Constructions of Normal Extended Functions for Elliptic Interface Problems

Guanghui Liu, Xiaoling Chen, Cunyun Nie, and Haiyuan Yu

Abstract—**It is requisite to construct the normal extended function for a given function defined on the interface. In this paper, the extended function is compulsory to satisfy some interface conditions. Firstly, we construct a proper normal extended correction function which can transfer the interface problem to some non-interface one. The correction function is designed in the form of power series which are helpful to theoretical analysis. Open and closed interface curves are considered respectively. Secondly, a simple but efficient algorithm is presented to obtain the extended function value at any given point not only on the interface, such as some Gaussian points. Finally, we employ the extended function into some interface problems and carry on with some numerical experiments by employing the linear finite element method. Numerical results confirm the validity of normal extended correction functions and the efficiency of the algorithm.**

Index Terms—**normal extended functions, jump conditions, interface problems.**

I. Introduction

IN many applications, the interface problem consists of the usual boundary value problem of the diffusion equation, plus jump conditions across the material interface required by pertinent physics. It is well-known that if the interface is smooth enough, the solution of the interface problem is also very smooth in individual regions where the coefficient is smooth, but due to the jump of the coefficient across the interface, the global regularity is usually low. Due to the low global regularity and the irregular geometry of the interface, achieving accuracy is difficult with standard finite element methods. Some researchers put up some popular and efficient methods for this problem, such as the immersed interface methods ([1], [2]), the average methods ([3], [4], [5]), the finite element methods ([6], [7]), the finite difference method and the mixed finite element method ([8], [9]), the asymptotic expansion method ([10], [11], [12], [13]), and so on. During the course of theoretical analysis, it is important and necessary that jump conditions across the material interface are properly dealt with. Sometimes, interface functions need be extended to some subregions where are convenient for error estimates. It stimulates us to consider how to construct the proper extended function for interface problems. The extended function should not only be satisfied with interface conditions, but also be in simple form and smooth enough to be helpful for theoretical analysis.

In this paper, the extended function is compulsory to satisfy some interface conditions, such as the nonhomogeneous solution and normal flux jumps. Firstly, we construct a proper normal extended correction function which can transfer general jump conditions to natural jump ones. It leads to the transformation from the interface problem to some non-interface one. The correction function is designed in the form of power series which are helpful to theoretical analysis. Open and closed interface curves are considered, respectively. Secondly, a simple but efficient algorithm is presented to obtain the function value at any given point not on the interface, such as some Gaussian points. This function value equals to that at some interface point from which there is an unique normal straight line and simultaneously the above given point is obligatorily on it. The interface point can be uniquely determined when the interface curve is convex as looked from the extended region. Finally, we use the above extended correction function to some interface problems and carry on with some numerical experiments where the linear finite element method is employed. Numerical results verify the validity of normal extended functions and the efficiency of the algorithm.

The remainders of this paper is organized as follows. In section 2, we introduce the demand of normal extended functions. In section 3, we present normal extended functions for two cases. In section 4, we employ the constructed functions to two interface problems and display numerical results to support our conclusions.

II. The background and demand of the normal extended function

In many fields, we focus on a type of elliptic equation as follows

$$-\nabla(\beta(\mathbf{x})\nabla u) = f, \tag{1}$$

together with Dirichlet conditions $u = g(\mathbf{x})$ on the boundary of the region $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$ (shown as Fig. 1), Γ is the intersection curve (called as interface) of the subregions Ω^+ and Ω^- where $\phi(\mathbf{x}) > 0$ and $\phi(\mathbf{x}) < 0$, respectively, and the equation of the interface is $\phi(\mathbf{x}) = 0$, $\beta(\mathbf{x})$ is a continuous function.

On the interface, jump conditions are as follows

$$[u]|_\Gamma = (u^+ - u^-)|_\Gamma = g_0(\mathbf{x}), \tag{2}$$

$$[\frac{\partial u}{\partial \mathbf{n}}]|_\Gamma = (u_n^+ - u_n^-)|_\Gamma = g_1(\mathbf{x}), \tag{3}$$

where u^+, u^- and u_n^+, u_n^- are the solutions on Ω^+, Ω^- and the outer normal derivatives on Γ, respectively.

