# High-accuracy Alternating Difference Scheme for the Fourth-order Diffusion Equation

Geyang Guo*, Shujuan Lü

***Abstract*—In this paper, a highly accurate parallel difference scheme for the fourth-order diffusion equation is studied. Based on a group of new Saul'yev type asymmetric difference schemes, a high-order, unconditionally stable and parallel alternating group explicit scheme is derived. The scheme is fourth-order truncation error in space, which is much more accurate than the known methods. Numerical experiments are performed to examine the convergence, unconditional stability and accuracy. A comparison of the accuracy of this scheme with the prior AGE methods is presented.**

***Index Terms*—fourth-order diffusion equation, alternating difference scheme, high accuracy, parallel computation, unconditional stability.**

## I. Introduction

WITH the development of the high-performance computer, the need to construct parallel algorithms has long been desired. In the past decade, the alternating schemes were widely studied. The pioneer work can be traced to 1983, Evans and Abdullah first developed the alternating group explicit (AGE) scheme $[1, 2]$ for parabolic equation. The AGE scheme uses the explicit scheme and the implicit scheme alternately in the time and space direction, which can implement the parallel computation and is unconditionally stable. Afterwards, the alternating segment explicit-implicit (ASE-I) scheme $[3]$ and the alternating segment Crank-Nicolson (ASC-N) scheme $[4]$ were proposed. Recently, the alternating schemes have been extended to two-dimensional diffusion systems $[5]$, dispersive equation $[6, 7, 8, 10, 11, 13, 16]$, nonlinear three-order KdV equation $[9]$, fourth-order diffusion equation $[12, 14]$ and Helmholtz equation $[17]$, respectively. The results of numerical examples show that these schemes have unconditional stability and intrinsic parallelism. Meanwhile, the introduction of the alternating schemes leads to the rapid development of the domain decomposition parallel methods $[18, 19, 20]$. However, the majority of the literature have focused their attentions on the parallelism, the major problem in the above algorithms is that the truncation error is only near second order in space. The construction of the highly accurate parallel difference scheme has been considered by only a limited number of investigators. In $[10, 11, 15]$, the fourth-order accurate AGE and ASC-N schemes have been constructed for the dispersive equation by a group of new high-order accurate asymmetric difference schemes.

In view of the limited information available of highly accurate parallel difference method, this paper undertakes a study of the construction of high-order accurate algorithm for the fourth-order diffusion equation. The numerical solving methods were widely studied $[12, 14, 21, 22, 23, 24]$. Although the unconditionally stable general schemes with intrinsic parallelism for fourth-order diffusion equation have been derived in $[12]$, the truncation error is only near second order in space. In this work, a group of new Saul'yev asymmetric difference schemes is constructed, basing on these schemes, we will derive a fourth-order accurate alternating group explicit (AGE) scheme, and it also has unconditional stability and intrinsic parallelism. Its numerical simulations show better accuracy than the AGE1 and AGE2 schemes in $[12]$. We hope the result of this paper makes an essential contribution in this direction.

We consider the following problem

$$Lu = \frac{\partial u}{\partial t} + \alpha \frac{\partial^4 u}{\partial x^4} = 0, x \in [0, l], t \in [0, T], \qquad (1)$$

with initial condition

$$u(x, 0) = u_0(x), x \in [0, l], \qquad (2)$$

and the boundary conditions

$$u(0, t) = u_{xx}(0, t) = u(l, t) = u_{xx}(l, t) = 0, t \in [0, T]. \qquad (3)$$

where $u_0(x)$ is a given function, $\alpha$ is a constant.

The rest of this paper is organized as follows. In section II, some basic schemes are given and the new AGE scheme is developed. In section III, the truncation errors and the unconditional stability are discussed. In section IV, numerical experiments are performed. At last, a brief conclusion is given.

