.31)$$

Similarly, we have

$$v_0 \geq k_4, \quad (3.32)$$

where $\upsilon^l = \min\{\alpha_2, \gamma_2\}$, $\upsilon^M = \max\{\alpha_2, \gamma_2\}$. In view of (3.29)-(3.32), we can easily obtain $\|z_0\|_{\mathbb{X}} = |u_0| + |v_0| < K$. Let $\Omega = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < K\}$, then Ω satisfies conditions (a) and (b) of Lemma 5.

Finally, we will show that condition (c) of Lemma 5 is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1-\iota)Fz, \ \ (\iota, z) \in [0,1] \times \mathbb{R}^2,$$

where

$$Fz = F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(r_1) - m(a_1)e^{\alpha_1 u} \\ m(r_2) - m(a_2)e^{\alpha_2 v} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$, $\forall \iota \in [0,1]$. By the invariance property of homotopy, direct calculation produces

$$\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) = \deg\big(F, \Omega \cap \mathrm{Ker}L, 0\big) \neq 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 5 are satisfied. Therefore, system (3.0) has one almost periodic solution, that is, system (1.2) has at least one positive almost periodic solution. This completes the proof. ■

IV. Global asymptotical stability

Lemma 6. ([33]) *Assume that $a > 0$, $b > 0$ and $\dot{x} \leq x(b - ax^{\alpha})$, where α is positive constant, then*

$$\lim_{t\to+\infty} \sup x(t) \leq \left[\frac{b}{a}\right]^{\frac{1}{\alpha}}.$$

Lemma 7. *Assume that (H_1) holds, then any positive solution $(x,y)^T$ of system (1.2) satisfies*

$$\limsup_{t\to\infty} x(t) \leq M, \quad \limsup_{t\to\infty} y(t) \leq N,$$

where

$$M = \left[\frac{r_1^+}{a_1^-}\right]^{\frac{1}{\alpha_1}}, \quad N = \left[\frac{r_2^+}{a_2^-}\right]^{\frac{1}{\alpha_2}}.$$

Proof: In view of the first equation of system (1.2), we have

$$\dot{x}(t) \leq x(t)\left[r_1^+ - a_1^- x^{\alpha_1}(t)\right],$$

which implies from Lemma 6 that

$$\limsup_{t\to\infty} x(t) \leq \left[\frac{r_1^+}{a_1^-}\right]^{\frac{1}{\alpha_1}} := M.$$

Further, from the second equation of system (1.2), it follows that

$$\dot{y}(t) \leq y(t)\left[r_2^+ - a_2^- y^{\alpha_2}(t)\right],$$

which implies from Lemma 6 that

$$\limsup_{t\to\infty} y(t) \leq \left[\frac{r_2^+}{a_2^-}\right]^{\frac{1}{\alpha_2}} := N.$$

This completes the proof. ■

Theorem 2. *Assume that (H_1)-(H_2) hold. Suppose further that*

$$a_1^- - \frac{c_1^+\gamma_1 K^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1\dot{\nu}_1^-} - \frac{c_2^+\gamma_2 K^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1\dot{\nu}_2^-} - \frac{b_2^+\beta_2 K^{\beta_2-\alpha_1}}{\alpha_1\dot{\mu}_2^-} > 0,$$

$$a_2^- - \frac{c_1^+\gamma_1 K^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2\dot{\sigma}_1^-} - \frac{c_2^+\gamma_2 K^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2\dot{\sigma}_2^-} - \frac{b_1^+\beta_1 K^{\beta_1-\alpha_2}}{\alpha_2} > 0,$$

where $K = \min\{M, N\}$. Then system (1.2) has a unique positive almost periodic solution, which is globally asymptotically stable.

Proof: By Theorem 1, we know that system (1.2) has at least one positive almost periodic solution $(x^*, y^*)^T$. Suppose that $(x,y)^T$ is another positive solution of system (1.2).

