Global Asymptotic Stabilization for a Class of High-Order Nonholonomic Systems with Time-Varying Delays

Yanling Shang and Ye Yuan

Abstract—**This paper addresses the global asymptotic stabilization for a class of high-order nonholonomic systems with time-varying delays. By employing input-state-scaling, and adding a power integrator techniques, and by choosing an appropriate Lyapunov-Krasoviskii functional, a state feedback controller is constructed. Based on switching strategy to eliminate the phenomenon of uncontrollability, the proposed controller ensures that the closed-loop system is globally asymptotically regulated at the origin. A simulation example is provided to demonstrate the effectiveness of the proposed scheme.**

Index Terms—**high-order nonholonomic systems, time-varying delays, state feedback, adding a power integrator nonholonomic systems, global asymptotic stabilization.**

I. Introduction

In this paper, we consider the following high-order nonholonomic systems with time-varying delays

$$\begin{cases} \dot{x}_0(t) = d_0(t)u_0^{p_0}(t) + f_0(t, x_0(t)) \\ \dot{x}_i(t) = d_i(t)x_{i+1}^{p_i}(t)u_0^{q_i}(t) \\ \qquad + f_i(t, x_0(t), x(t), x(t-d(t))) \\ \qquad i = 1, \cdots, n-1 \\ \dot{x}_n(t) = d_n(t)u_1^{p_n}(t) \\ \qquad + f_n(t, x_0(t), x(t), x(t-d(t))) \end{cases} \tag{1}$$

where $x_0 \in R$ and $x = (x_1, \cdots, x_n)^T \in R^n$ are system states, u_0 and u_1 are control inputs, respectively; $p_i \geq 1, i = 0, 1, \cdots, n$ are odd integers, and $q_k \geq 1$, $k = 1, 2, \cdots, n-1$ are integers; $d_i(t)$ are disturbed virtual control coefficients; f_0 and f_i, $i = 1, \cdots, n$ are unknown continuous functions with $f_0(t, 0) = 0$ and $f_i(t, 0, 0, 0) = 0$; $d(t) : R_+ \rightarrow [0, d]$ is the time-varying delay satisfying $\dot{d}(t) \leq \eta < 1$ for a known constant η;

As an important class of nonlinear systems, nonholonomic systems have attracted a great deal of attention over the past decades because they can be used to model numerous mechanical systems, such as mobile robots, car-like vehicle and under-actuated satellites, see, e.g., [1-4] and the references therein. However, from Brockett necessary condition [5], it is well known that no smooth (or even continuous) time-invariant static state feedback exists for the stabilization of nonholonomic systems. To overcome this difficulty, with the effort of many researchers a number of intelligent approaches have been proposed, which can mainly be classified into discontinuous time-invariant stabilization[6,7], smooth time-varying stabilization[8,9] and hybrid stabilization[10], see the recent survey paper [11] for more details. Using these valid approaches, the robust issue of nonholonomic systems has been extensively studied [12-16]. More meaningly, the high-order nonholonomic systems in power chained form, which can be viewed as the extension of the classical nonholonomic systems, have been achieved investigation[17-19].

However, the aforementioned results do not consider the effect of time delay. As a matter of fact, time-delay is actually widespread in state, input and output due to sensors, calculation, information processing or transport, and its emergence is often a significant cause of instability and serious deterioration in the system performance [20]. Therefore, how extending these methods to the systems with time delays is naturally regarded as an interesting research topic. Recently, [21] and [22] investigated the state-feedback stabilization problem for delayed nonholonomic systems with different structures. However, the control design for high-order delayed nonholonomic system (1) is extremely challenging because on the one hand, some intrinsic features of system (1), such as its Jacobian linearization being neither controllable nor feedback linearizable, lead to the existing design tools being hardly applicable, and on the other hand, the existence of time-delay effect will make the common assumption on the high-order systems nonlinearities infeasible and what conditions should be imposed on system (1) remain unanswered. To the best of the authors' knowledge, there are few results on the high-order delayed nonholonomic system (1).

Motivated the above observation, in this paper we focus our attention on solving the problem of global asymptotic stabilization by state feedback for high-order nonholonomic system (1) with time-varying delays. The contribution of this paper is highlighted as follows. First, motivated by the work in [23] and flexibly using the methods of adding a power integrator, a recursive design procedure for the time-delay independent state-feedback controller is given. Then, by employed an appropriate Lyapunov- Krasovskii functional, we show that the controller designed guarantees global asymptotic regulation of the resulting closed-loop system.