## II. The New Alternating Group Explicit Scheme

### A. The Basic Schemes

Divide the domain of definition $[0, l] \times [0, T]$ by parallel lines $x = x_j = jh (j = 0, 1, 2 \cdots, J)$, $t = t^n = n\tau (n = 0, 1, 2 \cdots, N)$, where $h = l/J$ is space mesh length, $\tau = T/N$ is time mesh length. $J$ and $N$ are positive integers. We use $U_j^n$ to represent the approximate solution of $u(x_j, t^n)$, where $u(x, t)$ represents the exact solution of (1). We first give six new asymmetric schemes $(4) - (9)$ (See Fig. 1).

$$-rU_{j+3}^{n+1} + 6rU_{j+2}^{n+1} - 6rU_{j+1}^{n+1} + (1 + r)U_j^{n+1} = -6rU_{j+2}^n +$$

Manuscript received August 26, 2016; revised December 06, 2016. This work was supported the Tianjin Research Program of Application Foundation and Advanced Technology (Youth Research Program) under Grant No: 15JCQNJC01600, the Research Abroad Fund of Tianjin University of Technology and Education under Grant Nos.: J10011060318 and J10011060321, the Youth Foundation of Tianjin University of Technology and Education under Grant No: KJ15-13, Tianjin High School Science and Technology Fund Planning Project under Grant No: JWK1603, and the National Natural Science Foundation of China under Grant Nos.: 11272024 and 11526157.

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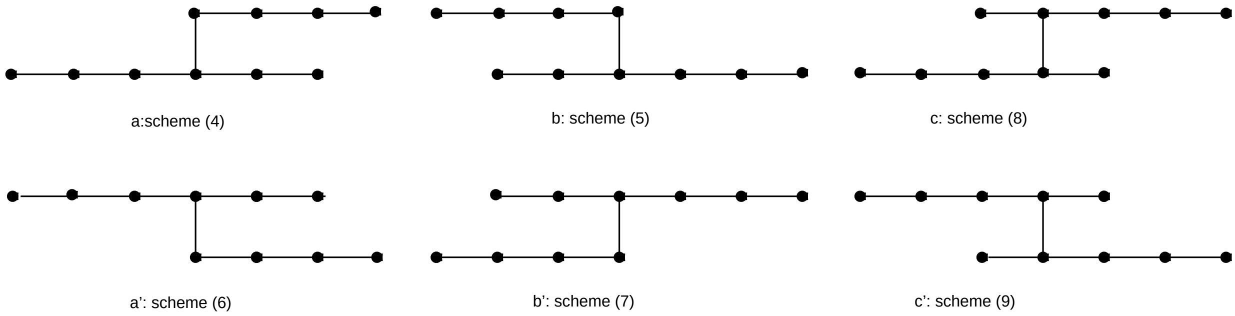


Fig. 1. The New Asymmetric Schemes (4)-(9)

$$33rU^n_{j+1}+(1-55r)U^n_j+39rU^n_{j-1}-12rU^n_{j-2}+rU^n_{j-3}, \quad (4)$$

$$(1+r)U^{n+1}_j-6rU^{n+1}_{j-1}+6rU^{n+1}_{j-2}-rU^{n+1}_{j-3}=-6rU^n_{j-2}+$$
$$33rU^n_{j-1}+(1-55r)U^n_j+39rU^n_{j+1}-12rU^n_{j+2}+rU^n_{j+3}, \quad (5)$$

$$6rU^{n+1}_{j+2}-33rU^{n+1}_{j+1}+(1+55r)U^{n+1}_j-39rU^{n+1}_{j-1}+12rU^{n+1}_{j-2}$$
$$-rU^{n+1}_{j-3}=rU^n_{j+3}-6rU^n_{j+2}+6rU^n_{j+1}+(1-r)U^n_j, \quad (6)$$

$$-rU^{n+1}_{j+3}+12rU^{n+1}_{j+2}-39rU^{n+1}_{j+1}+(1+55r)U^{n+1}_j-33rU^{n+1}_{j-1}$$
$$+6rU^{n+1}_{j-2}=(1-r)U^n_j+6rU^n_{j-1}-6rU^n_{j-2}+rU^n_{j-3}, \quad (7)$$

$$-rU^{n+1}_{j+3}+12rU^{n+1}_{j+2}-33rU^{n+1}_{j+1}+(1+28r)U^{n+1}_j-6rU^{n+1}_{j-1}=$$
$$6rU^n_{j+1}+(1-28r)U^n_j+33rU^n_{j-1}-12rU^n_{j-2}+rU^n_{j-3}, \quad (8)$$

$$-6rU^{n+1}_{j+1}+(1+28r)U^{n+1}_j-33rU^{n+1}_{j-1}+12rU^{n+1}_{j-2}-rU^{n+1}_{j-3}=$$
$$rU^n_{j+3}-12rU^n_{j+2}+33rU^n_{j+1}+(1-28r)U^n_j+6rU^n_{j-1}. \quad (9)$$

where $r=\alpha\tau/6h^4$.