From (H_3), there must exist $\epsilon > 0$ and $\theta > 0$ such that

$$a_1^- - \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1\dot{\nu}_1^-} - \frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1\dot{\nu}_2^-} - \frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1\dot{\mu}_2^-} > \theta,$$

$$a_2^- - \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2\dot{\sigma}_1^-} - \frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2\dot{\sigma}_2^-} - \frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2} > \theta.$$

From Lemma 7, there must exist $T_0 > 0$ such that

$$x(t) < K+\epsilon, \quad y(t) < K+\epsilon, \quad \forall t \geq T_0.$$

Define

$$V(t) = \sum_{i=1}^{3} V_i(t),$$

where

$$V_1(t) = |\ln x(t) - \ln x^*(t)| + |\ln y(t) - \ln y^*(t)|,$$

$$\begin{aligned} &V_2(t) \\ &= \int_{\nu_1(t)}^{t} \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1\dot{\nu}_1^-}|x^{\alpha_1}(s) - x^{*\alpha_1}(s)|\,\mathrm{d}s \\ &\quad + \int_{\nu_2(t)}^{t} \frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1\dot{\nu}_2^-}|x^{\alpha_1}(s) - x^{*\alpha_1}(s)|\,\mathrm{d}s \\ &\quad + \int_{\mu_2(t)}^{t} \frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1\dot{\mu}_2^-}|x^{\alpha_1}(s) - x^{*\alpha_1}(s)|\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} &V_3(t) \\ &= \int_{\sigma_1(t)}^{t} \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2\dot{\sigma}_1^-}|y^{\alpha_2}(s) - y^{*\alpha_2}(s)|\,\mathrm{d}s \\ &\quad + \int_{\sigma_2(t)}^{t} \frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2\dot{\sigma}_2^-}|y^{\alpha_2}(s) - y^{*\alpha_2}(s)|\,\mathrm{d}s \\ &\quad + \int_{\mu_1(t)}^{t} \frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2\dot{\mu}_1^-}|y^{\alpha_2}(s) - y^{*\alpha_2}(s)|\,\mathrm{d}s. \end{aligned}$$

For $t \geq T_0$, calculating the upper right derivative of $V_i (i = 1,2,3)$ along the solution of system (1.2), it follows that

$$D^+V_1(t)$$

$$
\begin{aligned}
\leq\ & -a_1^-|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& +b_1^+|y^{\beta_1}(\mu_1(t)) - y^{*\beta_1}(\mu_1(t))| \\
& +c_1^+y^{\delta_1}(\sigma_1(t))|x^{\gamma_1}(\nu_1(t)) - x^{*\gamma_1}(\nu_1(t))| \\
& +c_1^+x^{*\gamma_1}(\nu_1(t))|y^{\delta_1}(\sigma_1(t)) - y^{*\delta_1}(\sigma_1(t))| \\
& -a_2^-|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
& +b_2^+|x^{\beta_2}(\mu_2(t)) - x^{*\beta_2}(\mu_2(t))| \\
& +c_2^+y^{\delta_2}(\sigma_2(t))|x^{\gamma_2}(\nu_2(t)) - x^{*\gamma_2}(\nu_2(t))| \\
& +c_2^+x^{*\gamma_2}(\nu_2(t))|y^{\delta_2}(\sigma_2(t)) - y^{*\delta_2}(\sigma_2(t))| \\
\leq\ & -a_1^-|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& +\frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2}|y^{\alpha_2}(\mu_1(t)) - y^{*\alpha_2}(\mu_1(t))| \\
& +\frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\nu_1(t)) - x^{*\alpha_1}(\nu_1(t))| \\
& +\frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2}|y^{\alpha_2}(\sigma_1(t)) - y^{*\alpha_2}(\sigma_1(t))| \\
& -a_2^-|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
& +\frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\mu_2(t)) - x^{*\alpha_1}(\mu_2(t))| \\
& +\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\nu_2(t)) - x^{*\alpha_1}(\nu_2(t))| \\
& +\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2}|y^{\alpha_2}(\sigma_2(t)) - y^{*\alpha_2}(\sigma_2(t))|,
\end{aligned}
$$