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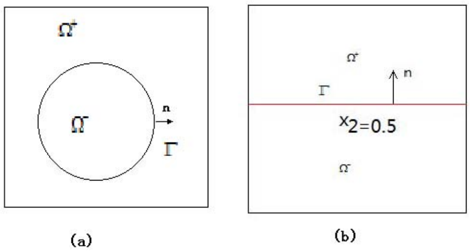

Fig. 1. The region and the interfaces.

If problem (1) satisfies the compatibility condition

$$-\int_\Gamma g_1 ds = \int_\Omega f(\mathbf{x}) d\mathbf{x},$$

then there exists unique solution for it.

To solve problem (1), one agreeable idea is to transform it to another elliptic problem with the natural jump condition

$$[q]|_\Gamma = 0, [q_n]|_\Gamma = 0, \tag{4}$$

where

$$q(\mathbf{x}) = u(\mathbf{x}) - \hat{u}(\mathbf{x}), \tag{5}$$

$$\hat{u}(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \Omega^- , \\ \tilde{u}(\mathbf{x}), & \mathbf{x} \in \Omega^+, \end{cases} \tag{6}$$

and $\tilde{u}(\mathbf{x})$ is the undetermined smooth function.

From (5) and (6), problem (1) can be transformed as follows

$$\begin{aligned} -\nabla \cdot (\nabla q) &= \tilde{f}, \quad \mathbf{x} \in \Omega/\Gamma, \\ [q](\mathbf{x}) &= 0, \quad \mathbf{x} \in \Gamma, \\ [q_n](\mathbf{x}) &= 0, \quad \mathbf{x} \in \Gamma, \\ q(\mathbf{x}) &= q_0, \quad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{7}$$

where $\tilde{f} = \begin{cases} f^-, & \mathbf{x} \in \Omega^-, \\ f^+ + \triangle \tilde{u}(\mathbf{x}), & \mathbf{x} \in \Omega^+, \end{cases}$
and $q_0 = (g - \tilde{u})|_{\partial\Omega}$.

One can see that (7) is an usual elliptic problem convenient to be solved by many numerical methods. Hence, it is the key point to construct the normal extended function $\tilde{u}(\mathbf{x})$ according to compulsory conditions (2) and (3) in interface problems. It is our main task to take in this paper.

III. Construction of the normal extended function

We will come to discuss how to extend the function $g_0(\mathbf{x})$ in (2) to some given subregion, such as Ω^+, in the normal direction. Let the extended function be $\tilde{u}(\mathbf{x})$. From (2),(3) and (4), we can obtain

$$\tilde{u}|_{\Gamma^+} = g_0(\mathbf{x}), \tag{8}$$

$$\tilde{u}_n|_{\Gamma^+} = g_1(\mathbf{x}), \tag{9}$$

where Γ^+ means the single side limit from subregion Ω^+.

In the following, we shall present how to construct $\tilde{u}(\mathbf{x})$ with the compulsory conditions (8) and (9) according to open and closed interface curves, respectively.

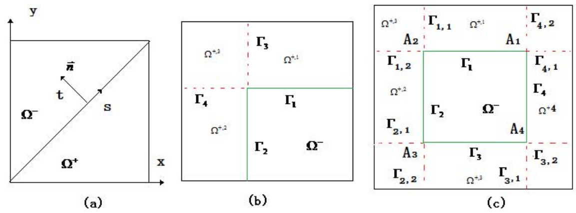

Fig. 2. A special interface.

A. Open interface curve

We firstly derive the normal extent function $\tilde{u}(\mathbf{x})$ for a simple but basic interface curve Γ expressed as $\phi(\mathbf{x}) = x_2 - x_1 = 0$. Let $s = \frac{x_2 - x_1}{\sqrt{2}}, \quad t = \frac{x_2 + x_1}{\sqrt{2}}$ and $\tilde{u}$ be in the form of power series

$$\tilde{u}(\mathbf{x}) = \tilde{u}(s,t) = \sum_{i=0}^{\infty} a_i(t)\ s^i, \tag{10}$$

where the interface $\Gamma : s = 0$ and variables s, t are shown in Fig. 2 (a).