The discrete initial-boundary value conditions are

$$U^0_j=u_0(x_j), j=0,1,2,\cdots,J$$
$$U^n_0=U^n_{-1}+U^n_1=U^n_{-2}+U^n_2=0,$$
$$U^n_J=U^n_{J-1}+U^n_{J+1}=U^n_{J-2}+U^n_{J+2}=0,$$
$$n=0,1,2,\cdots,N.$$

Let $L^{(4)}_h, L^{(5)}_h, L^{(6)}_h, L^{(7)}_h, L^{(8)}_h, L^{(9)}_h$ be the discretizied operators for $L$ based on schemes $(4)-(9)$. From the Taylor series expansion at $(x_j,t^n)$, we obtain the following truncation error expressions (10)-(15) for formulaes $(4)-(9)$:

$$L^{(4)}_h u^n_j-[Lu]^n_j=3rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]^n_j$$
$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j+\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4), \quad (10)$$

$$L^{(5)}_h u^n_j-[Lu]^n_j=-3rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]^n_j$$
$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j-\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4), \quad (11)$$

$$L^{(6)}_h u^n_j-[Lu]^n_j=-3rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]^n_j$$
$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j-\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4), \quad (12)$$

$$L^{(7)}_h u^n_j-[Lu]^n_j=3rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]^n_j$$
$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j+\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4), \quad (13)$$

$$L^{(8)}_h u^n_j-[Lu]^n_j=-6rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j+7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j$$
$$-3rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4), \quad (14)$$

$$L^{(9)}_h u^n_j-[Lu]^n_j=6rh[\frac{\partial^2 u}{\partial t\partial x}]^n_j-7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]^n_j$$
$$+3rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]^n_j+O(\tau+h^4). \quad (15)$$

### *B. The New Alternating Group Explicit Scheme*

The new parallel AGE scheme is constructed as follow. Assuming $J-1=6k$, $k\geq 1$ is a positive integer. we consider the model of the group at the $(n+1)$st and the $(n+2)$nd time levels, where $n$ is an even number. We divide the nodes of the (n+1)st time level into $k$ groups, each group contains 6 nodes in $x$ direction. Based on the alternating technique, we divide the nodes of the (n+2)nd time level into $k+1$ groups, the first and the $(k+1)$st groups contain 3 nodes in $x$ direction. The other groups contain 6 nodes in $x$ direction. The nodes in every group can be computed by the asymmetric difference schemes (a,c,b',a',c',b) according to the rule displayed in Fig. 2.

The new AGE scheme can be expressed as

$$(I+rG_1)U^{n+1}=(I-rG_2)U^n, \quad (16)$$

$$(I+rG_2)U^{n+2}=(I-rG_1)U^{n+1}, \quad (17)$$

$$n=0,2,4,6,\cdots.$$

where $U^n=(u^n_1,u^n_2,\cdots,u^n_{J-1})^T$, and the matrices $G_1$ and $G_2$ are given by

$$G_1=\begin{pmatrix} Q_{6\times 6} & & & & \\ & Q_{6\times 6} & & & \\ & & \ddots & & \\ & & & Q_{6\times 6} & \\ & & & & Q_{6\times 6}\end{pmatrix},$$

$$G_2=\begin{pmatrix} Q^l_{3\times 3} & & & & P_{3\times 3} \\ & Q_{6\times 6} & & & \\ & & \ddots & & \\ & & & Q_{6\times 6} & \\ P^T_{3\times 3} & & & & Q^r_{3\times 3}\end{pmatrix},$$

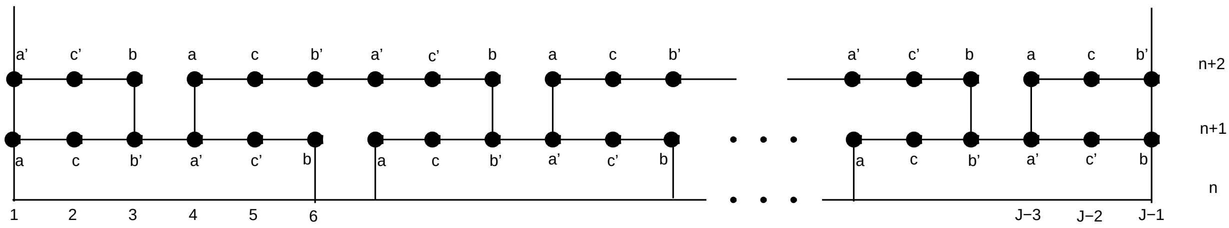


Fig. 2. The Diagram of the New AGE Scheme

$$Q_{6\times 6}=\begin{pmatrix} 1 & -6 & 6 & -1 & & \\ -6 & 28 & -33 & 12 & -1 & \\ 6 & -33 & 55 & -39 & 12 & -1 \\ -1 & 12 & -39 & 55 & -33 & 6 \\ & -1 & 12 & -33 & 28 & -6 \\ & & -1 & 6 & -6 & 1 \end{pmatrix}_{6\times 6},$$