$$
\begin{aligned}
& D^+V_2(t) \\
\leq\ & \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1\dot{\nu}_1^-}|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& -\frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\nu_1(t)) - x^{*\alpha_1}(\nu_1(t))| \\
& +\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1\dot{\nu}_2^-}|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& -\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\nu_2(t)) - x^{*\alpha_1}(\nu_2(t))| \\
& +\frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1\dot{\mu}_2^-}|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& -\frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1}|x^{\alpha_1}(\mu_2(t)) - x^{*\alpha_1}(\mu_2(t))|,
\end{aligned}
$$

$$
\begin{aligned}
& D^+V_3(t) \\
\leq\ & \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2\dot{\sigma}_1^-}|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
& -\frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2}|y^{\alpha_2}(\sigma_1(t)) - y^{*\alpha_2}(\sigma_1(t))| \\
& +\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2\dot{\sigma}_2^-}|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
& -\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2}|y^{\alpha_2}(\sigma_2(t)) - y^{*\alpha_2}(\sigma_2(t))| \\
& +\frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2}|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
& -\frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2\dot{\mu}_1^-}|y^{\alpha_2}(\mu_1(t)) - y^{*\alpha_2}(\mu_1(t))|.
\end{aligned}
$$

Together with the above inequalities, it follows that

$$
\begin{aligned}
& D^+V(t) \\
\leq\ & -\bigg\{a_1^- - \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_1}}{\alpha_1\dot{\nu}_1^-} \\
& -\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_1}}{\alpha_1\dot{\nu}_2^-} \\
& -\frac{b_2^+\beta_2(K+\epsilon)^{\beta_2-\alpha_1}}{\alpha_1\dot{\mu}_2^-}\bigg\}|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| \\
& -\bigg\{a_2^- - \frac{c_1^+\gamma_1(K+\epsilon)^{\delta_1+\gamma_1-\alpha_2}}{\alpha_2\dot{\sigma}_1^-} \\
& -\frac{c_2^+\gamma_2(K+\epsilon)^{\delta_2+\gamma_2-\alpha_2}}{\alpha_2\dot{\sigma}_2^-} \\
& -\frac{b_1^+\beta_1(K+\epsilon)^{\beta_1-\alpha_2}}{\alpha_2}\bigg\}|y^{\alpha_2}(t) - y^{*\alpha_2}(t)| \\
\leq\ & -\theta[|x^{\alpha_1}(t) - x^{*\alpha_1}(t)| + |y^{\alpha_2}(t) - y^{*\alpha_2}(t)|].
\end{aligned}
$$

Therefore, V is non-increasing. Integrating the last inequality from T_0 to t leads to

$$
\begin{aligned}
& V(t) + \theta\int_{T_0}^{t}[|x^{\alpha_1}(s) - x^{*\alpha_1}(s)| \\
& +|y^{\alpha_2}(s) - y^{*\alpha_2}(s)|]\,\mathrm{d}s \\
\leq\ & V(T_0) < +\infty, \quad \forall t \geq T_0,
\end{aligned}
$$

that is,

$$
\int_{T_0}^{+\infty}[|x^{\alpha_1}(s) - x^{*\alpha_1}(s)| + |y^{\alpha_2}(s) - y^{*\alpha_2}(s)|]\,\mathrm{d}s < +\infty,
$$

which implies that

$$
\lim_{s\to+\infty}[|x^{\alpha_1}(s) - x^{*\alpha_1}(s)| + |y^{\alpha_2}(s) - y^{*\alpha_2}(s)|] = 0.
$$

Thus, the almost periodic solution of system (1.2) is asymptotical stability. The global asymptotical stability implies that the almost periodic solution is unique. This completes the proof. ■

V. Some applications

Application V.1. Consider the following logistic equation

$$
\dot{N}(t) = N(t)\Big[r(t) - a(t)N(t) - b(t)N(t-\mu)\Big], \quad (5.1)
$$

where $\mu \geq 0$ is a constant, r, a and b are nonnegative almost periodic functions with $m(r) > 0$ and $m(a+b) > 0$.