To obtain $\tilde{u}$, we only need determine the coefficients $a_i(t), i = 0, 1, 2, 3, ...$

The conditions (8) and (10) lead to

$$a_0(t) = \tilde{u}|_{s=0} = g_0(\mathbf{x}(t)). \tag{11}$$

By some basic calculations and conditions (9) and (10),

$$a_1(t) = \frac{\partial \tilde{u}}{\partial s}|_{s=0} = u_{\mathbf{n}}|_{\Gamma^+} = g_1(\mathbf{x}(t)). \tag{12}$$

From some inductions, one can see that

$$-\triangle \tilde{u} = -\tilde{\triangle} \tilde{u}, \tag{13}$$

where $\tilde{\triangle}\tilde{u} = \frac{\partial^2 \tilde{u}}{\partial s^2} + \frac{\partial^2 \tilde{u}}{\partial t^2}$, $\triangle \tilde{u} = \frac{\partial^2 \tilde{u}}{\partial x_1^2} + \frac{\partial^2 \tilde{u}}{\partial x_2^2}$.

From (14) and the facts

$$\begin{aligned} -\tilde{\triangle}\tilde{u}|_\Gamma &= [-\sum_{i=0}^{\infty} a_i''(t) s^i + \sum_{i=2}^{\infty} a_i i(i-1) s^{i-2}]|_{s=0} \\ &= -a_0''(t) - 2a_2, \end{aligned}$$

and

$$-\triangle \tilde{u}|_\Gamma = -\triangle(u^+ - u^-)|_\Gamma = (f^+ - f^-)|_\Gamma,$$

one can obtain

$$a_2(t) = -\frac{1}{2} a_0''(t) - \frac{1}{2!}(f^+ - f^-)|_{s=0},$$

where $a_0''(t) = \frac{\partial^2 a_0}{\partial t^2}$, and $(f^+ - f^-)|_{s=0}$ is a function about variable t.

Similar derivation leads to

$$a_3(t) = -\frac{1}{6} a_1''(t) - \frac{1}{3!} \frac{\partial (f^+ - f^-)}{\partial s}|_{s=0}.$$

In general,

$$a_k(t) = -\frac{1}{k(k-2)} a_{k-2}''(t) - \frac{1}{k!} F^{(k-2)}(t), \ k \geq 2, \tag{14}$$

where $a_{k-2}''(t) = \frac{\partial^2 a_{k-2}}{\partial t^2}$,
and $F^{(k-2)}(t) := \frac{\partial^{k-2}(f^+ - f^-)}{\partial^{k-2} s}|_{s=0}$ is a function about variable t.

Hence, from (11), (12) and (15), we have

$$a_k(t) = \begin{cases} g_0(\mathbf{x}(t)), & k=0, \\ g_1(\mathbf{x}(t)), & k=1, \\ -\frac{1}{k(k-1)} a''_{k-2}(t) - \frac{1}{k!}F^{(k-2)}(t), & k \geq 2, \end{cases} \quad (15)$$

and the normal extended correction function (10) is determined.

Remark 1: We can obtain the corresponding extended function only by some rotation transformation as the interface is other straight line.

In the following, we will consider a general case. In general, we can consider the interface composed by several segment lines. To be convenient, we only discuss the case of two segment lines (shown as Fig. 2 (b)), where $\Gamma = \Gamma_1 \cup \Gamma_2,\ (x_1^0, x_2^0) = \Gamma_1 \cap \Gamma_2$, and

$$\Gamma_1: \quad x_2 = x_2^0, x_1 \geq x_1^0, \quad \Gamma_2: \quad x_1 = x_1^0, x_2 \leq x_2^0.$$

Hence, region Ω is divided into two subregions Ω^- and Ω^+, and Ω^+ is partitioned three parts $\Omega^{+,i}, i = 1, 2, 3$.

Assume that interface functions $g_0(\mathbf{x}), g_1(\mathbf{x}),\ \mathbf{x} \in \Gamma_1, \Gamma_2$, are smooth enough, respectively. In the following, we shall determine the corresponding normal extended functions $\tilde{u}_i(\mathbf{x}), i = 1, 2, 3$ in three subregions, respectively.

For the interface $\Gamma_1 : x_2 - x_2^0 = 0$,

$$\tilde{u}_1(\mathbf{x}) = \sum_{i=0}^{\infty} a_i(x_1)(x_2 - x_2^0)^i, \quad \mathbf{x} \in \Omega^{+,1}. \quad (16)$$

By some calculations similar to (15),

$$a_k(x_1) = \begin{cases} g_0(x_1), & k=0, \\ g_1(x_1), & k=1, \\ -\frac{1}{k(k-1)} a''_{k-2}(x_1) - \frac{1}{k!}F^{(k-2)}(x_1), & k \geq 2, \end{cases}$$

where

$$\begin{aligned} F^{(k-2)}(x_1) &= \frac{\partial^{k-2}(f^+ - f^-)}{\partial x_2^{k-2}}|_{x_2 = x_2^0} \\ &= \frac{\partial^{k-2}(f^+ - f^-)}{\partial x_2^{k-2}}(x_1). \end{aligned}$$

