A New Branch and Bound Method for Solving Sum of Linear Ratios Problem

Chun-Feng Wang, and Xin-Yue Chu

Abstract—**For globally solving sum of linear ratios problem (SLRP), this paper presents a new branch-and-bound method. In this method, a new linear relaxation technique is proposed firstly; then, the initial problem SLRP is solved by a sequence of linear programming problems. Meanwhile, to improve the convergence speed of our algorithm, two accelerating techniques are presented. The proposed algorithm is proved to be convergent, and some experiments are provided to show its feasibility and efficiency.**

Index Terms—**Linear relaxation; Global optimization; Sum of linear ratios; Accelerating technique; Branch and bound.**

I. INTRODUCTION

THIS paper considers the following sum of linear ratios problem (SLRP):

$$\text{SLRP}\begin{cases} v = \quad \min \quad \Phi(x) = \sum\limits_{i=1}^{p} \frac{\delta_i(x)}{\theta_i(x)} \\ \qquad\quad \text{s.t.} \quad Ax \le b, \end{cases}$$

where $p \ge 2$, $A \in R^{m\times n}$, $b \in R^m$ are arbitrary real numbers, $\delta_i(x) = \sum\limits_{j=1}^{n} c_{ij}x_j + d_i$, $\theta_i(x) = \sum\limits_{j=1}^{n} e_{ij}x_j + f_i$ are affine functions, $D = \{x \in R^n \mid Ax \le b\}$ is bounded with $intD \ne \emptyset$, and for $\forall x \in D$, $\delta_i(x) \ge 0$, $\theta_i(x) \ne 0$, $i = 1, \cdots, p$. In fact, if we use the method of [1] to preprocess problem SLRP, we also only request $\theta_i(x) \ne 0$, $i = 1, \cdots, p$.

Among fractional programming, the problem SLRP is a special class of optimization problems. Since its initial development, it has attracted the interest of practitioners and researchers for many years. There are two main reasons. One reason is that it frequently appears in a wide variety of applications, such as financial optimization [2], portfolio optimization [3,4], microeconomics [5], plant layout design [6], transportation problems [7], layered manufacturing problems [8,9], and so on. Another reason is that, it is well-known the problem SLRP is NP-hard [10,11], that is, it generally posses multiple local optima, many of which fail to be globally optimal.

During the past years, many algorithms have been developed to solve special cases of problem SLRP. For example, for $x \in D$, under the assumption that $\delta_i(x) \ge 0$, $\theta_i(x) > 0$, by using variable transformation, Charnes and Cooper [12] put forward an efficient elementary simplex method with $p = 1$. Based on the work of [12], Konno et al. [13] proposed a similar parametric elementary simplex method with $p = 2$, which can be used to solve large scale problem. When $p = 3$, Konno and Abe [14] developed a heuristic algorithm. When $p \ge 3$, by using the characteristics of exponential and logarithmic functions, Wang et al. [15] presented a double linearization technique. By using an equivalent transformation and a linearization technique, Shen and Wang [16] proposed a branch and bound algorithm for solving a sum of linear ratios problem with coefficients. Through solving an equivalent concave minimum problem of the original problem, Benson [17] put forward a new branch and bound algorithm. Depetrini and Locatelli [18] proposed a fully polynomial time approximate scheme (FPTAS) for the case where p is fixed. Through using suitable transformation, Benson [19] proposed a method, which has a potential to solve SLRP by some well known techniques. By using the theory of monotonic optimization, Hoaiphuong and Tuy [20] presented a unified method to solve a wider class of fractional programming problems. Under the assumption that $\delta_i(x) \ge 0$, $\theta_i(x) \ne 0$, Ji et al. [21] developed a branch and bound algorithm. Under the assumption that $\theta_i(x) > 0$, Carlsson and Shi [22] proposed a linear relaxation algorithm for solving the sum-of-linear-ratios problem with lower dimension. Under the assumption that $\theta_i(x) \ne 0$, three global optimization algorithms were developed [1,23,24]. In the case that $\delta_i(x)$ and $\theta_i(x)$ are nonlinear functions, several algorithms have been proposed [25-27].

The purpose of this paper is to develop a reliable and effective method for globally solving problem SLRP with lower dimension. In this method, firstly, by utilizing the characteristic of the problem SLRP, we present a new linearization technique, which can be embedded within a branch-and-bound algorithm. After that, two accelerating techniques are presented, which can be used to improve the convergence speed of our algorithm. Finally, numerical experiments show that the proposed algorithm is feasible and effective, and the computational advantages are demonstrated. Compared with [15,17,22], the model considered in this paper has a more general form; compared with [1,24], our algorithm is easier to implement, and does not need to add new variables and constraints.

This paper is organized as follows. In Section 2, the new linear relaxation technique is presented, which can be used to obtain the relaxed linear program (RLP) for problem SLRP. In order to improve the convergence speed of our algorithm, two accelerating techniques are presented in Section 3. In Section 4, the global optimization algorithm is described, and the convergence of this algorithm is established. Numerical results are reported to show the feasibility and efficiency of our algorithm in Section 5.