The rest of this paper is organized as follows. In Section II, preliminary knowledge and the problem formulation are given. Section III presents the input-state-scaling technique and the recursive design procedure, while Section V provides the switching control strategy and the main result. Section 5

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gives a simulation example to illustrate the theoretical finding of this paper. Finally, concluding remarks are proposed in Section VI.

II. Problem Formulation and Preliminaries

The control objective is to construct robust control laws of the form

$$u_0 = \mu_0(x_0), \quad u_1 = \mu(x_0, x) \tag{2}$$

such that all signals of the closed-loop system are bounded. Furthermore, global asymptotic regulation of the states are achieved, i.e. $\lim_{t\to\infty}(|x_0(t)| + |x(t)|) = 0$.

In order to achieve the above control objective, throughout the paper, the following assumptions regarding system (1) are imposed.

Assumption 1. For $i = 0, 1, \cdots, n$, there are positive constants c_{i1} and c_{i2} such that

$$c_{i1} \le d_i(t) \le c_{i2}$$

Assumption 2. For f_0, there is a positive constant c_{03} such that

$$|f_0(t, x_0(t))| \le c_{03}|x_0|$$

For $i = 1, \cdots, n$, there are $\mathcal{F}$ functions a_i such that

$$|f_i(\cdot)| \le a_i(x_0(t))\Big(|x_i(t)| + |x_i(t-d(t))|\Big)$$
$$+a_i(x_0(t))\sum_{j=1}^{i-1}\Big(|x_j(t)|^{\frac{1}{p_j\cdots p_{i-1}}} + |x_j(t-d(t))|^{\frac{1}{p_j\cdots p_{i-1}}}\Big)$$

To make the paper self-contained, we recall that a continuously differential function $f : R^m \to R$ is called a $\mathcal{F}$ function if it is nonnegative and monotone-nondecreasing on $[0, +\infty)$. It is worthwhile to point out that there exist many functions such as $f_1(x) \equiv c > 0$ and $f_2(x) = x^m$,where $m \ge 0$ being $\mathcal{F}$ functions. Furthermore, it can be showed that if f and g are $\mathcal{F}$ functions, then $f + g$, $f \cdot g$ and $f \circ g$ are also $\mathcal{F}$ functions.

Remark 1. Assumption 1 is common and similar to the ones usually imposed on the nonholonomic systems [12,13,19]. It is worth pointing out that Assumption 2 is somewhat stringent, and when $a_i(x_0(t)) = a$ (a is a positive constant) and $d(t) = 0$, it is the same as that in [24]. Particularly, $p_i = 1$,$i = 1, \cdots, n$ the assumption is equivalent to that in [20,25].

The following two lemmas can be found in [19,25], which serve as the basis of the key tools for the adding a power integrator technique.

Lemma 1. For $x \in R$, $y \in R$, and $p \ge 1$ is a constant, the following inequalities hold:

$$|x + y|^p \le 2^{p-1}|x^p + y^p|$$
$$(|x| + |y|)^{1/p} \le |x|^{1/p} + |y|^{1/p} \le 2^{(p-1)/p}(|x| + |y|)^{1/p}$$

If $p \ge 1$ is odd, then

$$|x - y|^p \le 2^{p-1}|x^p - y^p|$$
$$|x^{1/p} - y^{1/p}| \le 2^{(p-1)/p}|x - y|^{1/p}$$

Lemma 2. Let c, d be positive real numbers and $\pi(x, y) > 0$ be a real-valued function. Then,

$$|x|^c|y|^d \le \frac{c\pi(x, y)|x|^{c+d}}{c + d} + \frac{d\pi^{-c/d}(x, y)|y|^{c+d}}{c + d}$$

III. Robust Controller Design

In this section, we focus on designing robust controller provided that $x_0(t_0) \ne 0$. The case where the initial condition $x_0(t_0) = 0$ will be treated in Section IV. The inherently triangular structure of system (1) suggests that we should design the control inputs u_0 and u_1 in two separate stages.

A. Design u_0 for x_0-subsystem

For x_0-subsystem, the control u_0 can be chosen as

$$u_0^{p_0}(t) = -\lambda_0 x_0(t) \tag{3}$$

where λ_0 is a positive design constant and satisfies $\lambda_0 > 1 + (c_{03}/c_{01})$.

