Existence and Multiplicity of Solutions for p-Laplacian Equations without the AR Condition

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Abstract—**The Ambrosetti-Rabinowitz (AR) condition is crucial in variational methods. In this paper we consider a class of p-Laplacian equations without the AR condition. Using Mountain pass lemma and Ekeland variational principle, we obtain the existence and multiplicity of the solutions. These results complement some known results.**

Index Terms—**P-Laplacian equations, AR condition, Mountain pass theorem, Ekeland variational principle, existence and multiplicity.**

I. Introduction

In this paper, we study the existence and multiplicity of nontrivial weak solutions for the following nonlinear elliptic equation

$$\begin{cases} -\Phi_p(x(z)) = m(z)|x(z)|^{r-2}x(z) + f(z, x(z)), \\ \qquad\qquad\qquad\qquad\qquad a.e. \text{ on } Z, \\ x|_{\partial Z} = 0, \; m \in L^\infty(Z)_+, m \neq 0, 1 < r < p < \infty, \end{cases} \tag{1.1}$$

where $\Phi_p x = div(\|Dx\|_{R^N}^{p-2} Dx)$ is called p-Laplacian differential operator, $Z \subset R^N$ is a bounded domain with a $C^2(\partial Z)$, and the function f is a Carathédory function which is assumed to be $(p-1)$-superlinear (convex term) near infinity and doesn't satisfy the Ambrosetti-Rabinowitz condition (AR condition for short). Since the term $m(z)|x(z)|^{r-2}x(z)$ is $(p-1)$-superlinear (concave term) near zero for $r < p$, so the right-hand-side of (1.1) reflects the combined properties of "convex" and "concave" and which ensures the existence of multiple solutions for equations [1] similar to (1.1) .

As we have known that the AR condition is very important in variational methods, which not only ensures that the Euler-Lagrange function [17] associated with (1.1) has a mountain pass geometry, but also guarantees the boundedness of Palais-Smale sequences corresponding to the Euler-Lagrange function. But some nonlinearities do not always satisfy the AR condition, see [3], [5], [9], [10], [12], [16], [13], [14], [18], [19], [21], [22], [23], [24] for details.

We will use the Mountain pass theorem [8] and Ekeland variational principle [6], with Cerami condition [4] to overcome the above difficulties.

We suppose that $f(z, x)$ satisfies the following conditions without the AR condition.

(HF) the function $f(z, x)$ satisfies $f(z, 0) = 0$ a.e. on Z, $f(z, x) \geq 0$ for a.e. $z \in \bar{Z}, \forall\, x \geq 0$ and

(i) for all $x \in R, z \to f(z, x)$ is measurable;

(ii) for almost all $z \in Z, x \to f(z, x)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in R$, we have $|f(z, x)| \leq a(z) + c|x|^{\tau-1}$, where $\tau \in (p, p^*)$ and $p^* := \frac{Np}{N-p}$, if $N > p$ or $p^* := +\infty$, if $N \leq p$;

(iv) the function $f(z, x)$ is $(p-1)$-superlinear, i.e. $\lim\limits_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$ uniformly for almost all $z \in Z$;

(v) there exists $\beta \in L^1(Z)_+$ such that $G(z, x) \leq G(z, y) + \beta(z), z \in Z$ for all $0 \leq x \leq y$, where $G(z, x) := f(z, x)x - pF(z, x)$ and $F(z, x) = \int_0^x f(z, t)dt$;

(vi) there is $\theta \in L^\infty(Z)_+, \theta(z) \leq \lambda_1$ a.e. on $Z, \theta \not\equiv \lambda_1$, and $\lim\limits_{x\to 0^+} \frac{f(z,x)}{x^{(p-1)}} \leq \theta(z)$ uniformly for a.e. $z \in Z$.

$(HF)'$ In addition to the assumptions $(i), (ii), (iii)$, there are also some assumptions on $f(z, x)$:

$f(z, x)$ is a function such that $f(z, 0) = 0$ a.e. on Z, $f(z, x) \leq 0$ for a.e. $z \in Z, \forall\, x \leq 0$;

(iv) the function $f(z, x)$ is $(p-1)$-superlinear, i.e. $\lim\limits_{x\to-\infty} \frac{f(z,x)}{x^{(p-1)}} = +\infty$ uniformly for almost all $z \in Z$;

(v) there exists $\beta \in L^1(Z)_+$ such that $G(z, x) \leq G(z, y) + \beta(z), z \in Z$ for all $y \leq x \leq 0$, where $G(z, x) := f(z, x)x - pF(z, x)$ and $F(z, x) = \int_0^x f(z, t)dt$;

(vi) there is $\theta \in L^\infty(Z)_+, \theta(z) \leq \lambda_1$ a.e. on $Z, \theta \not\equiv \lambda_1$, and $\lim\limits_{x\to 0^-} \frac{f(z,x)}{x^{(p-1)}} \leq \theta(z)$ uniformly for a.e. $z \in Z$.