$$P_{3\times 3}=\begin{pmatrix} -1 & 12 & -39 \\ & -1 & 12 \\ & & -1 \end{pmatrix}_{3\times 3},$$

$$Q^{l}_{3\times 3}=\begin{pmatrix} -55 & 33 & -6 \\ 33 & -28 & 6 \\ -6 & 6 & -1 \end{pmatrix}_{3\times 3},$$

$$Q^{r}_{3\times 3}=\begin{pmatrix} -1 & 6 & -6 \\ 6 & -28 & 33 \\ -6 & 33 & -55 \end{pmatrix}_{3\times 3}.$$

## III. The Analysis of the Truncation Errors and Stability

### A. The Analysis of the Truncation Errors

Let us give out the error analysis for the AGE scheme. In the AGE scheme, there are three pairs of schemes (4) and (6), (5) and (7), (8) and (9) which are alternatingly used between two times levels. From the Taylor series expansion at $(x_j, t^n)$ for formula (4) and expansion at $(x_j, t^{n+1})$ for formula (6), we get the following truncation error expressions, respectively.

$$L_h^{(4)}u_j^n-[Lu]_j^{n+1}=3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$
$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}-\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4),\quad(18)$$

$$L_h^{(6)}u_j^n-[Lu]_j^{n+1}=-3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$
$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}+\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4),\quad(19)$$

Similarly, we obtain the truncation error expressions (20) – (23), respectively.

$$L_h^{(5)}u_j^n-[Lu]_j^{n+1}=-3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}+\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$
$$-\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}+\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4),\quad(20)$$

$$L_h^{(7)}u_j^n-[Lu]_j^{n+1}=3rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}-\frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t\partial x^2}]_j^{n+1}$$
$$+\frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}-\frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4),\quad(21)$$

$$L_h^{(8)}u_j^n-[Lu]_j^{n+1}=-6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}+7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$
$$+3rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4),\quad(22)$$

$$L_h^{(9)}u_j^n-[Lu]_j^{n+1}=6rh[\frac{\partial^2 u}{\partial t\partial x}]_j^{n+1}-7rh^3[\frac{\partial^4 u}{\partial t\partial x^3}]_j^{n+1}$$
$$-3rh\tau[\frac{\partial^3 u}{\partial t^2\partial x}]_j^{n+1}+O(\tau+h^4).\quad(23)$$

For the pairs of the asymmetrical schemes, by comparing the results (18) with (12), (19) with (10), (20) with (13), (21) with (11), (22) with (15),(23) with (14), we find that the leading terms have opposite signs at the adjacent time levels, the effect of the terms with $h, h^2, h^3$ can be canceled. Therefore, the truncation errors are $O(\tau + h^4)$. The above discussions proved that the truncation error of the new AGE method is approximately $O(h^4)$ in space.

### B. The Analysis of the Unconditional Stability

To prove the stability, we have to introduce the following Kellogg Lemma [25].

**Lemma 1.** *If $\rho > 0$, $C + C^T$ is nonnegative definite, then $(I + \rho C)^{-1}$ exists and there holds*

$$||(I+\rho C)^{-1}||_2 \le 1.$$

**Lemma 2.** *Under the conditions of Lemma 1, the following inequality holds*

$$||(I-\rho C)(I+\rho C)^{-1}||_2 \le 1.$$

**Theorem 1.** *For any real number $r$, the AGE scheme (16)-(17) is unconditionally stable.*

*Proof:* By eliminating $U^{n+1}$ from (16)-(17), we obtain $U^{n+2} = GU^n$. where $G$ is the growth matrix

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2).$$

For any even number $n$, there holds

$$G^n = (I+rG_2)^{-1}(I-rG_1)(I+rG_1)^{-1}[(I-rG_2)(I+rG_2)^{-1}$$
$$\cdot(I - rG_1)(I + rG_1)^{-1}]^{n-1}(I - rG_2).$$