As a result, the following lemma can be established by considering the Lyapunov function candidate $V_0 = x_0^2/2$ and by applying directly the Gronwall-Bellman inequality[10].

Lemma 3. For any initial $t_0 \ge 0$ and any initial condition $x_0(t_0) \in R$, the corresponding solution $x_0(t)$ exists for each $t \ge t_0$ and satisfies

$$\begin{aligned} x_0(t_0) \ge 0 \Rightarrow\ & x_0(t_0)e^{-(\lambda_0 c_{02} + c_{03})(t-t_0)} \\ &\le x_0(t) \le x_0(t_0)e^{-(\lambda_0 c_{01} - c_{03})(t-t_0)} \end{aligned} \tag{4}$$

$$\begin{aligned} x_0(t_0) < 0 \Rightarrow\ & x_0(t_0)e^{-(\lambda_0 c_{01} - c_{03})(t-t_0)} \\ &\le x_0(t) \le x_0(t_0)e^{-(\lambda_0 c_{02} + c_{03})(t-t_0)} \end{aligned} \tag{5}$$

Remark 2. From Lemma 3, we can see that $x_0(t)$ can be zero only at $t = t_0$, when $x(t_0) = 0$ or $t = \infty$. Consequently, it is concluded that x_0 does not cross zero for all $t \in (t_0, \infty)$ provided that $x_0(t_0) \ne 0$. Furthermore, from (3), it follows that the u_0 exists, does not cross zero for all $t \in (t_0, \infty)$ independent of the x-subsystem and satisfies $\lim_{t\to\infty} u_0(t) = 0$ provided that $x_0(t_0) \ne 0$.

B. Input-state-scaling transformation

From the above analysis, we can see the x_0-state in (1) can be globally exponentially regulated to zero via $u_0(t)$ in (3) as $t \to \infty$. It is troublesome in controlling the x-subsystem via the control input $u_1(t)$ because, in the limit (i.e. $u_0(t) = 0$), the x-subsystem is uncontrollable. This problem can be avoided by utilizing the following discontinuous input-state-scaling transformation

$$z_i(t) = \begin{cases} \dfrac{x_i(t)}{u_0^{r_i}(t)}, & t \ge t_0 \\ x_i(t), & t_0 - d \le t < t_0 \end{cases} \qquad i = 1, \cdots, n \tag{6}$$

where $r_i = q_i + p_i r_{i+1}, 1 \le i \le n-1$ and $r_n = 0$.

Remark 3. Note that (6) is a modified version of input-state-scaling transformation[18,19], and is used to deal with time delay terms. Because of the particular choice (6), for $i = 1, \cdots, n$, $z_i(t - d(t))$ are well-defined.

Under the new z-coordinates, the x-subsystem is transformed into

$$\begin{cases} \dot{z}_i(t) = d_i(t)z_{i+1}^{p_i}(t) + g_i(t, x_0(t), x(t), x(t-d(t))) \\ \quad i = 1, \cdots, n-1 \\ \dot{z}_n(t) = d_n(t)u_1^{p_n}(t) + g_n(t, x_0(t), x(t), x(t-d(t))) \end{cases} \tag{7}$$

where

$$\begin{aligned} g_i(\cdot) = & \frac{f_i(t, x_0(t), x(t), x(t-d(t)))}{u_0^{r_i}(t)} \\ & + r_i z_i(t)\frac{u_0^{p_0}(t) + f_0(t, x_0(t))}{p_0 x_0(t)} \end{aligned} \tag{8}$$

In order to obtain the estimation for the nonlinear function g_i, the following lemma can be derived by the assumption before.

Lemma 4. For $i = 1, \cdots, n$, there exist $\mathcal{F}$ functions ϕ_i such that

$$\begin{aligned} |g_i(\cdot)| \le\ & \phi_i(x_0(t))\Big(|z_i(t)| + |z_i(t-d(t))|\Big) \\ & + \phi_i(x_0(t)) \sum_{j=1}^{i-1} \Big(|z_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} \\ & + |z_j(t-d(t))|^{\frac{1}{p_j \cdots p_{i-1}}}\Big) \end{aligned} \tag{9}$$