Manuscript received Feb. 26, 2017. This paper is supported by the National Natural Science Foundation of China (U1404105), the Key Scientific and Technological Project of Henan Province(142102210058); the Youth Science Foundation of Henan Normal University(2013qk02); Henan Normal University National Research Project to Cultivate the Funded Projects (01016400105).

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(Advance online publication: 23 August 2017)

II. RELAXED LINEAR PROGRAM (RLP)

In problem SLRP, for $\forall x \in D$, since $\theta_i(x) \neq 0$, by the intermediate value theorem, we have $\theta_i(x) > 0$ or $\theta_i(x) < 0$. For convenience in expression, let

$$I_+ = \{i \mid \theta_i(x) > 0,\ i = 1, \cdots, p\},$$
$$I_- = \{i \mid \theta_i(x) < 0,\ i = 1, \cdots, p\}.$$

To solve problem SLRP, the principal task is the construction of lower bound for this problem and its partitioned subproblems. A lower bound of problem SLRP and its partitioned subproblems can be obtained by solving a relaxed linear program (RLP). For generating the problem RLP, the strategy proposed by this paper is to underestimate the objective function $\Phi(x)$ with a linear function.

Towards this end, we first solve $2n$ linear programming problems $l_j^0 = \min\limits_{x\in D} x_j,\ u_j^0 = \max\limits_{x \in D} x_j (j = 1, \cdots, n)$, and construct a rectangle $H^0 = \{x \in R^n \mid l_j^0 \leq x_j \leq u_j^0,\ j = 1, \cdots, n\}$. Then, the problem SLRP can be rewritten as the following form:

$$\text{SLRP}\left\{\begin{array}{lll} v = & \min & \Phi(x) = \sum\limits_{i=1}^{p} \frac{\delta_i(x)}{\theta_i(x)} \\ & \text{s.t.} & Ax \leq b, \\ & & x \in H^0. \end{array}\right.$$

Let $H = \{x \mid l \leq x \leq u\}$ be the initial box H^0 or modified box as defined for some partitioned subproblem in a branch and bound scheme. Now, we demonstrate how to derive the problem RLP for problem SLRP over H.

Compute $\underline{\xi}_i = \sum\limits_{j=1}^{n} \min\{c_{ij}l_j, c_{ij}u_j\} + d_i, \overline{\xi}_i = \sum\limits_{j=1}^{n} \max\{c_{ij}l_j, c_{ij}u_j\} + d_i,\ \underline{\eta}_i = \sum\limits_{j=1}^{n} \min\{e_{ij}l_j, e_{ij}u_j\} + f_i,\ \overline{\eta}_i = \sum\limits_{j=1}^{n} \max\{e_{ij}l_j, e_{ij}u_j\} + f_i$, obviously, we have $\underline{\xi}_i \leq \delta_i(x) \leq \overline{\xi}_i,\ \underline{\eta}_i \leq \theta_i(x) \leq \overline{\eta}_i,\ i = 1, \cdots, p.$

To derive the problem RLP of problem SLRP over H, we first consider the term $\frac{\delta_i(x)}{\theta_i(x)},\ i = 1, \cdots, p$.

For $i \in I_+$, since $\overline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x) \geq 0,\ \theta_i(x) - \overline{\eta}_i \leq 0$, we have

$$(\overline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x))(\theta_i(x) - \overline{\eta}_i) \leq 0,$$

that is

$$\overline{\eta}_i n_i(x)\theta_i(x) - \overline{\eta}_i^2\delta_i(x) - \underline{\xi}_i d_i^2(x) + \underline{\xi}_i\overline{\eta}_i d_i(x) \leq 0.$$

Furthermore, we have

$$\overline{\eta}_i^2\delta_i(x) \geq \overline{\eta}_i n_i(x)\theta_i(x) - \underline{\xi}_i d_i^2(x) + \underline{\xi}_i\overline{\eta}_i d_i(x). \quad (1)$$

Since $\overline{\eta}_i\underline{\eta}_i d_i(x) > 0$, dividing inequality (1) by $\overline{\eta}_i\underline{\eta}\theta_i(x)$, we have

$$\frac{\delta_i(x)}{\theta_i(x)} \geq \frac{\delta_i(x)}{\overline{\eta}_i} - \frac{\underline{\xi}_i}{\overline{\eta}_i^2}\theta_i(x) + \frac{\underline{\xi}_i}{\overline{\eta}_i}. \quad (2)$$

In addition, since $\overline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x) \geq 0,\ \theta_i(x) - \underline{\eta}_i \geq 0$, we have

$$(\overline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x))(\theta_i(x) - \underline{\eta}_i) \geq 0,$$

that is

$$\overline{\eta}_i n_i(x)\theta_i(x) - \overline{\eta}_i\underline{\eta}_i n_i(x) - \underline{\xi}_i d_i^2(x) + \underline{\xi}_i\underline{\eta}_i d_i(x) \geq 0. \quad (3)$$