Since $G_1$ and $G_2$ are all symmetric, for any real number $r$, we can obtain the following inequality from the Kellogg Lemma

$$\|G^n\|_2 \le \|(I + rG_2)^{-1}\|_2 \cdot \|(I - rG_1)(I + rG_1)^{-1}\|_2^n$$
$$\cdot\|(I - rG_2)(I + rG_2)^{-1}\|_2^{n-1} \cdot \|(I - rG_2)\|_2.$$

Hence

$$\|G^n\|_2 \le \|(I - rG_2)\|_2 \le \sqrt{\|(I - rG_2)\|_\infty \cdot \|(I - rG_2)\|_1}$$
$$\le \sqrt{1 + 146r}$$

This shows that the AGE scheme is unconditionally stable. ∎

## IV. Numerical Experiments

In this section, we perform numerical experiments for $(1)-(3)$ using the following model problem

$$u_0(x) = \sin x, \ \ \alpha = 1, \ \ l = \pi.$$

The exact solution of this problem is

$$u(x,t) = e^{-t}\sin x.$$

We first illustrate the convergence rates in space for the new AGE scheme. Let $v_j^n = u(x_j, t^n)$ be the exact solution of the problem $(1)-(3)$ and $u_j^n$ be the approximate solution. We introduce the following $L_\infty-$norm error and $L_2-$norm error

$$E_{\infty,h} = \max_j |v_j^n - u_j^n|,$$

$$E_{2,h} = \big(\sum_j |v_j^n - u_j^n|^2 h\big)^{\frac{1}{2}}.$$

Thus, we can calculate the rates of convergence by the following definitions

$$rate = \frac{\log(E_{\infty,h_1}/E_{\infty,h_2})}{\log(h_1/h_2)},$$

$$rate = \frac{\log(E_{2,h_1}/E_{2,h_2})}{\log(h_1/h_2)}.$$

where $h_1$ and $h_2$ are the space mesh steps.

Let 'AGE6' represents the new AGE 6-points scheme described above, 'AGE4' represents the AGE 4-points scheme in [12], and 'AGE8' represents the AGE 8-points scheme in [12]. For the AGE6, AGE4, AGE8 schemes, we give the $L_\infty-$norm errors, $L_2-$norm errors and the convergence rates in Tables I and II, respectively. We can see from these tables that the convergence rate of the new AGE6 scheme appears to be $O(h^4)$ in space, which is coincident with our theoretical analysis, while the AGE4 and the AGE8 schemes appear to be $O(h^2)$ in space [12].

Next, we compare the errors for the AGE6 scheme with the AGE4 and the AGE8 schemes at the same time $t$ in Tables III and IV, respectively, where the absolute error $ae = |u_j^n - u(x_j, t^n)|$, the relative error $pe = \dfrac{|u_j^n - u(x_j,t^n)|}{|u(x_j,t^n)|} \times \%$, and 'Exact' represents the values of the exact solution $u(x_j, t^n)$. The results show that the AGE6 scheme is more accurate than the AGE4 and the AGE8 schemes in [12]. In addition, from Figures 3-8, we can see clearly that the AGE6 solutions are more accurate than the AGE4 and AGE8 solutions.

Third, we verified the stability of the AGE6 method. From Tables V and VI, we can easily find that the high-accuracy AGE6 method is unconditionally stable.

Finally, based on the group of asymmetric difference schemes (4)-(9), the AGE6 scheme changes the global domain of definition into some small independent segments, and can be computed in parallel, the parallelism is clarity.

## V. Conclusion

In this paper, we first constructed a group of new asymmetric schemes, basing on the idea of the alternating schemes, we designed the new AGE scheme with high-order accuracy for the fourth-order diffusion equation. The theoretics analysis and the numerical simulations show that the new AGE