Hypothesis $(HF)(iv)$ implies the $(p-1)$-superlinear growth of $f(t, x)$ on x near ∞ , which is weaker than the well-known AR-condition and simplifies the verification of the PS-condition for the Euler functional of related problem. It should be pointed out that the hypothesis $(HF)(v)$ or $(HF)'(v)$ is different with the corresponding one in [11]. Hypothesis $(HF)(v)$ is a monotonicity condition [15], which is employed to study the multiplicity of positive solutions for nonlinear problems.

The rest of the paper is organized as follows. In Section II, we give some preliminaries. The main results for the existence and multiplicity of solutions to Eq.(1.1) are presented in Section III. Finally, we conclude this paper in Section IV.

II. Preliminaries

We firstly give some notations.

$r^\pm = \max\{\pm r, 0\}, \forall r \in R$, $m(\cdot)$ denotes the Lebesgue measure on R^N, an order Banach space $C_0^1(\bar{Z}) = \{x \in C^1(\bar{Z}) : x\,|_{\partial Z=0}\}$. $C_+ = \{x \in C^1(\bar{Z}) : x(z) \geq 0 \text{ for all } z \in \bar{Z}\}$ is a positive cone of $C_0^1(\bar{Z})$ with a nonempty interior given by

$$intC_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial Z}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\},$$

where $n(z)$ denotes the unit outward normal $z \in \partial Z$.

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We assume the Dirichlet p-Laplacian problem

$$\begin{cases} -\Phi_p(x(z)) = \lambda |x(z)|^{r-2} x(z), \ a.e. \text{ on } Z \\ x|_{\partial Z} = 0, \end{cases} \tag{2.1}$$

has a nontrivial solution and the symbol λ_1 denotes its principal eigenvalue. Obviously, from [8], the principal eigenvalue $\lambda_1 > 0$ is isolate and simple. From [2], we have

$$\lambda_1 = \inf \left\{ \frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \not\equiv 0 \right\} \tag{2.2}$$

and from [7], using (2.2), we obtain the following lemmas.

Lemma 2.1: If $\theta \in L^\infty(Z)_+$ satisfies $\theta(z) \le \lambda_1$ a.e. on Z and $\theta \ne \lambda_1$, then there exists $\xi_0 > 0$ such that $\|Dx\|_p^p - \int_Z \theta |x|^p dz \ge \xi_0 \|Dx\|_p^p$ for all $x \in W_0^{1,p}(Z)$.

In the following, the duality brackets $\langle \cdot, \cdot \rangle$ is listed for $(W^{-1,q}(Z), W_0^{1,p}(Z))$, where $W^{-1,q}(Z) \equiv W_0^{1,p}(Z)^*$ with $\frac{1}{p} + \frac{1}{q} = 1$. The nonlinear map $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is defined as

$$\langle A(x), y \rangle = \int_Z \|Dx\|_{R^N}^{p-2} (Dx, Dy)_{R^N} dz, \tag{2.3}$$

for all $x, y \in W_0^{1,p}(Z)$. Then we obtain the following lemma.

Lemma 2.2: If $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the map defined by (2.3), then A is bounded, continuous, strictly monotone (hence maximal monotone too) and if $x_n \rightharpoonup w$ and $\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \le 0$, then $x_n \to w$ in $W_0^{1,p}(Z)$ (i.e., A is of type $(S)_+$).

III. Main results

In this section, we discuss a class of p-Laplacian equations without the AR condition and give the existence and multiplicity of their solutions.

We define

$$f_+(z,x) = \begin{cases} 0, \ \text{if } x \le 0; \\ f(z,x), \ \text{if } x > 0. \end{cases} \tag{2.4}$$

and $F_+ = \int_0^x f_+(z,t)dt$. We also define the function $I_+ : W_0^{1,p}(Z) \to R$ as

$$I_+(x) = \frac{1}{p}\|Dx\|_p^p - \frac{1}{r}\int_Z m(x^+(z))^r dz - \int_Z F_+(z, x(z))dz$$

for all $x \in W_0^{1,p}(Z)$. Thus $I_+ \in C^1(W_0^{1,p}(Z))$. Then we get the following lemmas.