Dividing inequality (3) by $\overline{\eta}_i\underline{\eta}_i d_i(x)$, we can obtain

$$\frac{\delta_i(x)}{\theta_i(x)} \leq \frac{\delta_i(x)}{\underline{\eta}_i} - \frac{\underline{\xi}_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x) + \frac{\underline{\xi}_i}{\overline{\eta}_i}. \quad (4)$$

For $i \in I_-$, since $\underline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x) \leq 0,\ \theta_i(x) - \overline{\eta}_i \leq 0$, we have

$$(\underline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x))(\theta_i(x) - \overline{\eta}_i) \geq 0,$$

that is

$$\underline{\eta}_i n_i(x)\theta_i(x) - \underline{\eta}_i\overline{\eta}_i n_i(x) - \underline{\xi}_i d_i^2(x) + \underline{\xi}_i\overline{\eta}_i d_i(x) \geq 0. \quad (5)$$

Since $\overline{\eta}_i\underline{\eta}_i d_i(x) < 0$, dividing inequality (5) by $\overline{\eta}_i\underline{\eta}_i d_i(x)$, we have

$$\frac{\delta_i(x)}{\theta_i(x)} \geq \frac{\delta_i(x)}{\overline{\eta}_i} - \frac{\underline{\xi}_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x) + \frac{\underline{\xi}_i}{\underline{\eta}_i}. \quad (6)$$

Meanwhile, since $\underline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x) \leq 0,\ \theta_i(x) - \underline{\eta}_i \geq 0$, we can derive

$$(\underline{\eta}_i n_i(x) - \underline{\xi}_i\theta_i(x))(\theta_i(x) - \underline{\eta}_i) \leq 0,$$

Furthermore, we get

$$\underline{\eta}_i n_i(x)\theta_i(x) - \underline{\eta}_i^2\delta_i(x) - \underline{\xi}_i d_i^2(x) + \underline{\xi}_i\underline{\eta}_i d_i(x) \leq 0. \quad (7)$$

Dividing inequality (7) by $\overline{\eta}_i\underline{\eta}_i d_i(x)$, we have

$$\frac{\delta_i(x)}{\theta_i(x)} \leq \frac{\delta_i(x)}{\underline{\eta}_i} - \frac{\underline{\xi}_i}{\underline{\eta}_i^2}\theta_i(x) + \frac{\underline{\xi}_i}{\underline{\eta}_i}. \quad (8)$$

From (2),(4),(6) and (8), we have the following relations:

$$\begin{aligned} \Phi(x) &= \sum_{i=1}^{p} \tfrac{\delta_i(x)}{\theta_i(x)} = \sum_{i\in I_+} \tfrac{\delta_i(x)}{\theta_i(x)} + \sum_{i \in I_-} \tfrac{\delta_i(x)}{\theta_i(x)} \\ &\geq \sum_{i\in I_+} [\tfrac{\delta_i(x)}{\overline{\eta}_i} - \tfrac{\underline{\xi}_i}{\overline{\eta}_i^2}\theta_i(x) + \tfrac{\underline{\xi}_i}{\overline{\eta}_i}] \\ &\quad + \sum_{i\in I_-} [\tfrac{\delta_i(x)}{\overline{\eta}_i} - \tfrac{\underline{\xi}_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x) + \tfrac{\underline{\xi}_i}{\underline{\eta}_i}] = \Phi^l(x), \end{aligned}$$

$$\begin{aligned} \Phi(x) &= \sum_{i=1}^{p} \tfrac{\delta_i(x)}{\theta_i(x)} = \sum_{i\in I_+} \tfrac{\delta_i(x)}{\theta_i(x)} + \sum_{i \in I_-} \tfrac{\delta_i(x)}{\theta_i(x)} \\ &\leq \sum_{i\in I_+} [\tfrac{\delta_i(x)}{\underline{\eta}_i} - \tfrac{\underline{\xi}_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x) + \tfrac{\underline{\xi}_i}{\overline{\eta}_i}] \\ &\quad + \sum_{i\in I_-} [\tfrac{\delta_i(x)}{\underline{\eta}_i} - \tfrac{\underline{\xi}_i}{\underline{\eta}_i^2}\theta_i(x) + \tfrac{\underline{\xi}_i}{\underline{\eta}_i}] = \Phi^u(x), \end{aligned}$$

From the above discussion, the following relaxed linear program (RLP) can be established as follows:

$$\text{RLP}\left\{\begin{array}{ll} \min & \Phi^l(x) \\ \text{s.t.} & Ax \leq b, \\ & x \in H. \end{array}\right.$$

This problem provides a lower bound for the optimal value of problem SLRP over H.