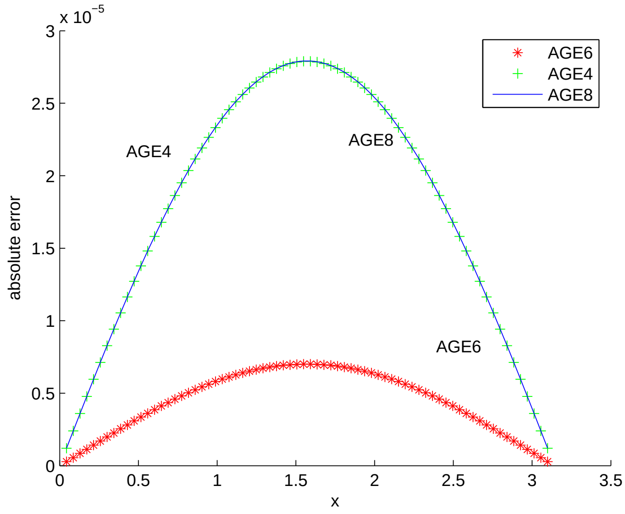


Fig. 3. Comparison of the absolute errors $t = 0.1, h = \pi/71, \tau = 10^{-6}$

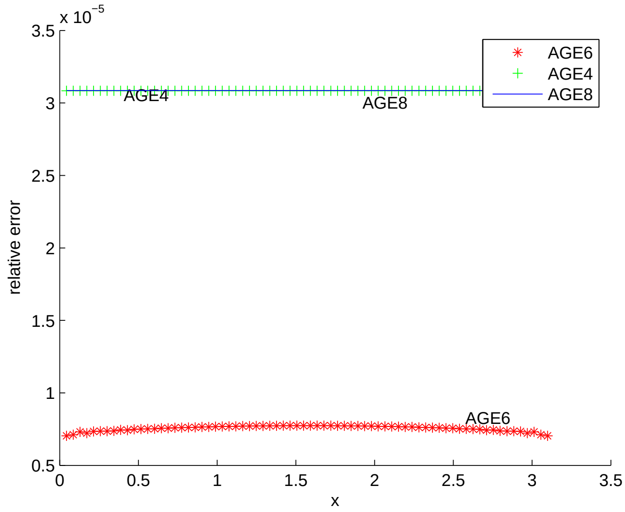


Fig. 4. Comparison of the relative errors $t = 0.1, h = \pi/71, \tau = 10^{-6}$

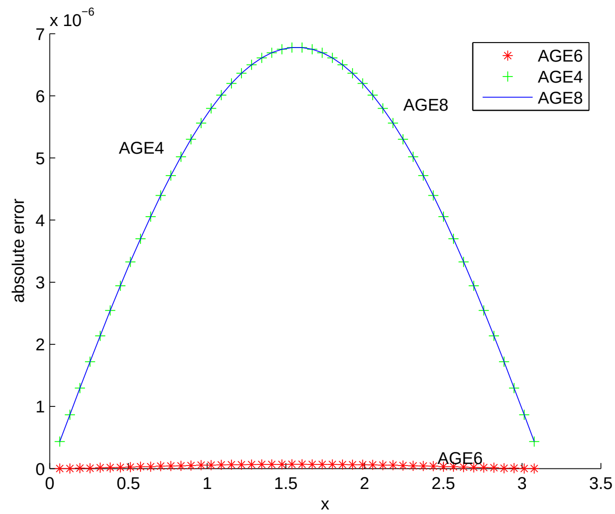


Fig. 5. Comparison of the absolute errors $t = 0.01, h = \pi/49, \tau = 10^{-6}$

TABLE I
THE CONVERGENCE RATE OF THE AGE6 SCHEME AT $t = 0.1, \tau = 1 \times 10^{-6}$

| $h$ | $\pi/7$ | $\pi/13$ | $\pi/19$ | $\pi/25$ |
|---|---|---|---|---|
| $L_\infty$ | 1.0134E-4 | 8.8584E-6 | 1.9557E-6 | 6.4410E-7 |
| Rate | – | 3.9470 | 3.9815 | 4.0462 |
| $L_2$ | 1.3027E-4 | 1.1184E-5 | 2.4597E-6 | 8.0928E-7 |
| Rate | – | 3.9660 | 3.9907 | 4.0511 |

TABLE II
THE CONVERGENCE RATE OF THE AGE6 SCHEME AT $t = 0.01, \tau = 1 \times 10^{-6}$

| $h$ | $\pi/7$ | $\pi/13$ | $\pi/19$ | $\pi/25$ |
|---|---|---|---|---|
| $L_\infty$ | 1.1088E-5 | 9.6926E-7 | 2.1392E-7 | 7.0340E-8 |
| Rate | – | 3.9373 | 3.9817 | 4.0525 |
| $L_2$ | 1.4254E-5 | 1.2237E-6 | 2.6914E-7 | 8.8550E-8 |
| Rate | – | 3.9660 | 3.9920 | 4.0507 |