For the interface $\Gamma_2 : x_1 - x_1^0 = 0$,

$$\tilde{u}_2(\mathbf{x}) = \sum_{i=0}^{\infty} b_i(x_2)(x_1 - x_1^0)^i, \quad \mathbf{x} \in \Omega^{+,2}. \quad (17)$$

Similarly, we have

$$b_k(x_2) = \begin{cases} g_0(x_2), & k=0, \\ g_1(x_2), & k=1, \\ -\frac{1}{k(k-1)} a''_{k-2}(x_2) - \frac{1}{k!}F^{(k-2)}(x_2), & k \geq 2, \end{cases}$$

where

$$\begin{aligned} F^{(k-2)}(x_2) &= \frac{\partial^{k-2}(f^+ - f^-)}{\partial x_1^{k-2}}|_{x_1 = x_1^0} \\ &= \frac{\partial^{k-2}(f^+ - f^-)}{\partial x_1^{k-2}}(x_2). \end{aligned}$$

For the remainder subregion $\Omega^{+,3}$, we can choose the simplest way that the normal extended function is a constant $g_0(x_1^0, x_2^0)$. This way naturally satisfies the consistency of two normal extended functions defined in subregions between $\Omega^{+,1}$ and $\Omega^{+,2}$.

$$\tilde{u}_3(\mathbf{x}) = g_0(x_1^0, x_2^0),\ \mathbf{x} \in \Omega^{+,3}. \quad (18)$$

Hence, the normal extended function $\tilde{u}(\mathbf{x}), \mathbf{x} \in \Omega^+$ is completely determined by (16), (17) and (18) for this general case.

Remark 2: We maybe encounter some difficulties when the reconstructions are needed for the first, even second order derivative at some non-interface point by $\tilde{u}_3(\mathbf{x}), \mathbf{x} \in \Omega^{+,3}$. These difficulties are from the facts that the derivative function is discontinuous on the intersection face between subregions $\Omega^{+,3}$ and $\Omega^{+,i}, i = 1, 2$. In practice, we can use some smooth curves to approximate the corner (x_1^0, x_2^0), which can relieve even eliminate it.

B. Closed interface curve

We firstly consider a simple but basic case that the interface is consisted of a rectangle (shown as 2 (c)). We shall not present the derivation and construction of the normal extended function $\tilde{u}(\mathbf{x}), \mathbf{x} \in \Omega^+$ because its procedure is only to repeat and resemble that in Section 3.1. As a matter of fact, we can deal with some convex closed polygon interface only by some rotation transformation.

In the following, we will consider how to construct the normal extended function $\tilde{u}(\mathbf{x}), \mathbf{x} \in \Omega^+$ when the interface is a circle

$$r = r_0,$$

where $r = \sqrt{x_1^2 + x_2^2}$ and r_0 is a constant.

Let $\tilde{u}(\mathbf{x}), \mathbf{x} \in \Omega^+$ be the form of power series in polar coordinates

$$\tilde{u}(\mathbf{x}) = \tilde{u}(r, \theta) = \sum_{i=0}^{\infty} a_i(\theta)\ (r - r_0)^i, \quad (19)$$

where $\theta = \arctan(\frac{x_2}{x_1})$.

Firstly, we have

$$a_0(\theta) = \tilde{u}|_{r=r_0} = g_0(r_0, \theta),$$
$$a_1(\theta) = \frac{\partial \tilde{u}}{\partial \mathbf{n}}|_{r=r_0} = \frac{\partial \tilde{u}}{\partial r}|_{r=r_0} = g_1(r_0, \theta).$$

Noticing that

$$\begin{aligned} \triangle \tilde{u}|_\Gamma &= (\frac{1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta})|_\Gamma \\ &= \frac{1}{r} a_1(r_0, \theta) + 2! a_2(r_0, \theta) + \frac{1}{r^2} a''_0(r_0, \theta) \\ &= (f^+ - f^-)|_\Gamma, \end{aligned}$$

we have

$$a_2(\theta) = -\frac{1}{2!}[(f^+ - f^-)|_\Gamma + \frac{1}{r_0} a_1(r_0, \theta) + \frac{1}{r_0^2} a''_0(r_0, \theta)].$$

In general, for $k \geq 2$,

$$\begin{aligned} a_k(\theta) \quad = \quad & -\frac{1}{r_0^2}\frac{1}{k(k-1)} a''_{k-2}(r_0, \theta) - \frac{1}{r_0}\frac{1}{k} a'_{k-1}(r_0, \theta) \\ & -\frac{1}{k!} F_{k-2}(r_0, \theta), \end{aligned}$$

where $F_k(r_0, \theta) = \frac{\partial^{k-2}(f^+ - f^-)}{\partial r^{k-2}}|_{\Gamma_1}$.