In [28], Xie and Li obtained that

Corollary 1. Eq. (5.1) *admits at least one positive almost periodic solution.*

Corollary 2. *Assume that all the coefficients in* Eq. (5.1) *are 2π-periodic, then* Eq. (5.1) *admits at least one positive 2π-periodic solution.*

As regards the existence of a periodic solution in Eq. (5.1), Freedman and Wu [29] proved an existence result for E-q. (5.1) by introducing a different assumption:

Theorem 3. *Assume that $r(t) > 0, a(t) > 0, b(t) \geq 0$ for all $t \in [0, 2\pi]$ and suppose that the delayed functional equation*

$$
r(t) - a(t)\psi(t) - b(t)\psi(t-\mu) = 0
$$

has a positive and continuously differentiable 2π-periodic solution, then Eq. (5.1) *has a positive 2π-periodic solution.*

Further, Lisena [30] obtained that

Theorem 4. *Assume that $m(r) > 0, a(t) > 0, b(t) \geq 0$ for all $t \in [0, 2\pi]$ and suppose that there exists a positive and continuously differentiable 2π-periodic function ϕ such that*

$$a(t)\phi(t) + b(t)\phi(t-\mu) = 0, \quad \forall t \in [0, 2\pi].$$

Then Eq. (4.1) *has a positive 2π-periodic solution.*

Remark 1. Obviously, Corollary 2 improves some conditions in Theorems 3-4.

Application V.2. Consider non-autonomous model of phytoplankton allelopathy as follows:

$$\begin{cases} \dot{x}(t) = x(t)\Big[r_1(t) - a_1(t)x(t) \\ \qquad -b_1(t)y(t) - c_1(t)x(t)y(t)\Big], \\ \dot{y}(t) = y(t)\Big[r_2(t) - a_2(t)y(t) \\ \qquad -b_2(t)x(t) - c_2(t)x(t-\mu)y(t)\Big], \end{cases} \tag{5.2}$$

where all the coefficients of (5.2) are nonnegative almost periodic functions.

Let

$$k_i^0 := \ln\frac{3\bar{r}_i}{\bar{a}_i} + \frac{3\bar{r}_i\omega}{2}, \ i = 1, 2,$$

$$\Delta_1^0(s) = r_1(s) - e^{k_2^0}b_1(s),$$

$$\Delta_2^0(s) = r_2(s) - e^{k_1^0}b_2(s), \ \forall s \in \mathbb{R},$$

where ω is defined as that in ($\Box$).

Similar to the proof of Theorem 1, we can easily show that

Corollary 3. *Assume that $m(\Delta_i^0) > 0\,(i = 1, 2)$, then system* (5.2) *admits at least one almost periodic solution.*

Remark 2. In [31], the authors obtained a existence result for the almost periodic solutions of system (5.2) on condition that r_1, r_2, a_1 and a_2 are strictly positive. However, Corollary 3 broaden such condition. Therefore, Corollary 3 improves the work in [31].

Application V.3. Consider the following a delayed Logistic equation

$$\dot{N}(t) = N(t)\Big[r(t) - a(t)N^p(t) - b(t)N^q(t-\mu(t))\Big] \tag{5.3}$$

where p, q are positive constants, μ, r, a, b are nonnegative 2π-periodic functions, $m(r) > 0$ and $m(a+b) > 0$.

Similar to the proof of Theorem 1, we can easily obtain that

Corollary 4. Eq. (5.3) *admits at least one positive 2π-periodic solution.*

In [32], Chen obtained that

Theorem 5. *Assume that $p \leq q$, then* Eq. (5.3) *admits at least one positive 2π-periodic solution.*

Remark 3. It is clear that Corollary 4 removes the condition $p \leq q$ in Theorem 5. Therefore, our result improves the work in [32].

VI. Examples and Simulations

Example 1. Consider the following Gilpin-Ayala competition model of phytoplankton allelopathy with time-varying delays with different periods:

$$\begin{cases} \dot{x}(t) = x(t)\Big[1 - |\sin\sqrt{3}t|x(t) \\ \qquad -\frac{\sin^2(\sqrt{2}t)}{e^2}y^{\frac{1}{20}}(\mu(t)) - x^{\frac{1}{2}}(\mu(t))y^{\frac{1}{3}}(\mu(t))\Big], \\ \dot{y}(t) = y(t)\Big[3 - \cos^2(\sqrt{2}t)y(t) \\ \qquad -\frac{|\cos\sqrt{3}t|}{e^2}x^{\frac{1}{7}}(\nu(t)) - 10x^4(\nu(t))y^3(\nu(t))\Big], \end{cases} \tag{6.1}$$

where $\mu(t) = t - 0.1\sin^2 t$ and $\nu(t) = t - 0.2|\sin t|, \forall t \in \mathbb{R}$.