Theorem 1 For all $x \in H$, let $\Delta x = u - l$, consider the functions $\Phi^l(x)$, $\Phi(x)$ and $\Phi^u(x)$. Then, we have $\lim\limits_{\Delta x \to 0}(\Phi(x) - \Phi^l(x)) = \lim\limits_{\Delta x \to 0}(\Phi^u(x) - \Phi(x)) \to 0.$

Proof We first prove $\lim\limits_{\Delta x\to 0}(\Phi(x)-\Phi^l(x))\to 0$. By the definitions $\Phi(x)$ and $\Phi^l(x)$, we have

$$\begin{aligned}
&\mid \Phi(x)-\Phi^l(x)\mid\\
=&\mid \sum_{i\in I_+}[(\tfrac{\delta_i(x)}{\theta_i(x)}-(\tfrac{\delta_i(x)}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i^2}\theta_i(x)+\tfrac{\xi_i}{\overline{\eta}_i})]\\
&+\sum_{i\in I_-}[\tfrac{\delta_i(x)}{\theta_i(x)}-(\tfrac{\delta_i(x)}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x)+\tfrac{\xi_i}{\underline{\eta}_i})]\mid\\
\le&\mid \sum_{i\in I_+}[(\tfrac{\delta_i(x)}{\theta_i(x)}-(\tfrac{\delta_i(x)}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i^2}\theta_i(x)+\tfrac{\xi_i}{\overline{\eta}_i})]\mid\\
&+\mid \sum_{i\in I_-}[\tfrac{\delta_i(x)}{\theta_i(x)}-(\tfrac{\delta_i(x)}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}\theta_i(x)+\tfrac{\xi_i}{\underline{\eta}_i})]\mid\\
\le&\sum_{i\in I_+}[\mid \delta_i(x)(\tfrac{1}{\theta_i(x)}-\tfrac{1}{\overline{\eta}_i})\mid+\mid\tfrac{\xi_i}{\overline{\eta}_i}(\tfrac{\theta_i(x)}{\overline{\eta}_i}-1)\mid]\\
&+\sum_{i\in I_-}[\mid \delta_i(x)(\tfrac{1}{\theta_i(x)}-\tfrac{1}{\overline{\eta}_i})\mid+\mid\tfrac{\xi_i}{\underline{\eta}_i}(\tfrac{\theta_i(x)}{\overline{\eta}_i}-1)\mid]\\
=&\sum_{i\in I_+}[\mid \delta_i(x)\tfrac{\overline{\eta}_i-\theta_i(x)}{\overline{\eta}_i d_i(x)}\mid+\mid\tfrac{\xi_i}{\overline{\eta}_i}\tfrac{\theta_i(x)-\overline{\eta}_i}{\overline{\eta}_i}\mid]\\
&+\sum_{i\in I_-}[\mid \delta_i(x)\tfrac{\overline{\eta}_i-\theta_i(x)}{\overline{\eta}_i d_i(x)}\mid+\mid\tfrac{\xi_i}{\underline{\eta}_i}\tfrac{\theta_i(x)-\overline{\eta}_i}{\overline{\eta}_i}\mid]\\
\le&\sum_{i\in I_+}[\overline{\xi}_i\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i\underline{\eta}_i}+\tfrac{\xi_i}{\overline{\eta}_i}\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i}]+\sum_{i\in I_-}[\overline{\xi}_i\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i\underline{\eta}_i}+\tfrac{\xi_i}{|\underline{\eta}_i|}\tfrac{\overline{\eta}_i-\underline{\eta}_i}{|\overline{\eta}_i|}]\\
=&\sum_{i\in I_+}[\overline{\xi}_i\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i\underline{\eta}_i}+\tfrac{\xi_i}{\overline{\eta}_i}\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i}]+\sum_{i\in I_-}[\overline{\xi}_i\tfrac{\overline{\eta}_i-\underline{\eta}_i}{\overline{\eta}_i\underline{\eta}_i}+\tfrac{\xi_i}{|\underline{\eta}_i|}\tfrac{\overline{\eta}_i-\underline{\eta}_i}{|\overline{\eta}_i|}].
\end{aligned}$$

By the definitions of $\underline{\eta}_i$ and $\overline{\eta}_i$, we know that, $\Delta s=\overline{\eta}_i-\underline{\eta}_i\to 0$ as $\Delta x\to 0$. From the above inequality, we have $\lim\limits_{\Delta x\to 0}(\Phi(x)-\Phi^l(x))=0$.

Similarly, we can prove $\lim\limits_{\Delta x\to 0}(\Phi^u(x)-\Phi(x))=0$, and the proof is complete.

From Theorem 1, it follows that $\Phi^l(x)$ and $\Phi^u(x)$ will approximate the function $\Phi(x)$ as $\Delta x\to 0$.

III. Accelerating techniques

As is well known, accelerating techniques are important for improving the convergence speed of an algorithm [26]. So, this section presents two accelerating techniques, which can be used to eliminate the region in which the global optimal solution of problem SLRP does not exist.

The accelerating techniques are derived as in the following theorems.