Hence, we have

$$a_k(\theta) = \begin{cases} g_0(r_0, \theta), & k=0, \\ g_1(r_0, \theta), & k=1, \\ -\frac{1}{r_0^2}\frac{1}{k(k-1)} a''_{k-2}(r_0, \theta) - \frac{1}{r_0}\frac{1}{k} a'_{k-1}(r_0, \theta) & \\ \qquad - \frac{1}{k!} F_{k-2}(r_0, \theta), & k \geq 2, \end{cases}$$

where

$$F^{(k-2)}(r_0,\theta) = \frac{\partial^{k-2}(f^+ - f^-)}{\partial r^{k-2}}|_\Gamma = \frac{\partial^{k-2}(f^+ - f^-)}{\partial r^{k-2}}(r_0,\theta).$$

Therefore, the normal extended function (19) is determined.

Remark 3: For the general interface curve $\Gamma:\phi(\mathbf{x})=0$, let $\tilde{u}$ be in the form of generalized power series as follows

$$\tilde{u}(\mathbf{x}) = \sum_{i=0}^{\infty} a_i(\mathbf{x})\phi^i(\mathbf{x}).$$

Due to the complicity, we only roundoff it several terms to approximate the normal extended function $\tilde{u}$. For instance, we have

$$\tilde{u}(\mathbf{x}) \approx a_0(\mathbf{x}) + a_1(\mathbf{x})\phi(\mathbf{x}) + a_2(\mathbf{x})\phi^2(\mathbf{x}),$$

where

$$a_0(\mathbf{x}) = g_0(\mathbf{x}),\ a_1(\mathbf{x}) = \frac{1}{\frac{\partial\phi}{\partial\mathbf{n}}}(g_1 - \frac{\partial g_0}{\partial\mathbf{n}})|_{\phi(\mathbf{x})=0},$$

$$a_2(\mathbf{x}) = \frac{(f^+ - f^-) - \triangle g_0 - g_1\triangle\phi - 2\nabla g_1\cdot\nabla\phi}{2!|\nabla\phi|^2}|_{\phi(\mathbf{x})=0}.$$

IV. An algorithm for the normal extended function

To construct the normal extended function, we present an algorithm about how to obtain the corresponding coordinates on the interface for any point in the extended region. After acquiring the coordinates, we can get the values at non-interface point.

Algorithm 1:
Input: any point $P(x_1,x_2)\in\Omega^+$ and the interface level-set function $\phi(x_1,x_2)$.
Output: $P_0(x_1^0,x_2^0)\in\Gamma$.
Conditions: $\overrightarrow{P_0P}\|\mathbf{n}$, where $\mathbf{n}$ is the out normal vector at P_0.

Step 1: find the equation of the line: $\overrightarrow{P_0P}$

$$\frac{x_1^0 - x_1}{\phi_{x_1}(x_1^0,x_2^0)} = \frac{x_2^0 - x_2}{\phi_{x_2}(x_1^0,x_2^0)}, \quad (20)$$

where

$$\phi_{x_1}(x_1^0,x_2^0) = \frac{\partial\phi(\mathbf{x})}{\partial x_1}|_{(x_1^0,x_2^0)},$$

$$\phi_{x_2}(x_1^0,x_2^0) = \frac{\partial\phi(\mathbf{x})}{\partial x_2}|_{(x_1^0,x_2^0)}.$$

Step 2: solve the equation (20) with the following one simultaneously,

$$\phi(x_1^0,x_2^0) = 0,$$

then we can get the corresponding coordinates (x_1^0,x_2^0) on the interface.

Remark 4: In above algorithm, the iterative methods, such as the fixed point method or Newton iterative method, are always introduced to solve the nonlinear equations.

