Initial Boundary Value Problem for a Generalized Zakharov Equations

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Abstract—**This paper considers the existence of the generalized solution to the initial boundary value problem for a class of generalized Zakharov equation in $(2+1)$ dimensions. By a priori integral estimates and the Galerkin method, the author establishes global in time existence of the solution to the problem.**

Index Terms—**generalized solution, generalized Zakharov equations, initial boundary value problem, Zakharov equations.**

I. Introduction

A Set of coupled nonlinear wave equations describing the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves was first derived by Zakharov [1]. The usual Zakharov system defined in space time $\mathbb{R}^{d+1}$ is expressed as

$$iE_t + \Delta E = nE,$$
$$n_{tt} - \Delta n = \Delta |E|^2,$$

where $E : \mathbb{R}^{d+1} \to \mathbb{C}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n : \mathbb{R}^{d+1} \to \mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium.

This system has been the subject of many studies [2], [3], [4], [5], [6], [7], [8], [9], [10]. The 1D Cauchy problem for the Zakharov system is shown to be locally well-posed for low regularity Schrödinger data and wave data under certain assumptions on the parameters [3]. The study [4] deals with the existence and uniqueness of smooth solution for a generalized Zakharov equation and establishes local in time existence and uniqueness in the case of dimensions 2 and 3. S. You studied a generalized Zakharov equation and obtained the existence and uniqueness of the global solutions to the initial value problem [10].

In this paper, we study the following systems of 2-dimensional generalized Zakharov type equations

$$iE_t + \Delta E - nE = \alpha |E|^2 E, \quad (1)$$

$$\partial_t n + \nabla \cdot V = 0, \quad (2)$$

$$V_t + \sum_{j=1}^{2} \frac{\partial \mathrm{grad}\varphi(V)}{\partial x_j} - \beta \Delta V_t + \nabla \left(n + |E|^2\right) = 0. \quad (3)$$

with initial boundary data as follows:

$$E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad V|_{t=0} = V_0(x), \quad (4)$$

$$E|_{\partial\Omega} = 0, \quad n|_{\partial\Omega} = 0, \quad V|_{\partial\Omega} = 0 \quad (5)$$

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where $E(x,t) = (E_1(x,t), E_2(x,t), \cdots, E_N(x,t))$ is an N-dimensional complex valued unknown functional vector, $V(x,t) = (V_1(x,t), V_2(x,t))$ is a 2-dimensional real-valued unknown functional vector, $n(x,t)$ is a real-valued unknown function, $\varphi(s)$ is a real function, and $\beta > 0$, $x \in \Omega \subset \mathbb{R}^2$, $t \geq 0$.

We study the generalized Zakharov system in dimension 2 with initial boundary data. First, a priori estimates of the problem is made. Next, using the Galerkin method, the existence of the global generalized solution of the problem is shown. In fact, nonlinear partial differential equations have also been studied by others using different approaches, as can be seen in [14]-[36].Now we state the main results of this paper.

Theorem 1. *Supposing that*

(1) $E_0(x) \in H^1(\Omega), V_0(x) \in H^1(\Omega), n_0(x) \in L^2(\Omega)$,

(2) $\varphi(v) \in C^2, \varphi(0) = 0$.

(3) $\|E_0(x)\|_{L^2}^2 < \dfrac{\varepsilon}{1+\varepsilon|\alpha|} \|\psi(x)\|_{L^2}^2$,

(4) $\left|\mathrm{grad}\varphi(v)\right| \leq C(|v|+1)$.

where $0 < \varepsilon < 1$, $\psi(x)$ *is a solution of the equation*

$$\Delta\psi - \psi + \psi^3 = 0.$$

Then there is the global generalized solution of the initial boundary problem (1)-(5).

$$E(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}),$$
$$V(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H_0^1),$$
$$n(x,t) \in L^\infty(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; L^2),$$

For the sake of convenience of the following contexts, we set some notations. For $1 \leq q \leq \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q times integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ or simply $\|\cdot\|_{L^q}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p=2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f,g) = \int_{\mathbb{R}^n} f(x)\cdot\overline{g(x)}dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. We use C to represent various constants that can depend on the initial data.