Lemma 3.1: If hypotheses $(i)-(v)$ of (HF) hold and $m \in L^\infty(Z)_+ \setminus \{0\}$, then I_+ satisfies the Cerami condition.

Proof: Let $\{x_n\} \subseteq W_0^{1,p}(Z)$ be a Cerami sequence [4], i.e.

$$I_+(x_n) \to c \in R \text{ and } (1 + \|x_n\|) I_+'(x_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

Our task now is to prove that $\{x_n\} \subseteq W_0^{1,p}(Z)$ is bounded. Firstly, we need to show that $\{x_n^-\} \subseteq W_0^{1,p}(Z)$ is bounded. From (2.5), we have

$$|\langle I_+'(x_n), u \rangle| \le \varepsilon_n, \ \forall \ u \in W_0^{1,p}(Z) \text{ with } \varepsilon_n \to 0. \tag{2.6}$$

Let $u = -x_n^- \in W_0^{1,p}(Z)$, then $\|Dx_n^-\|_p^p \le \varepsilon_n$. By Poincaré's inequality, $\{x_n^-\} \subseteq W_0^{1,p}(Z)$ is bounded. Secondly, we prove that $\{x_n^+\} \subseteq W_0^{1,p}(Z)$ is bounded. By contradiction, we suppose that $\|x_n^+\| \to \infty$ as $n \to \infty$. Let $y_n = \frac{x_n^+}{\|x_n^+\|}, n = 1, 2, \cdots$. Then $\|y_n\| = 1$ and $y_n \ge 0, n = 1, 2, \cdots$. We can choose a suitable subsequence $\{y_{n_k}\} \subseteq \{y_n\}$ (for the convenience, we still denote it as $\{y_n\}$) such that for a.e. $z \in Z, n = 1, 2, \cdots, y_n \rightharpoonup y \in W_0^{1,p}(Z)$, $y_n \to y \in L^\tau(Z)_+$, $y_n \to y$ $a.e.$ on Z, $|y_n(z)| \le h(z)$, with $h \in L^\tau(Z)_+$. Then $y \ge 0$. In (2.6), let $u = x_n^+ \in W_0^{1,p}(Z)$, then $|\langle I_+'(x_n), x_n^+ \rangle| \le \varepsilon_n$. Thus

$$\begin{aligned} &\left| \|Dx_n^+\|_p^p - \int_Z m(x_n^+)^r dz - \int_Z f_+(z,x_n) x_n^+ dz \right| \le \varepsilon_n, \\ &\left| \|Dy_n^+\|_p^p - \frac{1}{\|x_n^+\|^{p-r}} \int_Z m(z)(y_n^+)^r dz \right. \\ &\left. \quad - \frac{1}{\|x_n^+\|^{p-1}} \int_Z f_+(z, x_n^+) y_n dz \right| \le \frac{\varepsilon_n}{\|x_n^+\|^p}. \end{aligned} \tag{2.7}$$

The next thing is to show $y = 0$. Let $Z_+ = \{z \in Z : y(z) > 0\}$, then $x_n^+(z) \to +\infty$ a.e. $z \in Z_+$. By $(HF)(iv)$, for a.e. $z \in Z_+$, as $n \to \infty$, it follows that

$$\frac{f_+(z, x_n^+(z))}{(x_n^+(z))^{p-1}} \to +\infty, \tag{2.8}$$

Let $\chi_n(z) = \chi_{x_n^+ > 0}(z) = \chi_{y_n > 0}(z)$, then

$$\chi_n(z) y_n^p(z) \to \chi_{Z_+}(z) y_n^p(z), \ \text{a.e. on } Z. \tag{2.9}$$

If Z_+ has a positive Lebesgue measure $m(Z_+) > 0$, by (2.8),(2.9), as $n \to \infty$, we have

$$\chi_n(z) \frac{f(z, x_n^+(z))}{(x_n^+(z))^{p-1}} y_n^p(z) \to +\infty, \ \text{a.e. on } Z. \tag{2.10}$$

By Fatou's Lemma and (2.10), we get

$$\begin{aligned} &\int_Z \frac{f(z, x_n^+(z))}{\|(x_n^+(z))\|^{p-1}} y_n(z) dz \\ &= \int_Z \chi_n(z) \frac{f(z, x_n^+(z))}{(x_n^+(z))^{p-1}} y_n^p(z) dz \to +\infty, \ \text{as } n \to \infty. \end{aligned} \tag{2.11}$$

While in (2.7), as $n \to \infty$, by (2.11) and $r < p$, we get a contraction that $+\infty \le \frac{\varepsilon_n}{\|x_n^+\|^p} \to 0$. Thus $m(Z_+) = 0$. Then $y = 0$ for $y \ge 0$.