Theorem 2 Assume that UB is the current known upper bound of the optimal value v of the problem SLRP over $H\subseteq H^0$. If there exists some index $k\in\{1,2,\cdots,n\}$ such that $\alpha_k>0$, then there is no globally optimal solution of problem SLRP over H^1; if $\alpha_k<0$ for some k, then there is no globally optimal solution of problem SLRP over H^2, where

$$H^1=(H^1_j)_{n\times 1}\subseteq H,\text{ with } H^1_j=\begin{cases}H_j, & j\ne k,\\ (\frac{\gamma_k}{\alpha_k},u_k]\bigcap H_k, & j=k,\end{cases}$$

$$H^2=(H^2_j)_{n\times 1}\subseteq H,\text{ with } H^2_j=\begin{cases}H_j, & j\ne k,\\ [l_k,\frac{\gamma_k}{\alpha_k})\bigcap H_k, & j=k,\end{cases}$$

$$\begin{aligned}
\alpha_k&=\sum_{i\in I_+}(\tfrac{c_{ik}}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i^2}e_{ik})+\sum_{i\in I_-}(\tfrac{c_{ik}}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}e_{ik}),\\
\Lambda_1&=\sum_{i\in I_+}(\tfrac{d_i}{\overline{\eta}_i}+\tfrac{\xi_i}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i^2}f_i)+\sum_{i\in I_-}(\tfrac{d_i}{\overline{\eta}_i}+\tfrac{\xi_i}{\underline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}f_i),\\
\gamma_k&=UB-\sum_{j=1,j\ne k}^{n}\min\{\alpha_j l_j,\alpha_j u_j\}-\Lambda_1.
\end{aligned}$$

proof Consider the kth component x_k of x. From $x_k\in(\frac{\gamma_k}{\alpha_k},u_k]$, it follows that

$$\frac{\gamma_k}{\alpha_k}<x_k\le u_k.$$

Since $\alpha_k>0$, we have $\gamma_k<\alpha_k x_k$. For all $x\in H^1$, by the above inequality and the definition of γ_k, it implies that

$$UB-\sum_{j=1,j\ne k}^{n}\min\{\alpha_j l_j,\alpha_j u_j\}-\Lambda_1<\alpha_k x_k,$$

that is

$$\begin{aligned}
UB&<\sum_{j=1,j\ne k}^{n}\min\{\alpha_j l_j,\alpha_j u_j\}+\alpha_k x_k+\Lambda_1\\
&\le\sum_{j=1}^{n}\alpha_j x_j+\Lambda_1=\Phi^l(x).
\end{aligned}$$

Thus, for all $x\in H^1$, we have $\Phi(x)\ge\Phi^l(x)>UB\ge v$, i.e. for all $x\in H^1$, $\Phi(x)$ is always greater than the optimal value v of the problem SLRP. Therefore, there can not exist globally optimal solution of problem SLRP over H^1.

For all $x\in H^2$, if there exists some k such that $\alpha_k<0$, from arguments similar to the above, it can be derived that there is no globally optimal solution of problem SLRP over H^2

Theorem 3 Assume that LB is the current known lower bound of the optimal value v of the problem SLRP over $H\subseteq H^0$. If there exists some index $k\in\{1,2,\cdots,n\}$ such that $\beta_k>0$, then there is no globally optimal solution of problem SLRP over H^3; if $\beta_k<0$, for some k, then there is no globally optimal solution of problem SLRP over H^4, where

$$H^3=(H^3_j)_{n\times 1}\subseteq H,\text{ with } H^3_j=\begin{cases}H_j, & j\ne k,\\ [l_k,\frac{\rho_k}{\beta_k})\bigcap H_k, & j=k,\end{cases}$$

$$H^4=(H^4_j)_{n\times 1}\subseteq H,\text{ with } H^4_j=\begin{cases}H_j, & j\ne k,\\ (\frac{\rho_k}{\beta_k},u_k]\bigcap H_k, & j=k,\end{cases}$$

$$\begin{aligned}
\beta_k&=\sum_{i\in I_+}(\tfrac{c_{ik}}{\underline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}e_{ik})+\sum_{i\in I_-}(\tfrac{c_{ik}}{\underline{\eta}_i}-\tfrac{\xi_i}{\underline{\eta}_i^2}e_{ik}),\\
\Lambda_2&=\sum_{i\in I_+}(\tfrac{d_i}{\underline{\eta}_i}+\tfrac{\xi_i}{\overline{\eta}_i}-\tfrac{\xi_i}{\overline{\eta}_i\underline{\eta}_i}f_i)+\sum_{i\in I_-}(\tfrac{d_i}{\underline{\eta}_i}+\tfrac{\xi_i}{\underline{\eta}_i}-\tfrac{\xi_i}{\underline{\eta}_i^2}f_i),\\
\rho_k&=LB-\sum_{j=1,j\ne k}^{n}\max\{\beta_j l_j,\beta_j u_j\}-\Lambda_2.
\end{aligned}$$

proof Consider the kth component x_k of x. By the assumption and the definitions of β_k and ρ_k, we have

$$l_k\le x_k<\frac{\rho_k}{\beta_k}.$$

Note that $\beta_k>0$, we have $\rho_k>\beta_k x_k$. For all $x\in H^3$, by the above inequality and the definition of ρ_k, it implies that

$$\begin{aligned}
LB&>\sum_{j=1,j\ne k}^{n}\max\{\beta_j l_j,\beta_j u_j\}+\beta_k x_k+\Lambda_2\\
&\ge\sum_{j=1}^{n}\beta_j x_j+\Lambda_2=\Phi^u(x)\ge\Phi(x).
\end{aligned}$$

Thus, for all $x\in H^3$, we have $v\ge LB>\Phi(x)$. Therefore, there can not exist globally optimal solution of problem SLRP over H^3.

