Operational Matrix Method for the Variable Order Time Fractional Diffusion Equation Using Legendre Polynomials Approximation

Nanyu Chen, Jun Huang*, Yacong Wu, Qian Xiao

Abstract—**In this paper, a numerical method based on Legendre polynomials is proposed for solving the variable order time fractional diffusion equation. We adopt the Coimbra variable order time fractional operator, which can be viewed as a Caputo-type definition. Operational matrix of differentiation is also introduced. Combining this matrix with the properties of Legendre polynomials, we transform the initial problem into a Sylvester equation. Numerical example is provided to demonstrate the validity and applicability of the technique. Moreover, comparing the methodology with the known method shows that our approach is more efficient and more convenient.**

Keywords—**Variable order; Legendre polynomials; Operational matrix; Sylvester equation; Numerical solution**

I. Introduction

In science and engineering, many dynamical systems can be described by fractional-order equation [1-3]. These dynamical systems generally originates in the fields of electrode-electrolyte [4], dielectric polarization [5], electromagnetic waves [6] and viscoelastic systems [7] etc. Various materials and processes have been found to be described using fractional calculus. Anomalous diffusion has been discussed in various physical fields [8-10]. The features of anomalous diffusion include history dependence, long-range, correlation and heavy tail characteristics. These features can be accommodated well by using fractional calculus. In order to deal with the diffusion processes in which the diffusion behaviors depend on time evolution, space variation, the variable-order diffusion models were proposed. The concept of variable order operator was first introduced by Samko[11-12] in 1993 and received much attention in the fields of viscoelasticity, viscoelastic deformation, viscous fluid. Nowadays, it has been employed as a powerful tool in complex anomalous diffusion modeling.

Up until now, to the best of the authors knowledge, the main approach for solving the variable order time fractional diffusion equation is finite difference method. Lin et al. [13] applied an explicit finite difference method to investigate stability and convergence of approximation for the variable order nonlinear fractional diffusion equation. Zhuang et al. [14] proposed explicit and implicit Euler method for the variable order fractional advection-diffusion equation. Chen et al. [15] used two numerical methods to solve the variable order anomalous sub-diffusion equation.

Legendre polynomials play a prominent role in various areas of mathematics. These polynomials have frequently used in both the solution of differential equations and approximation theory [16-17]. Abbasbandy et.al. [18] presented the operational matrix method based on fractional order Legendre polynomials for the time fractional convection diffusion equations. Islam and Hossain [19] used the Bernstein and Legendre polynomials to solve the eighth order boundary value problem.

In this study, we consider the following variable order time fractional diffusion equation:

$$D_t^{q(x,t)}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\quad (x,t)\in\Omega=[0,1]\times[0,1] \tag{1}$$

with initial and boundary conditions

$$u(x,0)=g(x),\ \ 0\le x\le 1 \tag{2}$$

$$u(0,t)=u(1,t)=0,\ \ 0<t<1 \tag{3}$$

where $0<q(x,t)<1$, $f(x,t)$ and $g(x)$ are the known functions. $D_t^{q(x,t)}$ denotes the variable order time fractional derivative defined by Coimbra [20]:

$$D_t^{q(x,t)}u(x,t)=\frac{1}{\Gamma(1-q(x,t))}\int_0^t (t-\sigma)^{-q(x,t)}\frac{\partial u(x,\sigma)}{\partial\sigma}d\sigma+\frac{(u(x,0^+)-u(x,0^-))t^{-q(x,t)}}{\Gamma(1-q(x,t))} \tag{4}$$

For the sake of simplicity, assuming $u(x,0^+)=u(x,0^-)$, then the Coimbra definition can be viewed as the following Caputo-type definition:

$$D_t^{q(x,t)}u(x,t)=\frac{1}{\Gamma(1-q(x,t))}\int_0^t (t-\sigma)^{-q(x,t)}\frac{\partial u(x,\sigma)}{\partial\sigma}d\sigma \tag{5}$$