This paper is organized as follows. In Section II, we make a priori estimates of the problem (1)-(5). In Section III, we establish the global generalized solution of the problem (1)-(5) by the Galerkin method.

II. A PRIORI ESTIMATES

In this section, we will derive a priori estimates for the solution of the system (1)-(5).

Lemma 1. *Suppose that* $E_0(x) \in L^2(\Omega)$. *Then for the solution of problem* (1)-(5), *we have*

$$\|E(x,t)\|_{L^2(\Omega)}^2 = \|E_0(x)\|_{L^2(\Omega)}^2.$$

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Proof: Taking the inner product of (1) and E, it follows that

$$(iE_t + \Delta E - nE,\, E) = \left(\alpha\,|E|^2\, E,\, E\right) \tag{6}$$

Since

$$\mathrm{Im}\,(iE_t,\, E) = \frac{1}{2}\frac{d}{dt}\,\|E\|_{L^2}^2\,,$$
$$\mathrm{Im}\,(\Delta E - nE,\, E) = \mathrm{Im}\left(\alpha\,|E|^2\, E,\, E\right) = 0,$$

hence from (6), we get

$$\frac{d}{dt}\,\|E(x,t)\|_{L^2}^2 = 0.$$

We thus get Lemma 1. ■

Lemma 2. *Suppose that*

(1) $E_0(x) \in H^1(\Omega)$, $V_0(x) \in H^1(\Omega)$, $n_0(x) \in L^2(\Omega)$,
(2) $\varphi(v) \in C^2$, $\varphi(0) = 0$.

Then we have

$$\mathscr{M}(t) = \mathscr{M}(0).$$

where

$$\mathscr{M}(t) = \|\nabla E\|_{L^2}^2 + \int \left(n\,|E|^2 + \frac{\alpha}{2}\,|E|^4\right) dx + \frac{\beta}{2}\,\|\nabla V\|_{L^2}^2 + \frac{1}{2}\,\|n\|_{L^2}^2 + \frac{1}{2}\,\|V\|_{L^2}^2\,.$$

Proof: Taking the inner products of (1) and E_t, it follows that

$$(iE_t + \Delta E - nE,\, E_t) = \left(\alpha\,|E|^2\, E,\, E_t\right). \tag{7}$$

Since

$$\mathrm{Re}\,(iE_t,\, E_t) = 0,$$
$$\mathrm{Re}\,(\Delta E,\, E_t) = -\mathrm{Re}\,(\nabla E,\, \nabla E_t) = -\frac{1}{2}\frac{d}{dt}\,\|\nabla E\|_{L^2}^2\,,$$

$$\begin{aligned}
&\mathrm{Re}\left(-nE - \alpha\,|E|^2\, E,\, E_t\right)\\
&= -\frac{1}{2}\int \left(n\left(|E|^2\right)_t + \alpha|E|^2\left(|E|^2\right)_t\right) dx\\
&= -\frac{1}{2}\frac{d}{dt}\int \left(n|E|^2 + \frac{\alpha}{2}|E|^4\right) dx + \frac{1}{2}\int n_t|E|^2 dx.
\end{aligned}$$

thus from (7) it follows that

$$\frac{d}{dt}\left(\|\nabla E\|_{L^2}^2 + \int \left(n|E|^2 + \frac{\alpha}{2}|E|^4\right) dx\right) = \int n_t|E|^2 dx. \tag{8}$$

From (2)-(3), we obtain

$$\begin{aligned}
&\int n_t|E|^2 dx = -\int (\nabla\cdot V)\,|E|^2 dx = \int V\cdot\nabla|E|^2 dx\\
&= \int V\cdot\left(-V_t - \sum_{j=1}^{2}\frac{\partial \mathrm{grad}\varphi(V)}{\partial x_j} + \beta\Delta V_t - \nabla n\right) dx\\
&= -\beta\int \nabla V\cdot\nabla V_t dx + \int (\nabla\cdot V)\, n dx - \frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2\\
&\quad + \sum_{j=1}^{2}\int \mathrm{grad}\varphi(V)\cdot\frac{\partial V}{\partial x_j} dx\\
&= -\frac{\beta}{2}\frac{d}{dt}\|\nabla V\|_{L^2}^2 - \int n_t n dx - \frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 + \sum_{j=1}^{2}\int \frac{\partial\varphi(V)}{\partial x_j} dx\\
&= -\frac{1}{2}\frac{d}{dt}\left(\beta\|\nabla V\|_{L^2}^2 + \|n\|_{L^2}^2 + \|V\|_{L^2}^2\right).
\end{aligned} \tag{9}$$