Let $t \in [0,1], \{t_n\} \subseteq [0,1], n = 1, 2, \cdots$, such that

$$I_+(t_n x_n^+) = \max_{t \in [0,1]} I_+(t x_n^+). \tag{2.12}$$

Let

$$v_n = (2p\|x_k^+\|^p)^{\frac{1}{p}} y_n, \ k, n = 1, 2, \cdots.$$

According to the Lebesgue dominated convengence theorem and $y = 0$, it follows that

$$\lim_{n\to\infty} \int_Z F_+(z, v_n) dz = 0, \ \lim_{n\to\infty} \int_Z m(z)|v_n(z)|^r dz = 0. \tag{2.13}$$

Since $\|x_n^+\| \to \infty$ as $n \to \infty$, we choose $n_0 \ge k$ such that

$$\frac{(2p\|x_k^+\|^p)^{\frac{1}{p}}}{\|x_n^+\|} \le 1, \ n \ge n_0. \tag{2.14}$$

From (2.12) and (2.14), we obtain

$$\begin{aligned} &I_+(t_n x_n^+) \ge I_+(v_n) \\ &= \frac{1}{p}\|Dv_n\|_p^p - \frac{1}{r}\int_Z m(v_n)^r dz - \int_Z F_+(z, v_n) dz \\ &= 2\|x_k^+\|_p - \frac{1}{r}\int_Z m(v_n)^r dz - \int_Z F_+(z, v_n) dz. \end{aligned} \tag{2.15}$$

From (2.13), (2.15), for sufficiently $n \geq n_0 \geq k$, we get

$$I_+(t_n x_n^+) \geq \|x_k^+\|_p, \tag{2.16}$$

and $I_+(t_n x_n^+) \to +\infty$ as $n \to \infty$.

It is easy to get that $I_+(0) = 0$. From the choice of $\{x_n\} \subseteq W_0^{1,p}(Z)$ and the boundness of $\{x_n^-\} \subseteq W_0^{1,p}(Z)$, we get that $I_+(x_n^+)$ is bounded. Thus $t_n \in (0,1)$. From (2.12), we have

$$\begin{aligned}0 = t_n\left(\frac{d}{dt} I_+(t x_n^+)\mid_{t=t_n}\right) &= \langle I'_+(t_n x_n^+), t_n x_n^+\rangle \\ &= t_n^p \|Dx_n^+\|_p^p - t_n^r \int_Z m(x_n^+)^r dz \\ &\quad - \int_Z f_+(z, t_n x_n^+) t_n x_n^+ dz. \end{aligned}\tag{2.17}$$

Then for $r < p$, by $(HF)(v)$, it follows that

$$\begin{aligned}&\frac{1}{p}\int_Z \sigma(z, x_n^+)dz + \frac{1}{p}\|\beta\|_{L^1} \\ &\geq \frac{1}{p}\int_Z \sigma(z, x_n^+)dz + \frac{1}{p}\int_Z |\beta| dz \\ &\geq \frac{1}{p}\int_Z \sigma(z, t_n x_n^+)dz \\ &= \int_Z \Big(\frac{1}{p} f_+(z, t_n x_n^+)) t_n x_n^+ - F(z, t_n x_n^+)\Big) dz \\ &= \frac{t_n^p}{p}\|Dx_n^+\|_p^p - \frac{t_n^r}{p}\int_Z m(x_n^+)^r dz - \int_Z F_+(z, t_n x_n^+)dz \\ &\geq I_+(t_n x_n^+),\end{aligned}\tag{2.18}$$

from (2.16), for $n \geq n_0 \geq k$, which implies that

$$\frac{1}{p}\int_Z \sigma(z, x_n^+)dz + \frac{1}{p}\|\beta\|_{L^1} \geq I_+(t_n x_n^+) \geq \|x_k^+\|^p. \tag{2.19}$$