For all $x\in H^4$, if there exists some k such that $\beta_k<0$, from arguments similar to the above, it can be derived that there is no globally optimal solution of problem SLRP over H^4

IV. Algorithm and its convergence

In this section, based on the former results, we present the branch and bound algorithm to solve problem SLRP. In order to find a global optimal solution, this method needs to solve a sequence of relaxed linear programming problems over partitioned subsets of H^0.

A. Branching rule

During each iteration of the algorithm, the branching process is a critical element in guaranteeing convergence, which can be used to create a more refined partition that cannot yet be excluded from further consideration in searching for a global optimal solution for problem SLRP. This paper chooses a simple and standard bisection rule, which is sufficient to ensure convergence since it drives the intervals shrinking to a singleton for all the variables along any infinite branch of the branch and bound tree.

Consider any node subproblem identified by rectangle $H = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j = 1, \cdots, n\} \subseteq H^0$. The branching rule is as follows:

(i) let $k = \text{argmax}\{u_j - l_j \mid j = 1, \cdots, n\}$;

(ii) let $\pi_k = (l_k + u_k)/2$;

(iii) let

$$H^1 = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j \neq k, l_k \leq x_k \leq \pi_k\},$$
$$H^2 = \{x \in R^n \mid l_j \leq x_j \leq u_j,\ j \neq k, \pi_k \leq x_k \leq u_k\}.$$

Through using this branching rule, the rectangle H is partitioned into two subrectangles H^1 and H^2.

B. Branch and bound algorithm

Based upon the results and operations given above, this subsection summarizes the basic steps of the proposed algorithm.

Let $LB(H^k)$ be the optimal function value of RLP over the subrectangle $H = H^k$, and $x^k = x(H^k)$ be an element of the corresponding argmin.

Algorithm statement

Step 1. Choose $\epsilon \geq 0$. Find an optimal solution $x^0 = x(H^0)$ and the optimal value $LB(H^0)$ for problem RLP with $H = H^0$. Set $LB_0 = LB(H^0)$, and $UB_0 = \Phi(x^0)$. If $UB_0 - LB_0 \leq \epsilon$, then stop: x^0 is an ϵ-optimal solutions of problem SLRP. Otherwise, set $Q_0 = \{H^0\},\ F = \emptyset,\ k = 1$, and go to Step 2.

Step 2. Set $UB_k = UB_{k-1}$. Subdivide H^{k-1} into two subrectangles $H^{k,1},\ H^{k,2}$ via the branching rule. Set $F = F \bigcup \{H^{k-1}\}$.

Step 3. Set $t = t + 1$. If $t > 2$, go to Step 5. Otherwise, continue.

Step 4. If $LB(H^{k,t}) > UB_k$, set $F = F \bigcup \{H^{k,t}\}$, and go to Step 3. Otherwise, let $UB_k = \min\{UB_k, \Phi(x^{k,t})\}$. If $UB_k = \Phi(x^{k,t})$, set $x^k = x^{k,t}$, go to Step 3.

Step 5. Set

$$F = F \bigcup \{H \in Q_{k-1} \mid UB_k \leq LB(H)\},$$
$$Q_k = \{H \mid H \in (Q_{k-1} \bigcup \{H^{k,1}, H^{k,2}\}), H \notin F\}.$$

Step 6. Set $LB_k = \min\{LB(H) \mid H \in Q_k\}$. Let H^k be the subrectangle which satisfies that $LB_k = LB(H^k)$. If $UB_k - LB_k \leq \epsilon$, then stop: x^k is a global ϵ-optimal solution of problem SLRP. Otherwise, set $k = k + 1$, and go to Step 2.

C. Convergence analysis

In this subsection, we give the global convergence properties of the above algorithm.

Theorem 4 The above algorithm either terminates finitely with a globally ϵ-optimal solution, or generates an infinite sequence $\{x^k\}$ of iteration such that any accumulation point is a globally optimal solution of problem SLRP.

Proof If the algorithm terminates finitely, then it terminates at some step $k \geq 0$. Upon termination, by the algorithm, it follows that

$$UB_k - LB_k \leq \epsilon. \tag{9}$$

Furthermore, from Step 4 and (9), the following relation holds

$$\Phi(x^k) - LB_k \leq \epsilon. \tag{10}$$

By Section 2, we have

$$LB_k \leq v. \tag{11}$$

Meanwhile, since x^k is a feasible solution of problem SLRP,

$$\Phi(x^k) \geq v. \tag{12}$$

From (10)-(12), it implies that

$$v \leq \Phi(x^k) \leq LB_k + \epsilon \leq v + \epsilon.$$

So, x^k is a global ϵ-optimal solution of the problem SLRP over H^0 in the sense that