II. Legendre Polynomials and Their Some Properties

The Legendre basis polynomials of degree n in $[0,1]$ are defined by [16]

$$P_{i+1}(x)=\frac{(2i+1)(2x-1)}{(i+1)}P_i(x)-\frac{i}{i-1}P_{i-1}(x),\qquad i=1,2,\ldots \tag{6}$$

where $P_0(x)=1,\ P_1(x)=2x-1$. The Legendre polynomials of degree i can be also given by

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$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2} \tag{7}$$

Let

$$\boldsymbol{\Phi}(x) = \left[P_0(x), P_1(x), \cdots, P_n(x)\right]^{\mathrm{T}} \tag{8}$$

The Legendre polynomials given by Eq.(6) can be expressed in the matrix form

$$\boldsymbol{\Phi}(x) = \boldsymbol{A}\Delta_n(x) \tag{9}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & (-1)^2 2! & \cdots & 0 \\ (-1)^2 & (-1)^3 \frac{3!}{1!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n & (-1)^{n+1}\frac{(n+1)!}{(n-1)!} & \cdots & (-1)^{2n}\frac{(2n)!}{n!} \end{bmatrix} \tag{10}$$

Obviously

$$\Delta_n(x) = \mathbf{A}^{-1}\boldsymbol{\Phi}(x) \tag{11}$$

A function $u(x,t) \in L^2([0,1]\times[0,1])$ can be expressed in terms of the Legendre basis. In practice, only the first $(n+1)\times(n+1)$ term of Legendre polynomials are considered. Hence

$$u(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij} P_n(x) P_n(t) = \boldsymbol{\Phi}^T(x)\boldsymbol{C}\boldsymbol{\Phi}(t) \tag{12}$$

where $\boldsymbol{C} = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ c_{10} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nn} \end{bmatrix}$, c_{ij} are called Legendre coefficients.

Theorem 1. For any $x_i, t_j \in [0,1]$, suppose that the function $\frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}$ obtained by using Legendre polynomials are the approximation of $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$, $\alpha = q(x_i, t_j)$, and $u(x,t)$ has bounded mixed fractional partial derivative $\left|\frac{\partial^{4+\alpha+\beta}u(x,t)}{\partial x^{2+\alpha}\partial t^{2+\beta}}\right| \le \hat{M}$, then we have

$$\left\|\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}\right\|_E \le \frac{\hat{M}}{8}\left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)}\right)'''$$

where $\|u(x,t)\|_E = \left(\int_{-1}^{1}\int_{-1}^{1} u^2(x,t)dxdt\right)^{1/2}$ and

$$u_{ij} = \left(\frac{2i+1}{2}\right)\left(\frac{2j+1}{2}\right)\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}P_i(x)P_j(t)dxdt.$$

Proof. The property of the $\{P_i(x)\}$ on $[-1,1]$ implies that

$$\int_{-1}^{1} P_i(x)P_j(x)dx = \begin{cases} \frac{2}{2i+1}, & i=j; \\ 0, & i \ne j \end{cases},$$

then

$$\begin{aligned}\left\|\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}\right\|_E^2 &= \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}\right]^2 dxdt \\ &= \int_{-1}^{1}\int_{-1}^{1}\left[\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} u_{ij}P_i(x)P_j(t)\right]^2 dxdt \\ &= \int_{-1}^{1}\int_{-1}^{1}\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} u_{ij}^2 P_i^2(x)P_j^2(t)dxdt \\ &= \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} u_{ij}^2 \int_{-1}^{1} P_i^2(x)dx \int_{-1}^{1} P_j^2(t)dt \\ &= \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} u_{ij}^2 \frac{2}{2i+1}\frac{2}{2j+1}\end{aligned}$$

The Legendre polynomials coefficients of function $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ are expressed as

$$u_{ij} = \left(\frac{2i+1}{2}\right)\left(\frac{2j+1}{2}\right)\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}P_i(x)P_j(t)dxdt.$$