Combining (8) with (9) we obtain

$$\frac{d}{dt}\left(\|\nabla E\|_{L^2}^2 + \int \left(n|E|^2 + \frac{\alpha}{2}|E|^4\right) dx\right) + \frac{1}{2}\frac{d}{dt}\left(\beta\|\nabla V\|_{L^2}^2 + \|n\|_{L^2}^2 + \|V\|_{L^2}^2\right) = 0.$$

Letting

$$\mathscr{M}(t) = \|\nabla E\|_{L^2}^2 + \int \left(n|E|^2 + \frac{\alpha}{2}|E|^4\right) dx + \frac{1}{2}\left(\beta\|\nabla V\|_{L^2}^2 + \|n\|_{L^2}^2 + \|V\|_{L^2}^2\right).$$

It follows that

$$\mathscr{M}(t) = \mathscr{M}(t).$$

■

Lemma 3 (Gagliardo-Nirenberg inequality [11])**.** *Assume that $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty, 0 \le j \le m$, we have the estimations*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \le C\|D^m u\|_{L^r(\mathbb{R}^n)}^{\alpha}\|u\|_{L^q(\mathbb{R}^n)}^{1-\alpha},$$

where C is a positive constant, $0 \le \frac{j}{m} \le \alpha \le 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$$

Lemma 4 (Sobolev's best constant estimates [12])**.** *Suppose that $f(x) \in H^1(\mathbb{R}^N)$. Then we have*

$$\|f\|_{L^{2p+2}(\mathbb{R}^N)}^{2p+2} \le C_{p,N}^{2p+2}\|\nabla f\|_{L^2(\mathbb{R}^N)}^{pN}\|f\|_{L^2(\mathbb{R}^N)}^{2+p(2-N)},$$
$$0 < p < \frac{2}{N-2},\ N \ge 2,$$

where the constant

$$C_{p,N} = \left(\frac{p+1}{\|\psi\|_{L^2(\mathbb{R}^N)}^{2p}}\right)^{\frac{1}{2p+2}}$$

and $\psi(x)$ is a ground state solution for the equation

$$\frac{pN}{2}\Delta\psi - \left(1 + \frac{p}{2}(2-N)\right)\psi + \psi^{2p+1} = 0. \tag{10}$$

Obviously, the solution of equation (10) exists, and $\psi(x) \neq 0$.

Lemma 5. *Suppose that the conditions of Lemma 2 are satisfied and*

$$\|E_0(x)\|_{L^2}^2 < \frac{\varepsilon}{1+\varepsilon|\alpha|}\,\|\psi(x)\|_{L^2}^2\,,$$

where $0 < \varepsilon < 1$, ψ is a solution of the equation

$$\Delta\psi - \psi + \psi^3 = 0.$$

Then we have

$$\|\nabla E\|_2^2 + \|\nabla V\|_2^2 + \|n\|_2^2 + \|V\|_2^2 \le C.$$

Proof: By Hölder inequality, Young inequality, there holds

$$\left|\int_\Omega n\,|E|^2\, dx\right| \le \|n\|_{L^2}\|E\|_{L^4}^2 \le \frac{\varepsilon}{2}\|n\|_{L^2}^2 + \frac{1}{2\varepsilon}\|E\|_{L^4}^4. \tag{11}$$

Using Gagliardo-Nirenberg inequality and Lemma 4, we express as

$$\|E\|_{L^4}^4 \le \frac{2}{\|\psi\|_{L^2}^2}\|\nabla E\|_{L^2}^2\|E\|_{L^2}^2. \tag{12}$$