On the other hand, from (2.5) and $\{x_n^-\} \subseteq W_0^{1,p}(Z)$ is bounded, we can choose $M_i > 0, i = 1, 2$ such that

$$\Big|\frac{1}{p}\|Dx_n^+\|_p^p - \frac{1}{r}\int_Z m(x_n^+)^r dz - \int_Z F_+(z, x_n^+)dz\Big| \leq M_1, \tag{2.20}$$

$$\begin{aligned}\big|\langle I'_+(x_n), x_n^+\rangle\big| &= \Big|\frac{1}{p}\|Dx_n^+\|_p^p - \frac{1}{p}\int_Z m(x_n^+)^r dz \\ &\quad - \frac{1}{p}\int_Z f_+(z, x_n^+)x_n^+ dz\Big| \\ &\leq M_2.\end{aligned}\tag{2.21}$$

From (2.20) and (2.21), we have

$$\begin{aligned}-M_1 - M_2 &\leq \frac{1}{p}\int_Z \sigma(z, x_n^+)dz - \frac{p-r}{p}\int_Z m(x_n^+)^r dz \\ &\leq M_1 + M_2.\end{aligned}$$

Combining (2.19), which implies that

$$\|x_k^+\|^p - \frac{p-r}{p}\|x_n^+\|^r \leq M_1 + M_2 + \frac{1}{p}\|\beta\|_{L^1}, \tag{2.22}$$

for $n \geq n_0 \geq k$. Recall that $k \geq 1$ was an arbitrary integer and let $k \to \infty$. Since $r < p$, from (2.22), we get a contradiction. This proves that $\{x_n^+\} \subseteq W_0^{1,p}(Z)$ is bounded and so $\{x_n\} \subseteq W_0^{1,p}(Z)$ is also bounded.

Then we may assume that $x_n \rightharpoonup x \ \in W_0^{1,p}(Z)$, $x_n \to x \ \in L^\tau(Z)_+$. Since that

$$\begin{aligned}&\Big|\langle I'_+(x_n), x_n - x\rangle\Big| \\ &= \Big|\langle A(x_n), x_n - x\rangle - \int_Z m(x_n^+)^r (x_n - x)dz \\ &\qquad - \int_Z f_+(z, x_n^+)(x_n - x)dz\Big| \\ &\leq \varepsilon_n,\end{aligned}\tag{2.23}$$

and $\int_Z m(x_n^+)^r (x_n - x)dz \to 0$, and $\int_Z f_+(z, x_n^+)(x_n - x)dz \to 0$. Then $\langle A(x_n), x_n - x\rangle \to 0$, as $n \to +\infty$, which shows that $x_n \to x \in W_0^{1,p}(Z)$ by Lemma 2.2. In all, I_+ satisfies the Cerami condition. This completes the proof. ■

Lemma 3.2: If hypotheses (HF) hold and $m \in L^\infty(Z)_+ \setminus \{0\}$, then there is $\zeta > 0$ such that $\|m\|_\infty \leq \zeta$. There also exist $\rho = \rho(\|m\|_\infty) > 0, \delta > 0$ such that $\inf_{\partial B_\rho} I_+(x) \geq \delta$, where $B_\rho = \{x \in W_0^{1,p}(Z) : \|x\| < \rho\}$.

Proof: By $(HF)(vi)$, $\forall \varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for a.e. $z \in Z, x \leq \delta(\varepsilon)$,

$$0 \leq f_+(z, x) \leq (\theta(z) + \varepsilon)(x^+)^{p-1}. \tag{2.24}$$

From $(HF)(iii)$, $\forall \varepsilon > 0$, there is $\delta(\varepsilon) > 0, c_3(\varepsilon) > 0$ such that for a.e. $z \in Z, x \leq \delta(\varepsilon)$,

$$0 \leq f_+(z, x) \leq c_3 (x^+)^{\tau - 1}. \tag{2.25}$$

Combining hypotheses $(HF)(vi)$ with $(HF)(iii)$, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0, c_3(\varepsilon) > 0$ such that for a.e. $z \in Z, x \in R$,

$$0 \leq f_+(z, x) \leq (\theta(z) + \varepsilon)(x^+)^{p-1} + c_3 (x^+)^{\tau-1}, \tag{2.26}$$

$$F_+(z, x) \leq \frac{1}{p}(\theta(z) + \varepsilon)(x^+)^p + \frac{c_3}{\tau}(x^+)^\tau. \tag{2.27}$$