Therefore, we get

$$\begin{aligned} u_{ij} &= \frac{2j+1}{4}\int_{-1}^{1}\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}[P_{i+1}(x) - P_{i-1}(x)]P_j(t)\Big|_{-1}^{1} dt - \frac{2j+1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\alpha+1}u(x,t)}{\partial x^{\alpha+1}}[P_{i+1}(x)-P_{i-1}(x)]P_j(t)dxdt \\ &= -\frac{2j+1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\alpha+1}u(x,t)}{\partial x^{\alpha+1}}[P_{i+1}(x)-P_{i-1}(x)]P_j(t)dxdt \\ &= -\frac{2j+1}{4}\int_{-1}^{1}\frac{\partial^{\alpha+1}u(x,t)}{\partial x^{\alpha+1}}\left[\frac{P_{i+2}(x)-P_i(x)}{2i+3} - \frac{P_i(x)-P_{i-2}(x)}{2i-1}\right]P_j(t)\Big|_{-1}^{1} dt + \\ &\quad \frac{2j+1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\alpha+2}u(x,t)}{\partial x^{\alpha+2}}\left[\frac{P_{i+2}(x)-P_i(x)}{2i+3} - \frac{P_i(x)-P_{i-2}(x)}{2i-1}\right]P_j(t)dxdt \\ &= \frac{2j+1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{\alpha+2}u(x,t)}{\partial x^{\alpha+2}}\left[\frac{P_{i+2}(x)-P_i(x)}{2i+3} - \frac{P_i(x)-P_{i-2}(x)}{2i-1}\right]P_j(t)dxdt\end{aligned}$$

Now, let

$$\tau_i(x) = (2i-1)P_{i+2}(x) - 2(2i+1)P_i(x) + (2i+3)P_{i-2}(x)$$

then, we obtain

$$u_{ij}=\frac{2j+1}{4(2i-1)(2i+3)}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{2+\alpha}u(x,t)}{\partial x^{2+\alpha}}\tau_i(x)P_j(t)dxdt\,.$$

By solving this equation, we get

$$u_{ij}=\frac{1}{4(2i-1)(2i+3)(2j-1)(2j+3)}\cdot$$

$$\int_{-1}^{1}\int_{-1}^{1}\frac{\partial^{4+\alpha+\beta}u(x,t)}{\partial x^{2+\alpha}\partial t^{2+\beta}}\tau_i(x)\tau_j(t)dxdt$$

So we have

$$\left|u_{ij}\right|\le\frac{1}{4(2i-1)(2i+3)(2j-1)(2j+3)}\cdot$$

$$\int_{-1}^{1}\int_{-1}^{1}\left|\frac{\partial^{4+\alpha+\beta}u(x,t)}{\partial x^{2+\alpha}\partial t^{2+\beta}}\right|\left|\tau_i(x)\right|\left|\tau_j(t)\right|dxdt$$

$$\le\frac{\hat{M}}{4(2i-1)(2i+3)(2j-1)(2j+3)}\cdot$$

$$\int_{-1}^{1}\left|\tau_i(x)\right|dx\int_{-1}^{1}\left|\tau_j(t)\right|dt$$

Moreover, it is obtained that

$$\int_{-1}^{1}\left|\tau_m(t)\right|dt\le\sqrt{24}\frac{2i+3}{\sqrt{2i-3}}\,,$$

thus, we have

$$\left|u_{ij}\right|\le\frac{24\hat{M}}{4(2i-1)(2i+3)(2j-1)(2j+3)}\frac{(2i+3)}{\sqrt{2i-3}}\frac{(2j+3)}{\sqrt{2j-3}}$$

$$\le\frac{6\hat{M}}{(2i-3)^{3/2}(2j-3)^{3/2}}$$

Namely

$$\left|u_{ij}\right|^2\le\frac{36\hat{M}^2}{(2i-3)^3(2j-3)^3}\,.$$

Therefore, we have

$$\left\|\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}-\frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}\right\|_E^2\le\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}u_{ij}^2\frac{2}{2i+1}\frac{2}{2j+1}$$

$$\le\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\frac{144\hat{M}^2}{(2i-3)^3(2j-3)^3(2i+1)(2j+1)}$$

$$=\left[\frac{\hat{M}}{8}\left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)}\right)^{m}\right]^2$$

thus

$$\left\|\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}-\frac{\partial^\alpha u_n(x,t)}{\partial x^\alpha}\right\|_E\le\frac{\hat{M}}{8}\left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)}\right)^{m}\,.$$

This completes the proof.