Corresponding to system (1.2), we have $\bar{r}_1 = 1$, $\bar{r}_2 = 3$, $\bar{a}_1 = \frac{2}{\pi}$, $b_1^M = b_2^M = \frac{1}{e^2}$, $\bar{a}_2 = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = \frac{1}{20}$, $\beta_2 = \frac{1}{7}$. Further, for $\forall a \in \mathbb{R}$, we can choose $\omega = \frac{2\sqrt{3}\pi}{3}$ so that ($\Box$) holds, that is,

$$\frac{1}{T}\int_a^{a+T} a_1(s)\,\mathrm{d}s \in \left[\frac{1}{\pi}, \frac{3}{\pi}\right],$$

$$\frac{1}{T}\int_a^{a+T} a_2(s)\,\mathrm{d}s \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall T \geq \omega = \frac{2\sqrt{3}\pi}{3}.$$

By a easy calculation, we obtain that

$$k_1 := \ln\frac{3\pi}{2} + \sqrt{3}\pi \approx 6.99159,$$

$$k_2 := \ln\frac{9\pi}{2} + 3\sqrt{3}\pi \approx 18.97300.$$

Hence

$$\bar{\Delta}_1 \geq 1 - \frac{1}{e} > 0, \quad \bar{\Delta}_2 \geq 3 - \frac{1}{e} > 0.$$

Therefore, all the conditions of Theorem 1 are satisfied. By Theorem 1, system (6.1) admits at least one positive almost periodic solution (see Figures 1-2).

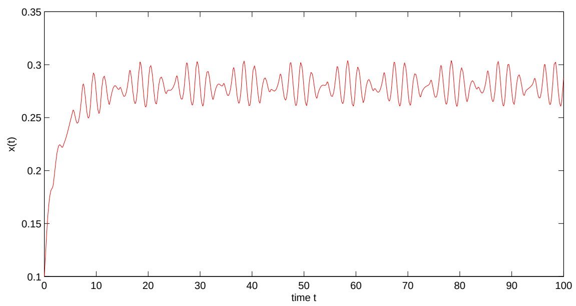

Fig. 1 State variable x of system (5.1)

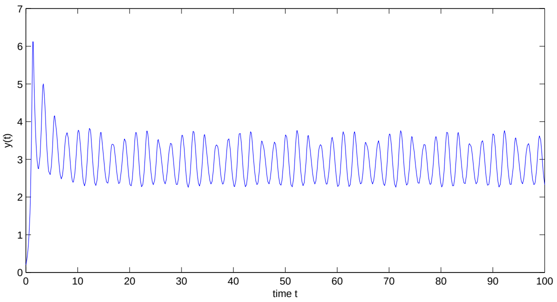

Fig. 2 State variable y of system (5.1)

Remark 4. In system (6.1), $|\sin\sqrt{3}t|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function and $\cos^2(\sqrt{2}t)$ is $\frac{\sqrt{2}\pi}{2}$-periodic function. So system (6.1) is with incommensurable periods. Through all the coefficients of system (6.1) are periodic functions, the positive periodic solutions of system (6.1) could not possibly exist. However, by Theorem 1, the positive almost periodic solutions of system (6.1) exactly exist.

VII. Conclusion

In this paper, some sufficient conditions are established for the existence, uniqueness and global asymptotical stability of almost periodic solution for a Gilpin-Ayala competition model of phytoplankton allelopathy. The main result obtained in this paper are completely new. Besides, the method used in this paper may be used to study the almost periodic dynamics of many other biological models such as predator-prey models, facultative mutualism models, and so on.

Acknowledgments

This work are supported by the Funding for Applied Technology Research and Development of Panzhihua City under Grant 2015CY-S-14 and Natural Science Foundation of Education Department of Sichuan Province under Grant 15ZB0419.

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