Thus, from (2.27) and $x^+(z) \leq |x(z)|$ a.e. on Z, we have

$$\begin{aligned}&I_+(x) \\ &= \frac{1}{p}\|Dx\|_p^p - \frac{1}{r}\int_Z m(x^+(z))^r dz - \int_Z F_+(z, x(z))dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{r}\|m\|_\infty \|x\|_r^r - \frac{1}{p}\int_Z \theta(z)|x|^p dz \\ &\qquad - \frac{\varepsilon}{p}\|x\|_p^p - \frac{c_3}{\tau}\|x\|_\tau^\tau,\end{aligned}\tag{2.28}$$

for $\forall x \in W_0^{1,p}(Z)$. By $W_0^{1,p}(Z)$ is embedded continuously and compactly into $L^r(Z)$ and $L^\tau(Z)$ for $r < p < \tau < p^*$, using Poincaré's inequality, (2.2) and Lemma 2.1, there are $c_4(\varepsilon) > 0, c_5 > 0, c_6 = \frac{1}{p}(\xi_0 - \frac{\varepsilon}{\lambda_1}) > 0$ such that

$$\begin{aligned}&I_+(x) \\ &= \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta(z)|x|^p dz - c_4\|Dx\|_p^\tau \\ &\qquad - c_5\|m\|_\infty \|Dx\|_p^r - \frac{\varepsilon}{p\lambda_1}\|Dx\|_p^p \\ &\geq \frac{1}{p}\big(\xi_0 - \frac{\varepsilon}{\lambda_1}\big)\|Dx\|_p^p - c_4\|Dx\|_p^\tau \\ &\qquad - c_5\|m\|_\infty \|Dx\|_p^r - \frac{\varepsilon}{p\lambda_1}\|Dx\|_p^p \\ &= (c_6 - c_4(\varepsilon)\|Dx\|_p^{\tau-p} - c_5\|m\|_\infty \|Dx\|_p^{r-p})\|Dx\|_p^p.\end{aligned}\tag{2.29}$$

Considering the auxiliary function

$$f(t) = c_4(\varepsilon)t^{\tau-p} + c_5\|m\|_\infty t^{r-p}, t > 0. \quad (2.30)$$

Since $r < \tau < p$, then $\lim_{t\to 0^+} f(t) = \lim_{t\to+\infty} f(t) = +\infty$. According to the continuity and differentiability of f, there exists $t_0 > 0$ such that $0 < f(t_0) = \min_{t\geq 0} f(t)$ and $0 = f'(t_0) = c_4(\tau-p)t_0^{\tau-p-1} + c_5\|m\|_\infty(r-p)t_0^{r-p-1}$. Thus we have $t_0 = \sqrt[(\tau-r)]{\left(\frac{c_5\|m\|_\infty(p-r)}{c_4(\tau-p)}\right)}$. From (2.30), there exists $\zeta > 0$ such that if $\|m\|_\infty \leq \zeta$, then $f(t_0) < c_6$. From (2.29), there exist $\|x\| = t_0 = \rho = \rho(\|m\|_\infty) > 0, \delta > 0$ such that $\inf_{\partial B_\rho} I_+(x) \geq \delta$, where $B_\rho = \{x \in W_0^{1,p}(Z) : \|x\| < \rho\}$. ■

Lemma 3.3: If hypotheses (HF) hold and $m \in L^\infty(Z)_+ \setminus \{0\}$ and $y \in C_+ \setminus \{0\}$ with $\|y\|_p = 1$. Then $I_+(\lambda y) \to -\infty$ as $\lambda \to \infty$.

Proof: According to $(HF)(iv)$, for $\forall\varepsilon > 0, \exists\, M(\varepsilon) > 0$ such that for a.e. $z \in Z$ and $x \geq M(\varepsilon)$, we have

$$f_+(z, x) \geq \frac{x^{p-1}}{\varepsilon}. \quad (2.31)$$

Let $c(\varepsilon) = \frac{1}{\varepsilon}M(\varepsilon)^{p-1}$, then $f_+(z, x) \geq \frac{x^{p-1}}{\varepsilon} - c(\varepsilon)$ for a.e. $z \in Z$ and $x \geq 0$. Thus we have

$$F_+(z, x) \geq \frac{x^p}{p\varepsilon} - c(\varepsilon)x, \quad \text{for } a.e.\ z \in Z \text{ and } x \geq 0. \quad (2.32)$$

If let $y \in C_+ \setminus \{0\}$ with $\|y\|_p = 1, \lambda > 0$, then for a.e. $z \in Z$ and $\bar{C}(\varepsilon) = c(\varepsilon)\|y\|_1$, by (2.32), we have

$$F_+(z, \lambda y(z)) \geq \frac{\lambda^p y^p(z)}{p\varepsilon} - c(\varepsilon)\lambda y(z), \quad (2.33)$$

$$\frac{F_+(z,\lambda y(z))}{\lambda^p} \geq \frac{y^p(z)}{p\varepsilon} - \frac{c(\varepsilon)}{\lambda^{p-1}}y(z), \quad (2.34)$$

$$\int_Z \frac{F_+(z, \lambda y(z))}{\lambda^p}dz \geq \int_Z \frac{y^p(z)}{p\varepsilon}dz - \int_Z \frac{c(\varepsilon)}{\lambda^{p-1}}y(z)dz = \frac{1}{p\varepsilon} - \frac{\bar{c}(\varepsilon)}{\lambda^{p-1}}. \quad (2.35)$$