III. The Operational Matrix of the Derivative

3.1 The Legendre polynomials operational matrix of integer order differentiation Figures

The differentiation of vector $\Phi(x)$ in Eq.(9) can be expressed as

$$\Phi'(x)=\mathbf{D}\Phi(x) \tag{13}$$

where $\mathbf{D}$ is the $(n+1)\times(n+1)$ operational matrix of derivatives for Legendre polynomials.

From Eq.(9), we have

$$\Phi'(x)=\boldsymbol{A}\cdot[0,1,2x,\cdots,nx^{n-1}]^T \tag{14}$$

Define the $(n+1)\times n$ matrix $\boldsymbol{V}$ and vector Δ_n^* as

$$\boldsymbol{V}=\begin{bmatrix}0&0&\cdots&0\\1&0&\cdots&0\\0&2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n\end{bmatrix},\quad \Delta_n^*=[1,x,x^2,\ldots x^{n-1}]^T \tag{15}$$

Eq.(14) may then be restated as

$$\Phi'(x)=\boldsymbol{A}\boldsymbol{V}\Delta_n^* \tag{16}$$

Because $x^k=\boldsymbol{A}_{[k+1]}^{-1}\Phi(x)$, where $\boldsymbol{A}_{[k+1]}^{-1}$ is the $k+1th$ row of $\boldsymbol{A}^{-1}$ for $k=0,1,\ldots,n$, so we have

$$\Delta_n^*=\boldsymbol{B}^*\Phi(x) \tag{17}$$

where $\boldsymbol{B}^*=[\boldsymbol{A}_{[1]}^{-1},\boldsymbol{A}_{[2]}^{-1},\boldsymbol{A}_{[3]}^{-1},\ldots\boldsymbol{A}_{[n]}^{-1}]^T$.

Therefore

$$\Phi'(x)=\boldsymbol{A}\boldsymbol{V}\boldsymbol{B}^*\Phi(x) \tag{18}$$

and we have the operational matrix of the derivative as

$$\mathbf{D}=\boldsymbol{A}\boldsymbol{V}\boldsymbol{B}^* \tag{19}$$

If we approximate $g(x)\cong\boldsymbol{g}^T\Phi(x)$, then for $n\ge2$ (n is the order of derivatives), we obtain

$$g^{(n)}(x)\cong\boldsymbol{g}^T\Phi^{(n)}(x)=\boldsymbol{g}^T\mathbf{D}^n\Phi(x) \tag{20}$$

3.2 The Legendre polynomials operational matrix of fractional order differentiation

Now, we derive Legendre polynomials operational matrix of fractional order differentiation.

Let

$$D_t^{q(x,t)}\left(\Phi(t)\right)\cong\mathbf{D}^{q(x,t)}\Phi(t) \tag{21}$$

where matrix $D_t^{q(x,t)}$ is called Legendre polynomials operational matrix of fractional order differentiation.

For this purpose, we use Eq.(9) and the definition of Caputo-type Eq.(5), as following

$$\begin{aligned}D_t^{q(x,t)}\left(\Phi(t)\right)&=D_t^{q(x,t)}\left(\boldsymbol{A}\Delta_n(t)\right)\\&=\boldsymbol{A}D_t^{q(x,t)}\left(\Delta_n(t)\right)\\&=\boldsymbol{A}\Psi(x,t)\end{aligned} \tag{22}$$

where $\Psi(x,t)=[\psi_0(x,t),\psi_1(x,t),\psi_2(x,t),\ldots\psi_n(x,t)]^T$,

$$\psi_i(x,t)=\frac{1}{\Gamma(1-q(x,t))}\int_0^t(t-\sigma)^{-q(x,t)}i\sigma^{i-1}d\sigma\,.$$