Note that from Lemma 2 and Equations (11), (12), one has

$$\left(1-\frac{1+\varepsilon|\alpha|}{\varepsilon\|\psi\|_{L^2}^2}\|E_0\|_{L^2}^2\right)\|\nabla E\|_{L^2}^2+\frac{\beta}{2}\|\nabla V\|_{L^2}^2 + \frac{1-\varepsilon}{2}\|n\|_{L^2}^2+\frac{1}{2}\|V\|_{L^2}^2 \le |\mathscr{M}(0)|.$$

Note that

$$\|E_0(x)\|_{L^2}^2 < \frac{\varepsilon}{1+\varepsilon|\alpha|}\|\psi(x)\|_{L^2}^2$$

and $0<\varepsilon<1$, we thus get Lemma 5. ■

Lemma 6. *Suppose that the conditions of Lemma 5 are satisfied and* $\left|\mathrm{grad}\varphi(v)\right| \le C\,(|v|+1)$. *Then we have*

$$\|E_t\|_{H^{-1}}+\|V_t\|_{H_0^1}+\|n_t\|_{L^2}\le C.$$

Proof: Taking the inner product of Eq. (1) and Φ, Eq. (2) and φ, it follows that

$$(iE_t+\Delta E-nE,\,\Phi)=\left(\alpha\,|E|^2\,E,\,\Phi\right), \tag{13}$$

$$(\partial_t n+\nabla\cdot V,\,\varphi)=0, \tag{14}$$

where $\varphi,\varphi_j\in H_0^2$ $(j=1,\cdots,N)$, $\Phi=(\varphi_1,\cdots,\varphi_N)$.

By Hölder inequality, it follows from Eq. (13) that

$$\begin{aligned}|(E_t,\Phi)| &\le |(\Delta E,\Phi)|+|(nE,\Phi)|+\left|\left(\alpha|E|^2E,\Phi\right)\right|\\ &= |(\nabla E,\nabla\Phi)|+|(nE,\Phi)|+\left|\left(\alpha|E|^2E,\Phi\right)\right|\\ &\le \|\nabla E\|_{L^2}\,\|\nabla\Phi\|_{L^2}+\|n\|_{L^2}\,\|E\|_{L^4}\,\|\Phi\|_{L^4}\\ &\quad+|\alpha|\,\|E\|_{L^6}^3\,\|\Phi\|_{L^2}\,.\end{aligned} \tag{15}$$

By Gagliardo-Nirenberg inequality, we know that

$$\begin{aligned}&\|E\|_{L^4}\le C\|\nabla E\|_{L^2}^{\frac12}\|E\|_{L^2}^{\frac12}\le C,\\ &\|E\|_{l^6}\le C\|\nabla E\|_{L^2}^{\frac23}\|E\|_{L^2}^{\frac13}\le C,\\ &\|\Phi\|_{L^4}\le C\|\nabla\Phi\|_{L^2}^{\frac12}\|\Phi\|_{L^2}^{\frac12}\le C\left(\|\nabla\Phi\|_{L^2}+\|\Phi\|_{L^2}\right).\end{aligned}$$

Hence from Eq. (15) we get

$$|(E_t,\Phi)|\le C\|\Phi\|_{H_0^1}. \tag{16}$$

Using Hölder inequality, from Eq. (14), there is

$$|(n_t,\varphi)|=|(\nabla\cdot V,\varphi)|\le\|\nabla V\|_{L^2}\,\|\varphi\|_{L^2}\le C\|\varphi\|_{L^2}. \tag{17}$$

Taking the inner product of Eq. (3) and V_t, it follows that

$$\left(V_t+\sum_{j=1}^{2}\frac{\partial\mathrm{grad}\varphi(V)}{\partial x_j}-\beta\Delta V_t+\nabla\left(n+|E|^2\right),\,V_t\right)=0. \tag{18}$$