TABLE III
THE ERRORS OF NUMERICAL SOLUTION AT $J = 24, \tau = 1 \times 10^{-6}, t = 0.1$

| scheme | error | j=4 | j=8 | j=12 | j=15 | j=22 |
|---|---|---|---|---|---|---|
| AGE6 | $ae(10^{-7})$ | 3.1179 | 5.4533 | 6.4411 | 6.1396 | 2.3782 |
| | $pe(10^{-7})$ | 7.1527 | 7.1381 | 7.1326 | 7.1345 | 7.1398 |
| AGE4[12] | $ae(10^{-4})$ | 1.1459 | 2.0082 | 2.3740 | 2.2623 | 0.8754 |
| | $pe(10^{-4})$ | 2.6291 | 2.6291 | 2.6291 | 2.6291 | 2.6291 |
| AGE8[12] | $ae(10^{-4})$ | 1.1461 | 2.0086 | 2.3742 | 2.2625 | 0.8757 |
| | $pe(10^{-4})$ | 2.6291 | 2.6291 | 2.6291 | 2.6291 | 2.6291 |
| Exact | $(10^{-1})$ | 4.3591 | 7.6398 | 9.0305 | 8.6055 | 3.3309 |

TABLE IV
THE ERRORS OF NUMERICAL SOLUTION AT $J = 48, \tau = 1 \times 10^{-6}, t = 0.01$

| scheme | error | j=8 | j=16 | j=24 | j=32 | j=40 |
|---|---|---|---|---|---|---|
| AGE6 | $ae(10^{-8})$ | 2.2930 | 5.4578 | 6.8330 | 5.9426 | 3.1638 |
| | $pe(10^{-8})$ | 4.7197 | 6.4465 | 6.9053 | 6.7701 | 5.8579 |
| AGE4[12] | $ae(10^{-6})$ | 3.3272 | 5.7981 | 6.7768 | 6.0114 | 3.6989 |
| | $pe(10^{-6})$ | 6.8484 | 6.8484 | 6.8484 | 6.8484 | 6.8484 |
| AGE8[12] | $ae(10^{-6})$ | 3.3274 | 5.7985 | 6.7772 | 6.0117 | 3.6991 |
| | $pe(10^{-6})$ | 6.8488 | 6.8488 | 6.8488 | 6.8488 | 6.8488 |
| Exact | $(10^{-1})$ | 4.5585 | 8.4667 | 9.8954 | 8.7773 | 5.4015 |

TABLE V
THE ERRORS OF NUMERICAL SOLUTION AT $J = 120, \tau = 1 \times 10^{-9}, t = 0.001, r = \tau/6h^4$

| $r$ | error | j=10 | j=30 | j=50 | j=70 | j=90 | j=110 |
|---|---|---|---|---|---|---|---|
| $r_1 = r$ | $ae(10^{-11})$ | 1.0193 | 2.3443 | 3.2153 | 3.2361 | 2.4046 | 1.1853 |
| | $pe(10^{-11})$ | 3.4621 | 3.3404 | 3.3419 | 3.3404 | 3.3397 | 3.4412 |
| $r_2 = 10r$ | $ae(10^{-10})$ | 1.0077 | 1.0854 | 1.4387 | 1.4688 | 1.0925 | 2.1654 |
| | $pe(10^{-10})$ | 1.0024 | 1.5466 | 1.4954 | 1.5162 | 1.5173 | 1.0036 |
| $r_3 = 100r$ | $ae(10^{-8})$ | 1.0044 | 1.1747 | 1.5611 | 1.5921 | 1.1841 | 1.0020 |
| | $pe(10^{-8})$ | 1.0406 | 1.6739 | 1.6226 | 1.6434 | 1.6445 | 1.0421 |

TABLE VI
THE ERRORS OF NUMERICAL SOLUTION AT $J = 36, \tau = 1 \times 10^{-7}, t = 1, r = \tau/6h^4$

| $r$ | error | j=5 | j=10 | j=15 | j=20 | j=25 | j=30 | j=35 |
|---|---|---|---|---|---|---|---|---|
| $r_1 = r$ | $ae(10^{-7})$ | 2.2759 | 4.1477 | 5.2859 | 5.4804 | 4.7049 | 3.0941 | 1.0013 |
| | $pe(10^{-6})$ | 1.5020 | 1.5019 | 1.5018 | 1.5018 | 1.5019 | 1.5020 | 1.5021 |
| $r_2 = 10r$ | $ae(10^{-8})$ | 3.6756 | 6.5949 | 8.2352 | 8.5414 | 7.3896 | 4.9830 | 1.5292 |
| | $pe(10^{-7})$ | 2.4257 | 2.3881 | 2.3412 | 2.3408 | 2.3589 | 2.4189 | 2.5620 |
| $r_3 = 100r$ | $ae(10^{-6})$ | 1.9044 | 3.4810 | 4.5504 | 4.6169 | 3.9578 | 2.5904 | 7.7288 |
| | $pe(10^{-5})$ | 1.2568 | 1.2605 | 1.2652 | 1.2653 | 1.2634 | 1.2574 | 1.2431 |