When $i\ge1$, let $\sigma=\varepsilon t$, then

$$\psi_i(x,t)=\frac{it^{i-q(x,t)}}{\Gamma(1-q(x,t))}\int_0^1(1-\varepsilon)^{-q(x,t)}\varepsilon^{i-1}d\varepsilon$$

$$=\frac{\Gamma(i+1)t^{i-q(x,t)}}{\Gamma(i+1-q(x,t))}$$

where B is the beta function which is defined as follows

$$B(m,n)=\int_0^1\tau^{m-1}(1-\tau)^{n-1}d\tau,\qquad \mathrm{Re}(m)>0,\mathrm{Re}(n)>0$$

when $i=0$, $\psi_0(x,t)=0$, $\Psi(x,t)$ can be expressed as

$$\Psi(x,t)=\Pi(x,t)\Delta_n(t) \tag{23}$$

where

$$\boldsymbol{\Pi}(t)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{\Gamma(2)t^{1-q(x,t)}}{\Gamma(2-q(x,t))} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(3)t^{2-q(x,t)}}{\Gamma(3-q(x,t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(n+1)t^{n-q(x,t)}}{\Gamma(n+1-q(x,t))} \end{bmatrix}$$

Eq.(22) will be

$$\begin{aligned} D_t^{q(x,t)}\left(\boldsymbol{\Phi}(t)\right) &= \boldsymbol{A}\boldsymbol{\Pi}(x,t)\Delta_n(t) = \boldsymbol{A}\boldsymbol{\Pi}(x,t)\boldsymbol{A}^{-1}\boldsymbol{A}\Delta_n(t) \\ &= \boldsymbol{A}\boldsymbol{\Pi}(x,t)\boldsymbol{A}^{-1}\boldsymbol{\Phi}(t) \end{aligned} \tag{24}$$

From Eq.(21) and Eq.(24), the Legendre polynomials operational matrix of fractional order differentiation $\mathbf{D}^{q(x,t)}$ is given by

$$\mathbf{D}^{q(x,t)} = \boldsymbol{A}\boldsymbol{\Pi}(x,t)\boldsymbol{A}^{-1} \tag{25}$$

IV. 4. LEGENDRE MATRICES FOR THE NUMERICAL SOLUTION OF THE VARIABLE ORDER TIME FRACTIONAL DIFFUSION EQUATION.

Consider Eq.(1), Eq.(2) and Eq.(3), by previous section, the function $u(x,t)$ can be approximated as Eq.(12). Then we get

$$\begin{aligned} D_t^{q(x,t)}u(x,t) &\cong D_t^{q(x,t)}\left(\boldsymbol{\Phi}^T(x)\boldsymbol{C}\boldsymbol{\Phi}(t)\right) = \boldsymbol{\Phi}^T(x)\boldsymbol{C}D_t^{q(x,t)}\left(\boldsymbol{\Phi}(t)\right) \\ &= \boldsymbol{\Phi}^T(x)\boldsymbol{C}\mathbf{D}^{q(x,t)}\boldsymbol{\Phi}(t) \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &\cong \frac{\partial^2\left(\boldsymbol{\Phi}^T(x)\boldsymbol{C}\boldsymbol{\Phi}(t)\right)}{\partial x^2} = \left(\frac{d^2\boldsymbol{\Phi}(x)}{dx^2}\right)^T \boldsymbol{C}\boldsymbol{\Phi}(t) \\ &= \left(\mathbf{D}^2\boldsymbol{\Phi}(x)\right)^T \boldsymbol{C}\boldsymbol{\Phi}(t) \\ &= \boldsymbol{\Phi}^T(x)\left(\mathbf{D}^2\right)^T \boldsymbol{C}\boldsymbol{\Phi}(t) \end{aligned} \tag{27}$$