Since

$$(V_t-\beta\Delta V_t,\,V_t)=\|V_t\|_{L^2}^2+\beta\,\|\nabla V_t\|_{L^2}^2\,,$$

$$\begin{aligned}\left|\left(\sum_{j=1}^{2}\frac{\partial\mathrm{grad}\varphi(V)}{\partial x_j},\,V_t\right)\right| &= \left|\left(\sum_{j=1}^{2}\mathrm{grad}\varphi(V),\,\frac{\partial V_t}{\partial x_j}\right)\right|\\ &\le C\left(\|V\|_{L^2}+1\right)\|\nabla V_t\|_{L^2}\\ &\le C+\frac{\beta}{4}\|\nabla V_t\|_{L^2}^2,\end{aligned}$$

$$|(\nabla n,V_t)|=|(n,\nabla\cdot V_t)|\le\|n\|_{L^2}\|\nabla V_t\|_{L^2}\le C+\frac{\beta}{4}\|\nabla V_t\|_{L^2}^2,$$

$$\begin{aligned}\left|\left(\nabla|E|^2,V_t\right)\right| &= \left|\left(|E|^2,\nabla\cdot V_t\right)\right|\le\|E\|_{L^4}^2\|\nabla V_t\|_{L^2}\\ &\le C+\frac{\beta}{4}\|\nabla V_t\|_{L^2}^2.\end{aligned}$$

from Eq. (18) we get

$$\|V_t\|_{L^2}^2+\frac{\beta}{4}\|\nabla V_t\|_{L^2}^2\le C. \tag{19}$$

Hence from (16), (17) and (19), we obtain

$$\|E_t\|_{H^{-1}}+\|V_t\|_{H_0^1}+\|n_t\|_{L^2}\le C.$$

■

III. THE EXISTENCE OF GLOBAL GENERALIZED SOLUTION

In this section, we formulate the proof of Theorem 1. First we give the definition of generalized solution for problems (1)-(5).

Definition 1. *The functions*

$$\begin{aligned}&E_m(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H^{-1}),\quad m=1,2,\cdots,N,\\ &V_\lambda(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H_0^1),\quad \lambda=1,2,\\ &n(x,t)\in L^\infty(\mathbb{R}^+;L^2)\cap W^{1,\infty}(\mathbb{R}^+;L^2)\end{aligned}$$

are called generalized solution of problems (1)-(5)*, if for any* $\xi\in H_0^1$ *they satisfy the integral equality*

$$\begin{aligned}&(iE_{mt},\xi)-(\nabla E_m,\nabla\xi)-(nE_m,\xi)=\left(\alpha\,|E|^2\,E_m,\xi\right),\\ &\qquad m=1,2,\cdots,N,\\ &(\partial_t n,\xi)-(V,\nabla\xi)=0,\\ &(V_{\lambda t},\xi)-\left(\sum_{j=1}^{2}\frac{\partial\varphi(V)}{\partial V_\lambda},\frac{\partial\xi}{\partial x_j}\right)+(\beta\nabla V_{\lambda t},\nabla\xi)\\ &\quad-\left(n+|E|^2,\frac{\partial\xi}{\partial x_\lambda}\right)=0,\ \ \lambda=1,2.\end{aligned}$$

with initial boundary data as follows:

$$\begin{aligned}&E|_{t=0}=E_0(x),\quad n|_{t=0}=n_0(x),\quad V|_{t=0}=V_0(x),\\ &E|_{\partial\Omega}=0,\quad n|_{\partial\Omega}=0,\quad V|_{\partial\Omega}=0\end{aligned}$$

Next, we give two lemmas recalled in [13].

Lemma 7. *Let* B_0,B,B_1 *be three reflexive Banach spaces and assume that the embedding* $B_0\to B$ *is compact. Let*

$$W=\left\{V\in L^{p_0}((0,T);B_0),\ \frac{\partial V}{\partial t}\in L^{p_1}((0,T);B_1)\right\},\quad T<\infty,\ \ 1<p_0,p_1<\infty.$$

W is a Banach space with norm

$$\|V\|_W=\|V\|_{L^{p_0}((0,T);B_0)}+\|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W\to L^{p_0}((0,T);B)$ *is compact.*

Lemma 8. *Let* Ω *be an open set of* $\mathbb{R}^n$ *and let* $g,g_\varepsilon\in L^p(\mathbb{R}^n)$, $1<p<\infty$, *such that*

$$g_\varepsilon\to g\quad\text{a.e. in }\Omega\qquad\text{and}\qquad\|g_\varepsilon\|_{L^p(\Omega)}\le C.$$

Then $g_\varepsilon\to g$ *weakly in* $L^p(\Omega)$.