$$v \leq \Phi(x^k) \leq v + \epsilon.$$

If the algorithm terminates infinitely, then an infinite sequence $\{x^k\}$ will be generated. Since the feasible region of SLRP is bounded, there exists a convergence subsequence in $\{x^k\}$. Without loss of generality, this subsequence is still represented by $\{x^k\}$ and set $\lim_{k\to\infty} x^k = x^*$. By the algorithm, we have

$$\lim_{k\to\infty} LB_k \leq v. \tag{13}$$

Since x^* is a feasible solution of problem SLRP,

$$v \leq \Phi(x^*). \tag{14}$$

From (13) and (14), we have

$$\lim_{k\to\infty} LB_k \leq v \leq \Phi(x^*). \tag{15}$$

On the other hand, by the algorithm and the continuity of $\Phi^l(x)$, we have

$$\lim_{k\to\infty} LB_k = \lim_{k\to\infty} \Phi^l(x^k) = \Phi^l(x^*). \tag{16}$$

By Theorem 1, we have

$$\Phi(x^*) = \Phi^l(x^*). \tag{17}$$

Therefore, from (16) and (17), we have $v = \Phi(x^*)$, that is x^* is a global optimal solution of problem SLRP.

V. Numerical experiments

To validate the performance of the proposed algorithm, some examples taken from the optimization literatures [1,15,16,23,29,30] were solved. Numerical results are reported, and compared with that of these references. The algorithm is implemented in Matlab 7.1, and all test problems are carried out on a Pentium IV (3.06 GHZ) microcomputer. The linear relaxation programming problems are solved by using simplex method.

The results of problems 1-7 are summarized in Table I, where the following notations have been used in row headers: ϵ: convergence error; Iter: number of algorithm iterations.

For Examples 1-7, we also used two algorithms to solve them, which are the algorithm (named Algorithm 1) proposed by this paper and the algorithm proposed by this paper but without using pruning techniques(named Algorithm 2), respectively. The comparison results are given in Table II. In Table II, $Time$ denotes execution time in seconds. For this test, ϵ is set to $1e-5$.

Table III summarizes our computational results of Example 8. For this test problem, ϵ is set to $1e-2$. In Table II, $\mathrm{Ave.Iter}$ represents the average number of iterations; $\mathrm{Ave.Time}$ stands for the average CPU time of the algorithm in seconds, which are obtained by randomly running our algorithm for 10 test problems.

Example 1[1]

$$\begin{array}{ll} \max & 0.9\times\dfrac{-x_1+2x_2+2}{3x_1-4x_2+5}+(-0.1)\times\dfrac{4x_1-3x_2+4}{-2x_1+x_2+3} \\ \mathrm{s.t.} & x_1+x_2\le 1.5, \\ & x_1-x_2\le 0, \\ & 0\le x_1\le 1,\ 0\le x_2\le 1. \end{array}$$

Example 2[1,15,30]

$$\begin{array}{ll} \max & \dfrac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50}+\dfrac{3x_1+4x_3+50}{4x_1+4x_2+5x_3+50} \\ & +\dfrac{x_1+2x_2+5x_3+50}{x_1+5x_2+5x_3+50}+\dfrac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50} \\ \mathrm{s.t.} & 2x_1+x_2+5x_3\le 10, \\ & x_1+6x_2+3x_3\le 10, \\ & 5x_1+9x_2+2x_3\le 10, \\ & 9x_1+7x_2+3x_3\le 10, \\ & x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0,\ x_4\ge 0. \end{array}$$

Example 3[30]

$$\begin{array}{ll} \min & \dfrac{-x_1+2x_2+2}{3x_1-4x_2+5}+\dfrac{4x_1-3x_2+4}{-2x_1+x_2+3} \\ \mathrm{s.t.} & x_1+x_2\le 1.5, \\ & x_1-x_2\le 0, \\ & 0\le x_1\le 1,\ 0\le x_2\le 1. \end{array}$$

Example 4[29,30]

$$\begin{array}{ll} \max & \dfrac{3x_1+5x_2+3x_3+50}{3x_1+4x_2+5x_3+50}+\dfrac{3x_1+4x_2+50}{4x_1+3x_2+2x_3+50} \\ & +\dfrac{4x_1+2x_2+4x_3+50}{5x_1+4x_2+3x_3+50} \\ \mathrm{s.t.} & 6x_1+3x_2+3x_3\le 10, \\ & 10x_1+3x_2+8x_3\le 10, \\ & x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0. \end{array}$$

Example 5[16]

$$\begin{array}{ll} \max & \dfrac{63x_1-18x_2+39}{13x_1+26x_2+13}+\dfrac{13x_1+26x_2+13}{37x_1+73x_2+13} \\ & +\dfrac{37x_1+73x_2+13}{13x_1+13x_2+13}+\dfrac{13x_1+13x_2+13}{63x_1-18x_2+39} \\ \mathrm{s.t.} & 5x_1-3x_2=3, \\ & 1.5\le x_1\le 3. \end{array}$$