Similarly, the function $f(x,t)$ can be given by

$$f(x,t) \cong \boldsymbol{\Phi}^T(x)\boldsymbol{F}\boldsymbol{\Phi}(t) \tag{28}$$

where $\boldsymbol{F}=\begin{bmatrix} f_{00} & f_{01} & \cdots & f_{0n} \\ f_{10} & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n0} & f_{n1} & \cdots & f_{nn} \end{bmatrix}$. Substituting Eq.(26), Eq.(27) and Eq.(28) into Eq.(1), we obtain

$$\boldsymbol{\Phi}^T(x)\boldsymbol{C}\mathbf{D}^{q(x,t)}\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}^T(x)\left(\mathbf{D}^2\right)^T \boldsymbol{C}\boldsymbol{\Phi}(t) + \boldsymbol{\Phi}^T(x)\boldsymbol{F}\boldsymbol{\Phi}(t) \tag{29}$$

Dispersing Eq.(29) by the points (x_i,t_j), $i=0,1,2,\cdots,n$ and $j=0,1,2,\cdots,n$, we have

$$\left(\mathbf{D}^2\right)^T \boldsymbol{C} - \boldsymbol{C}\mathbf{D}^{q(x,t)} = -\boldsymbol{F} \tag{30}$$

Eq.(30) is a Sylvester equation. Solving it, we can get the matrix of $\boldsymbol{C}$.

From the Eq.(2), we have $\boldsymbol{\Phi}^T(x)\boldsymbol{C}\boldsymbol{\Phi}(0)=g(x)$, then we may calculate c_{i1}, $i=0,1,2\ldots,n$.

According to the condition Eq.(3), we can take $c_{1j}=0$, $j=0,1,2\ldots,n$.

V. NUMERICAL EXAMPLE

Example 1: Consider the following variable order time fractional diffusion equation [21]

$$\begin{cases} D_t^{q(x,t)}u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \\ u(x,0)=10x^2(1-x) \qquad 0\le x\le 1 \\ u(0,t)=u(1,t)=0 \qquad 0<t<1 \end{cases} \tag{31}$$

where $q(x,t)=\dfrac{2+\sin(xt)}{4}$ (satisfies $0<q(x,t)<1$) and

$$f(x,t)=20x^2(1-x)\left[\frac{t^{2-q(x,t)}}{\Gamma(3-q(x,t))}+\frac{t^{1-q(x,t)}}{\Gamma(2-q(x,t))}\right]- 20(t+1)^2(1-3x) \tag{32}$$

The exact solution is

$$u(x,t)=10x^2(1-x)(t+1)^2 \tag{33}$$

We applied the Legendre polynomials method to solve this problem for various values of n. The absolute errors of the numerical solutions and the exact solution for $n=3$, $n=4$, $n=4$ are shown in Table I. The numerical solutions for $n=3$ and the exact solution are shown in Fig. 1 and Fig. 2.

Table I.
THE ABSOLUTE ERRORS FOR DIFFERENT n

(x,t)	Present method			Method in [21]
	$n=3$	$n=4$	$n=5$	
(0,1)	0	0	0	0
(0.1, 1)	1.0954e-013	7.4496e-012	2.0938e-011	2.9962e-005
(0.2, 1)	2.9088e-013	4.5400e-011	5.8094e-010	5.9723e-005
(0.3, 1)	3.3595e-013	1.2077e-010	3.0756e-009	8.8035e-005
(0.4, 1)	4.7518e-014	2.2054e-010	1.1654e-008	1.1251e-004
(0.5, 1)	7.7360e-013	3.0961e-010	3.8213e-008	1.2981e-004
(0.6, 1)	2.3288e-012	3.2339e-010	1.1230e-007	1.3595e-004
(0.7, 1)	4.8051e-012	1.5515e-010	2.9866e-007	1.2705e-004
(0.8, 1)	8.3951e-012	3.6181e-010	7.2617e-007	1.0048e-004
(0.9, 1)	1.3241e-011	1.4777e-009	1.6334e-006	5.6431e-005