Now, one can estimate Theorem 1.

Proof: By using the Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$-\Delta\omega_j(x)=\lambda_j\omega_j(x),\ \ \omega_j(x)\in H_0^1(\Omega),\ \ j=1,2,\cdots,l.$$

The approximate solution of problem (1)-(5) can be written as

$$E^l(x,t)=\sum_{j=1}^{l}\alpha_j^l(t)\omega_j(x),\quad V^l(x,t)=\sum_{j=1}^{l}\beta_j^l(t)\omega_j(x),$$
$$n^l(x,t)=\sum_{j=1}^{l}\gamma_j^l(t)\omega_j(x),$$

where

$$E^l=\left(E_1^l,\cdots,E_N^l\right),\quad \alpha_j^l(t)=\left(\alpha_{j1}^l(t),\cdots,\alpha_{jN}^l(t)\right),$$
$$V^l=\left(V_1^l,V_2^l\right),\quad \beta_j^l(t)=\left(\beta_{j1}^l(t),\beta_{j2}^l(t)\right).$$

According to Galerkin's method, these undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ must satisfy the following initial value problem of the system of ordinary differential equations:

$$\left(iE_{mt}^l,\omega_\kappa\right)-\left(\nabla E_m^l,\nabla\omega_\kappa\right)-\left(n^lE_m^l,\omega_\kappa\right)$$
$$=\left(\alpha\left|E^l\right|^2E_m^l,\omega_\kappa\right),\quad m=1,2,\cdots,N,\tag{20}$$
$$\left(\partial_tn^l,\omega_\kappa\right)-\left(V^l,\nabla\omega_\kappa\right)=0,\quad \kappa=1,2,\cdots,l,\tag{21}$$
$$\left(V_{\lambda t}^l,\omega_\kappa\right)-\left(\sum_{j=1}^{2}\frac{\partial\varphi(V^l)}{\partial V_\lambda^l},\frac{\partial\omega_\kappa}{\partial x_j}\right)+\left(\beta\nabla V_{\lambda t}^l,\nabla\omega_\kappa\right)$$
$$-\left(n^l+\left|E^l\right|^2,\nabla\omega_\kappa\right)=0,\ \lambda=1,2.\tag{22}$$

with initial boundary data:

$$E^l|_{t=0}=E_0(x),\quad n^l|_{t=0}=n_0(x),\quad V^l|_{t=0}=V_0(x),\tag{23}$$
$$E^l|_{\partial\Omega}=0,\quad n^l|_{\partial\Omega}=0,\quad V^l|_{\partial\Omega}=0\tag{24}$$

Supposing

$$E_0^l(x)\xrightarrow{H^1}E_0(x),\quad V_0^l(x)\xrightarrow{H^1}V_0(x),$$
$$n_0^l(x)\xrightarrow{L^2}n_0(x),\quad l\to\infty.$$

Similar to the proof of Lemma 1-6, for the solution $E^l(x,t)$, $V^l(x,t)$ and $n^l(x,t)$ of problem (20)-(24), we can establish the following estimations:

$$\left\|E^l\right\|_{H^1}+\left\|V^l\right\|_{H^1}+\left\|n^l\right\|_{L^2}\le C\tag{25}$$
$$\left\|E_t^l\right\|_{H^{-1}}+\left\|V_t^l\right\|_{H_0^1}+\left\|n_t^l\right\|_{L^2}\le C\tag{26}$$

where the constant C is independent of l and D. By compact argument, some subsequence of $\left(E^l,V^l,n^l\right)$, also labeled as l, has a weak limit (E,V,n). More precisely,

$$E^l(x,t)\to E(x,t)\text{ in }L^\infty(\mathbb{R}^+;H^1)\text{ weakly star,}\tag{27}$$
$$V^l(x,t)\to V(x,t)\text{ in }L^\infty(\mathbb{R}^+;H^1)\text{ weakly star,}\tag{28}$$
$$n^l(x,t)\to n(x,t)\text{ in }L^\infty(\mathbb{R}^+;L^2)\text{ weakly star.}\tag{29}$$