Thus

$$\liminf_{\lambda\to\infty} \int_Z \frac{F_+(z, \lambda y(z))}{\lambda^p}dz \geq \frac{1}{p\varepsilon}. \quad (2.36)$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{\lambda\to+\infty} \int_Z \frac{F_+(z, \lambda y(z))}{\lambda^p}dz = +\infty. \quad (2.37)$$

Then by (2.37), by $r < p$, we have $\lim_{\lambda\to+\infty} \frac{I_+(\lambda y)}{\lambda^p} = -\infty$, and $I_+(\lambda y) \to -\infty$ as $\lambda \to +\infty$. It follows that there exists $\lambda_0, \eta > 0$ such that $\lambda_0 y \in W_0^{1,p}(Z), \|\lambda_0 y\|_p > \eta > 0$ and $I_+(\lambda_0 y) < 0$. ■

Theorem 3.1: Let (HF) hold and $m \in L^\infty(Z)_+ \setminus \{0\}$. If there is $\zeta > 0$ such that $\|m\|_\infty \leq \zeta$. Then (1.1) has at least two positive solutions $x_1, x_2 \in intC_+$.

Proof: By Lemma 2.3-2.5, we have proved that I_+ satisfies a mountain pass geometry [8]. Thus there exists $x_1 \in W_0^{1,p}(Z)$ such that

$$I_+(0) = 0 < \eta \leq I_+(x_1), \text{ and } I'_+(x_1) = 0. \quad (3.1)$$

From (3.1), it follows that $x_1 \not\equiv 0$. From (3.1), we also have

$$A(x_1) = m(x_1^+)^{r-1} + N_+(x_1), \quad (3.2)$$

where $N_+(u)(z) = f_+(z, u(z)), u \in W_0^{1,p}(Z)$. From (3.2), for $-x_1^- \in W_0^{1,p}(Z)$, we have $\|Dx_1^-\|_p^p = 0$ since $f_+(z, z) = 0$ for a.e. $z \in Z$ and $x \leq 0$, which shows that $x_1 \geq 0$ and $x_1 \not\equiv 0$.

From (3.2), we get

$$-\Phi_p x_1(z) = m(z)x_1^{r-1}(z) + f(z, x_1(z)), \quad a.e. \text{ on } Z \text{ and } z\,|_{\partial Z} = 0. \quad (3.3)$$

By nonlinear regularity theory [8], we have $x_1 \in C_+ \setminus \{0\}$. From (2.30), we obtain $\Phi_p x_1(z) \leq 0$ a.e. on Z. By the nonlinear strong maximum principle of [20], we show that $x_1 \in intC_+$.

According to Lemma 2.4, there is $\zeta > 0$ such that $\|m\|_\infty \leq \zeta$. There also exists $\rho = \rho(\|m\|_\infty) > 0, \delta > 0$ such that $\inf_{\partial B_\rho} I_+(x) \geq \delta > 0$, where $B_\rho = \{x \in W_0^{1,p}(Z) : \|x\| < \rho\}$.

Next, we will show that $-\infty < \inf_{\bar{B}_\rho} I_+ < 0$. From (2.29), we have $-\infty < \inf_{\bar{B}_\rho} I_+$. Let $u \in \bar{C}^1 = \{u \in C^1(Z) : u \text{ has support in Z}\}$ with $u \geq 0, u \not\equiv 0$ and $\lambda > 0$, then for $F_+ \geq 0$,

$$\begin{aligned} &I_+(\lambda u) \\ &= \frac{\lambda^p}{p}\|Du\|_p^p - \frac{\lambda^r}{r}\int_Z \theta(z)u^r dz - \int_Z F_+(z, \lambda u)dz \\ &\leq \frac{\lambda^p}{p}\|Du\|_p^p - \frac{\lambda^r}{r}\int_Z \theta(z)u^r dz, \end{aligned} \quad (3.4)$$