Example 6[16]

$$\begin{array}{ll} \max & \dfrac{3x_1+4x_2+50}{3x_1+5x_2+4x_3+50}-\dfrac{3x_1+5x_2+3x_3+50}{5x_1+5x_2+4x_3+50} \\ & -\dfrac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50}-\dfrac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50} \\ \mathrm{s.t.} & 6x_1+3x_2+3x_3\le 10, \\ & 10x_1+3x_2+8x_3\le 10, \\ & x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0. \end{array}$$

Example 7[23]

$$\begin{array}{ll} \max & \dfrac{37x_1+73x_2+13}{13x_1+13x_2+13}+\dfrac{63x_1-16x_2+39}{13x_1+26x_2+13} \\ \mathrm{s.t.} & 5x_1-3x_2=3, \\ & 1.5\le x_1\le 3. \end{array}$$

TABLE I: Computational results of Examples 1-7

Example	ϵ	Methods	Optimal solution	Optimal value	Iter
1	1e-9	[1]	(0.0, 1.0)	3.575	1
	1e-9	$ours$	(0.0, 1.0)	3.575	1
2	1e-9	[1]	(1.1111, 0.0, 0.0)	4.0907	1289
	1e-6	[15]	(1.1111, 1.365e-5, 1.351e-5)	4.081481	39
	1e-5	[30]	(0.0013, 0.0, 0.0)	4.087412	1640
	1e-9	$ours$	(1.1111, 0.0, 0.0)	4.0907	18
3	1e-8	[30]	(0.0, 0.283935547)	1.623183358	71
	1e-8	$ours$	(0.0, 0.283935547)	1.623183358	47
4	1e-5	[29]	(0.0, 1.6725, 0.0)	3.0009	1033
	1e-8	[30]	(0.0, 3.3333, 0.0)	3.00292	119
	1e-8	$ours$	(0.0, 3.3333, 0.0)	3.00292	50
5	1e-6	[16]	(3.0, 4.0)	3.2917	9
	1e-6	$ours$	(3.0, 4.0)	3.2917	8
6	1e-6	[16]	(-1.838e-16, 3.3333, 0.0)	1.9	8
	1e-6	$ours$	(0.0, 3.3333, 0.0)	1.9	6
7	1e-4	[23]	(3.0, 4.0)	5.0	32
	1e-4	$ours$	(3.0, 4.0)	5.0	17

TABLE II: Computational results of Algorithm 1 and Algorithm 2 for Examples 1-7

Example	Methods	Optimal solution	Optimal value	Iter	$Time$
1	Algorithm 1	(0.0, 1.0)	3.575	1	0.016
	Algorithm 2	(0.0, 1.0)	3.575	1	0.016
2	Algorithm 1	(1.1111, 0.0, 0.0)	4.0907	9	0.281
	Algorithm 2	(1.1111, 0.0, 0.0)	4.081481	20	0.672
3	Algorithm 1	(0.0, 0.283935547)	1.623183358	31	1.0
	Algorithm 2	(0.0, 0.283935547)	1.623183358	50	1.719
4	Algorithm 1	(0.0, 3.3333, 0.0)	3.0009	40	1.328
	Algorithm 2	(0.0, 3.3333, 0.0)	3.00292	77	2.562
5	Algorithm 1	(3.0, 4.0)	3.2917	8	0.187
	Algorithm 2	(3.0, 4.0)	3.2917	10	0.203
6	Algorithm 1	(0, 3.3333, 0.0)	1.9	5	0.172
	Algorithm 2	(0.0, 3.3333, 0.0)	1.9	28	0.922
7	Algorithm 1	(3.0, 4.0)	5.0	19	0.568
	Algorithm 2	(3.0, 4.0)	5.0	35	1.253

From Table I, it can be seen that, for Examples 1-7, our algorithm can determine the global optimal solution effectively than that of the references [1,15,16,23,29,30].

The comparison results of Table 2 show that the pruning techniques are very good at improving the convergence speed of our algorithm.

Example 8

$$\begin{array}{ll} \min & \sum_{i=1}^{p} \frac{\sum_{j=1}^{n} c_{ij}x_j + d_i}{\sum_{j=1}^{n} e_{ij}x_j + f_i} \\ \text{s.t.} & x \in D = \{x \in R^n \mid Ax \leq b\}, \end{array}$$

where the elements of the matrix $A \in R^{m\times n}$, c_{ij}, $e_{i,j} \in R$ are randomly generated in the interval [0,1]. All constant terms of denominators and numerators are the same number, which randomly generated in [50,100]. The elements of $b \in R^m$ are equal to 1. This agrees with the way random numbers are generated in [22].

From Table III, the computational results show that our algorithm performs well on the test problems, and can solve them in a reasonable amount of time.

The results in Tables I-III show that our algorithm is both feasible and efficient.

TABLE III: Computational results of Example 8

(p, m, n)	Ave.Time	Ave.Iter
(2,20,20)	0.0264	1
(2,20,30)	0.029	1
(5,20,20)	0.6154	9.7
(5,30,20)	0.6389	10.22
(7,20,20)	2.2765	14.5
(7,30,20)	4.3327	18.8
(10,20,20)	31.1031	61.8
(10,30,20)	38.4108	95.3

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