Eq. (26) implies that

$$E_t^l\to E_t\text{ in }L^\infty(\mathbb{R}^+;H^{-1})\text{ weakly star,}\tag{30}$$
$$V_t^l\to V_t\text{ in }L^\infty(\mathbb{R}^+;H_0^1)\text{ weakly star,}$$
$$n_t^l\to n_t\text{ in }L^\infty(\mathbb{R}^+;L^2)\text{ weakly star.}$$

Moreover, it should be noted that the following maps are continuous.

$$H^1\to L^4,\quad u\mapsto u,$$
$$H^1\times L^2\to L^2,\quad (u,v)\mapsto uv.$$

It then follows from Eq. (27) and (29) that

$$\left|E^l\right|^2\to w\text{ in }L^\infty(\mathbb{R}^+;L^2)\text{ weakly star,}\tag{31}$$
$$n^lE^l\to z\text{ in }L^\infty(\mathbb{R}^+;L^2)\text{ weakly star.}\tag{32}$$

First, we prove $w=|E|^2$. Let Π be any bounded subdomain of Ω. We notice that

the embedding $H^1(\Pi)\to L^4(\Pi)$ is compact.

and for any Banach space X,

the embedding $L^\infty(\mathbb{R}^+;X)\to L^2(0,T;X)$ is continuous.

Hence, according to Eq. (27), (31) and Lemma 7, applied to $B_0=H^1(\Pi), B=L^4(\Pi), B_1=H^{-1}(\Pi)$, and says that some subsequence of $E^l|_\Pi$ (also referred to as l) converges strongly to $E|_\Pi$ in $L^2(0,T;L^4(\Pi))$. So we can assume that

$$E^l\to E\text{ strongly in }L^2(0,T;L_{loc}^4(\Pi)),$$

and thus

$$E^l\to E\text{ a.e. in }[0,T]\times\Pi.$$

Then, using Lemma 8 and Eq. (31) imply that $w=|E|^2$

Second, we prove $z=nE$. Let χ be a test function in $L^2(0,T;H^1)$, $\operatorname{supp}\chi\subset\Omega\subset\mathbb{R}^2$.

$$\int_0^T\int_\Omega\left(n^lE^l-nE\right)\chi dxdt$$
$$=\int_0^T\int_\Omega n^l\left(E^l-E\right)\chi dxdt+\int_0^T\int_\Omega\left(n^l-n\right)E\chi dxdt.$$

On one hand

$$\left|\int_0^T\int_\Omega n^l\left(E^l-E\right)\chi dxdt\right|$$
$$\le\left\|n^l\right\|_{L^\infty(0,T;L^2(\Omega))}\left\|E^l-E\right\|_{L^2(0,T;L^4(\Omega))}\|\chi\|_{L^2(0,T;L^4(\Omega))},$$

Since Ω is bounded, we deduce from Eq. (27) and (29) that

$$\left|\int_0^T\int_\Omega n^l\left(E^l-E\right)\chi dxdt\right|\to0\quad(l\to+\infty).$$

On the other hand, it should be noted that $E\chi\in L^1(0,T;L^2)$. In fact,

$$\|E\chi\|_{L^1(0,T;L^2)}\le\|E\|_{L^2(0,T;L^4)}\|\chi\|_{L^2(0,T;L^4)}<\infty.$$

Therefore we deduce from Eq. (29) that

$$\int_0^T\int_\Omega\left(n^l-n\right)E\chi dxdt\to0\quad(l\to+\infty).$$

Thus $n^lE^l\to nE$ in $L^2(o,T;H^{-1})$. So $z=nE$.

Hence taking $l\to\infty$ from Eq. (20)-(24), by using the density of ω_j in $H_0^1(\Omega)$ we get the existence of local generalized solution for the periodic initial boundary value problem (1)-(5); letting $D\to\infty$, the existence of local solution for the initial boundary value problem (1)-(5) can be obtained. By the continuation extension principle and a priori estimates, we can get the existence of a global generalized solution for problem (1)-(5).

We thus complete the proof of Theorem 1. ■

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