Since $r < p$, from (3.4) and λ small enough, we have $I_+(\lambda u) < 0$ and $-\infty < \inf_{\bar{B}_\rho} I_+ < 0$. Let $\epsilon \in (0, \inf_{\partial\bar{B}_\rho} I_+ - \inf_{\bar{B}_\rho} I_+)$ and consider the function $I_+ : \bar{B}_\rho \to R$. By using Ekeland variational principle [6], we obtain that there exists $x(\epsilon) \in \bar{B}_\rho$ such that

$$I_+(x(\epsilon)) \leq \inf_{\bar{B}_\rho} I_+ + \epsilon, \quad (3.5)$$

$$I_+(x(\epsilon)) \leq I_+(y) + \epsilon\|y - x(\epsilon)\|, \ \forall\ y \in \bar{B}_\rho. \quad (3.6)$$

Due to (3.5) and we choose suitable $\epsilon > 0$ such that

$$I_+(x(\epsilon)) \leq \inf_{\bar{B}_\rho} I_+ + \epsilon < \inf_{\partial\bar{B}_\rho} I_+. \quad (3.7)$$

From (3.7), it is easy to see $x(\epsilon) \in B_\rho$. Define the following function

$$\varphi_\epsilon(y) = I_+(y) + \epsilon\|y - x(\epsilon)\|. \quad (3.8)$$

From (3.6), it follows that $x(\epsilon) \in B_\rho$ is a minimizer of φ_ϵ on $\bar{B}_\rho$. Therefore for all $\lambda > 0$ and $k \in W_0^{1,p}(Z)$ with $\|k\| = 1$, we have $\frac{\varphi_\epsilon(x(\epsilon)+\lambda k)-\varphi_\epsilon(x(\epsilon))}{\lambda} \geq 0$, then $\frac{I_+(x(\epsilon)+\lambda k)-I_+(x(\epsilon))}{\lambda} + \epsilon\|k\| \geq 0$, and

$$\langle I'_+(x(\epsilon)), k\rangle \geq -\epsilon\|k\|, \text{ and } \|\langle I'_+(x(\epsilon))\rangle\| \leq \epsilon. \quad (3.9)$$

Let $\epsilon_n = \frac{1}{n}$ and choose $x_n \equiv x_{\epsilon_n} \in B_\rho$. Then by (3.7), $I_+(x_n) \to \inf_{\bar{B}_\rho} I_+$ and $I'_+(x_n) \to 0$. By Lemma 2.3, we can assume that $x_n \to \bar{x} \in W_0^{1,p}(Z)$. Thus $I_+(\bar{x}) = \inf_{\bar{B}_\rho} I_+ < 0 = I_+(0)$, which implies that $\bar{x} \not\equiv 0$. Recall that $I_+(\bar{x}) = \inf_{\bar{B}_\rho} I_+ < \inf_{\partial\bar{B}_\rho} I_+ \leq I_+(x_1)$, then $\bar{x} \neq x_1$.

Since $I'_+(x_n) \to 0$, we have $I'_+(\bar{x}) \to 0$ and $A(\bar{x}) = m(\bar{x}^+)^{r-1} + N_+(\bar{x})$. Then, for $-\bar{x}^- \in W_0^{1,p}(Z)$, we have $\|D\bar{x}^-\|_p^p = 0$ since $f_+(z, x) = 0$ for a.e. $z \in Z$ and $x \leq 0$, which shows that $\bar{x} \geq 0$ and $\bar{x} \not\equiv 0$. Also, $\bar{x}$ is a solution of problem (1.1). Then, in a similar way as we did for x_1, via the nonlinear regularity theory and the nonlinear strong maximum principle, we show that $\bar{x} \in intC_+$. ■

Similar to the proof of Theorem 3.1, we state the theorems as follows but omit the proof.

Theorem 3.2: Let $(HF)'$ hold and $m \in L^\infty(Z)_+ \setminus \{0\}$. If there is $\zeta > 0$ such that $\|m\|_\infty \leq \zeta$. Then (1.1) has at least two negative solutions $x_3, x_4 \in -intC_+$.

Theorem 3.3: Let (HF) and $(HF)'$ hold and $m \in L^\infty(Z)_+ \setminus \{0\}$. If there is $\zeta > 0$ such that $\|m\|_\infty \leq \zeta$. Then (1.1) has at least four solutions $x_1, x_2 \in -intC_+$ and $x_3, x_4 \in -intC_+$.

IV. Conclusions

In this paper, we use Mountain pass lemma and Ekeland variational principle to obtain the existence and multiplicity of the solutions of p-Laplacian equations without the AR condition, and our hypothesis condition is weaker than the AR condition.

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