V. Numerical experiments

In this section, we will present two typical numerical examples for normal extended functions when the interface curves are open and closed, respectively, as solving interface problem (1). In numerical tests, the linear finite element method is employed, we only choose some parts of the power series which are enough to obtain the saturated convergent order of finite element approximations for interface problem (1) in our tests.

Example 1 In problem (1), we choose $\beta^+=1,\beta^-=2$, $\Omega=(0,1)^2$, the interface $x_2-0.5=0$ and

$$u(\mathbf{x}) = \begin{cases} u^+ = 2sin(\pi x_1)sin(\pi x_2)(x_2-0.5), & \mathbf{x}\in\Omega^+, \\ u^- = sin(\pi x_1)sin(\pi x_2)(x_2-0.5)+2, & \mathbf{x}\in\Omega^-. \end{cases}$$

One can see that

$$[u]|_\Gamma = -2, \quad [u_n]|_\Gamma = 0.$$

We choose the interceptive (roundoffed) normal extended function $\tilde{u}(\mathbf{x})=-2$ in this problem and carry on with the numerical tests as follows. Numerical results are shown as Tab. 1, where $\|\cdot\|_\varsigma,\varsigma=0,1,\infty$ denote the norms L^2,H^1,L^∞, and γ denotes the ratio of the errors $u-u^h$ between the step sizes h and $\frac{h}{2}$. From the results in this table, one can see that the errors of u^h in norms L^2,H^1,L^∞ are saturated-convergence orders, respectively.

TABLE I
The errors of u_h in norms L^∞,L^2,H^1 for the open interface curve.

$N_1\times N_2$	$\|u-u_h\|_0$	γ	$\|u-u_h\|_\infty$	γ	$\|u-u_h\|_1$	γ
8×8	3.3005e-3		6.95e-2		2.55	
16×16	8.7057e-4	4.00	2.23e-2	3.12	1.32	1.93
32×32	2.2394e-4	4.00	5.69e-3	3.92	6.69e-1	1.97
64×64	5.6827e-5	4.00	1.43e-3	3.98	3.39e-1	1.97

Example 2 In problem (1), we choose (as reference [2]) $\beta^+=\beta^-=1$, the interface $x_1^2+x_2^2=r_0^2$, $r_0=0.5$, $\Omega=(-1,1)^2$ and

$$u(\mathbf{x}) = \begin{cases} 2r^3+5, & \phi(\mathbf{x})\le 0, \\ r^3, & \phi(\mathbf{x})>0, \end{cases} \qquad r=\sqrt{x_1^2+x_2^2}.$$

One can see that

$$[u]|_\Gamma = -(r_0^3+5), \quad [u_n]|_\Gamma = -3r_0^2.$$

We choose the interceptive (roundoffed) normal extended function is

$$\tilde{u}(\mathbf{x}) = -(r_0^3+5) - 3r_0^2\,\frac{r^2-r_0^2}{2r},$$

in this example and carry on numerical tests as follows. Numerical results are shown as Tab. 2 . From the results in this table, one can see that the errors of u^h in norms L^2,H^1,L^∞ are saturated-convergence orders, respectively.

VI. Summary and conclusions

In this paper, we construct the normal extended correction function in the form of power series which are helpful to theoretical analysis, for open and closed interface curves, respectively. A simple but basic and efficient algorithm is designed to obtain the extended function value of

TABLE II
THE ERRORS OF u_h IN NORMS L^∞, L^2, H^1 FOR THE CLOSED INTERFACE CURVE.

N	$\|u-u_h\|_0$	γ	$\|u-u_h\|_\infty$	γ	$\|u-u_h\|_1$	γ
64	3.989e-5		1.636e-4		1.506e-1	
128	1.204e-5	3.23	4.779e-5	3.43	7.533e-2	1.99
256	2.872e-6	4.19	1.162e-5	4.11	3.767e-2	1.99
512	7.261e-7	3.95	2.973e-6	3.91	1.884e-2	1.99
1024	1.769e-7	4.10	7.327e-7	4.06	9.421e-3	1.99

any non-interface point projected to the interface in the direction of normal direction. We apply the extended function to some interface problems and carry on some numerical experiments. Numerical results confirm the validity of normal extended functions and the efficiency of the algorithm. In the future, we will consider the following two problems: one is how to reconstruct the corresponding first and second order derivatives by the normal extended function values, the other is how to construct the normal extended function for some interface curves which are not convex. For the second one, maybe we can only design some algorithm for some neighbor region near the interface curve or else the projected interface point can not uniquely be determined. These are main tasks in the future.

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