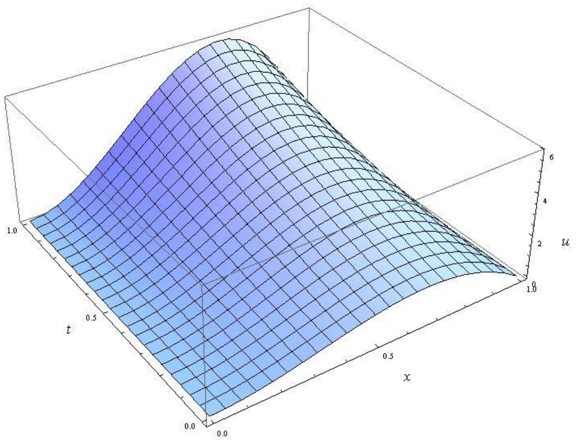

Fig.1. The numerical solutions for $n=3$.

In Table I, we list the results obtained by the Legendre polynomials method proposed in this paper together with the finite difference method [21] results. The displayed results show that our method is more convenient and more accurate than the finite difference method. From Fig. 1 and Fig. 2, we can see clearly that the numerical solutions are very good agreement with the exact solution.

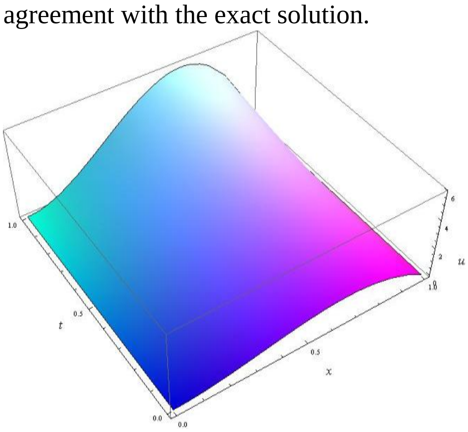

Fig.2. The exact solution.

Example 2: Consider the following

$$\begin{aligned} &D_t^{(1-x)/3}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \\ &u(x,0) = 5x\left(x^2-1\right) \\ &u(0,t) = 0, \\ &(x,t) \in \Omega = [0,1]\times[0,1] \end{aligned} \tag{34}$$

where

$$f(x,t) = \frac{3\left(5+t^2+t^3\right)(-1+x)x^{\frac{2+x}{3}}(40+53x)}{(8+x)(5+x)(2+x)\Gamma\left(\frac{4}{3}+\frac{1}{3}x\right)} - 6x(5+t^2+t^3)$$

The exact solution of the above equation is $u(x,t) = x\left(x^2-1\right)\left(5+t^2+t^3\right)$.The absolute errors between the exact solution and the numerical solution are displayed as follows:

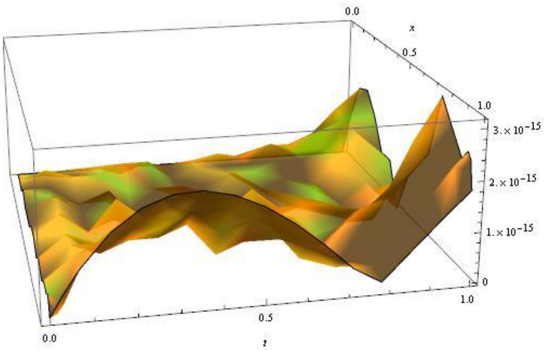

Fig. 3. The absolute error for Example 2 of $n = 3$.

From Fig. 3, we can find that the absolute errors are very tiny and only a small number of Legendre polynomials are needed when $n \geq 3$.

VI. Conclusion

In this paper, we have proposed a numerical approach for solving the variable order time fractional diffusion equation by using Legendre operational matrix. The operational matrix of differentiation $\mathbf{D}$ and $\mathbf{D}^{q(x,t)}$ have been used for transforming the variable order time fractional diffusion equation into a Sylvester equation that can be solved easily. Finally, numerical example reveals that the present method is very accurate and convenient for solving this problem.

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