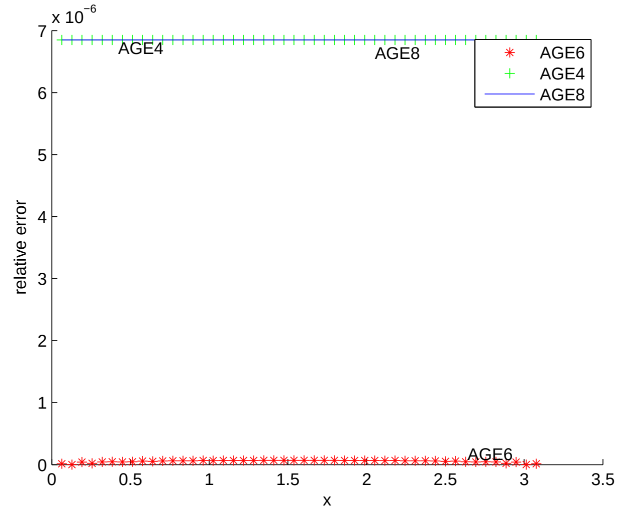


Fig. 6. Comparison of the relative errors $t = 0.01, h = \pi/49, \tau = 10^{-6}$

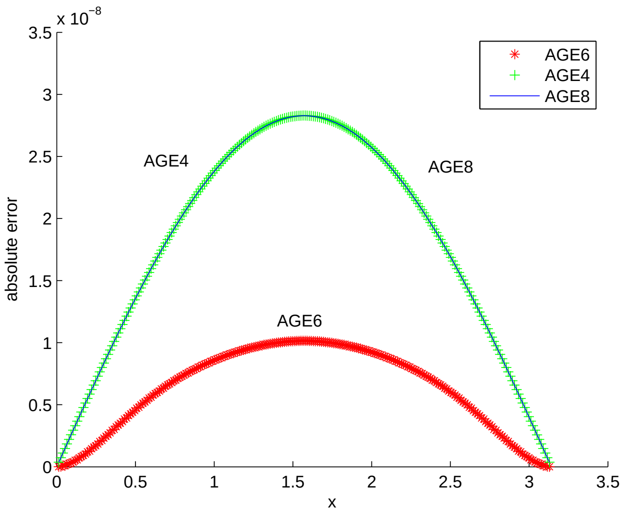


Fig. 7. Comparison of the absolute errors $t = 0.001, h = \pi/241, \tau = 10^{-7}$

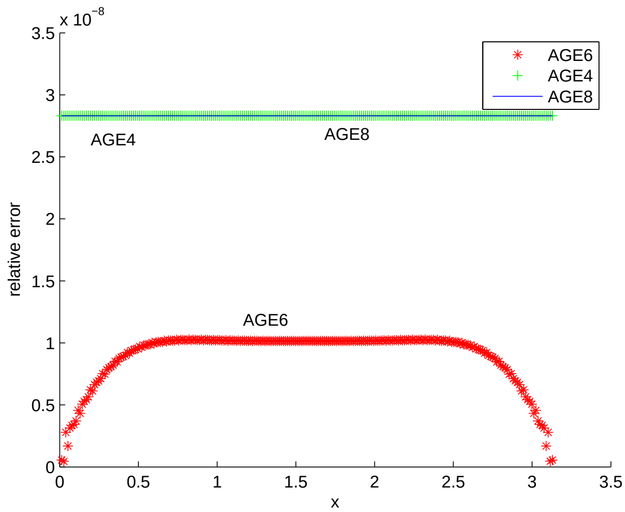


Fig. 8. Comparison of the relative errors $t = 0.001, h = \pi/241, \tau = 10^{-7}$

scheme constructed in the paper has fourth-order accuracy, which is more accurate than the AGE1, AGE2 schemes in [12].

## ACKNOWLEDGMENT

The author would like to thank the referees very much for their valuable suggestions on improving this paper.

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