Proof. In view of (3)-(6), (8)and Assumption 2, we get

$$\begin{aligned} & |g_i(t, x_0(t), x(t), x(t-d(t)))| \\ \le\ & a_i\Big(\frac{|x_i(t)|}{|u_0^{r_i}(t)|} + \frac{|x_i(t-d(t))|}{|u_0^{r_i}(t)|}\Big) \\ & + \frac{r_i(\lambda_0 + c_{03})}{p_0}|z_i(t)| \\ & + a_i \sum_{j=1}^{i-1}\Big(\frac{|x_j(t)|^{\frac{1}{p_j\cdots p_{i-1}}}}{|u_0^{r_i}(t)|} + \frac{|x_j(t-d(t))|^{\frac{1}{p_j\cdots p_{i-1}}}}{|u_0^{r_i}(t)|}\Big) \\ \le\ & \Big(a_i + \frac{r_i(\lambda_0 + c_{03})}{p_0}\Big)|z_i(t)| \\ & + \Big(a_i \frac{|u_0^{r_i}(t-d(t))|}{|u_0^{r_i}(t)|}\Big)|z_i(t-d(t))| \\ & + a_i \sum_{j=1}^{i-1}\Big(|z_j(t)|^{\frac{1}{p_j\cdots p_{i-1}}} |u_0(t)|^{\frac{r_j}{p_j\cdots p_{i-1}} - r_i} \\ & + |z_j(t-d(t))|^{\frac{1}{p_j\cdots p_{i-1}}} \frac{|u_0(t-d(t))|^{\frac{r_j}{p_j\cdots p_{i-1}}}}{|u_0^{r_i}(t)|}\Big) \\ \le\ & b_{i1}|z_i(t)| + b_{i2}|z_i(t-d(t))| \\ & + b_{i3}\sum_{j=1}^{i-1}|z_j(t)|^{\frac{1}{p_j\cdots p_{i-1}}} \\ & + b_{i4}\sum_{j=1}^{i-1}|z_j(t-d(t))|^{\frac{1}{p_j\cdots p_{i-1}}} \\ \le\ & \phi_i\Big(|z_i(t)| + |z_i(t-d(t))|\Big) \\ & + \phi_i \sum_{j=1}^{i-1}\Big(|z_j(t)|^{\frac{1}{p_j\cdots p_{i-1}}} + |z_j(t-d(t))|^{\frac{1}{p_j\cdots p_{i-1}}}\Big) \end{aligned} \tag{10}$$

where $\phi_i = \max\{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$, $b_{i1} = a_i + (r_i(\lambda_0 + c_{03})/p_0)$, $b_{i2} = a_i exp\Big\{r_i d(\lambda_0 c_{01} - c_{03})/p_0\Big\}$, $b_{i3} = a_i|\lambda_0 x_0(t)|^{(r_1+\cdots+r_{i-1}-r_i)/p_0}$ and $b_{i4} = a_i|\lambda_0 x_0(t)|^{(r_1+\cdots+r_{i-1}-r_i)/p_0} \times \max\Big\{1, exp\{r_i d(r_1 + \cdots + r_{i-1} - r_i)(\lambda_0 c_{01} - c_{03})/p_0\}\Big\}$ are $\mathcal{F}$ functions. Thus, the inequality (9) follows.

C. Design u_1 for x-subsystem

In this subsection, we proceed to design the control input u_1 by using adding a power integrator technique. To simplify the deduction procedure, we sometimes denote $\chi(t)$ by χ, for any variable $\chi(t)$.

Step 1. Introduce the Lyapunov-Krasoviskii functional $V_1 = \frac{n}{2}x_0^2 + \frac{1}{2}z_1^2 + \frac{n}{1-\eta}\int_{t-d(t)}^{t} z_1^2(s)ds$. With the help of (7) and (9), it can be verified that

$$\begin{aligned} \dot{V}_1 \le\ & -nx_0^2 + d_1 z_1 z_2^{p_1} \\ & + \phi_1(x_0)|z_1|\Big(|z_1| + |z_1(t-d(t))|\Big) \\ & + \frac{n}{1-\eta}z_1^2 - \frac{n(1-\dot{d}(t))}{1-\eta}z_1^2(t-d(t)) \\ \le\ & -nx_0^2 + d_1 z_1 z_2^{p_1} \\ & + z_1^2\Big(\frac{n}{1-\eta} + \phi_1(x_0) + \frac{1}{4}\phi_1^2(x_0)\Big) \\ & -(n-1)z_1^2(t-d(t)) \end{aligned} \tag{11}$$

Obviously, the first virtual controller

$$\begin{aligned} z_2^{*p_1} =\ & -\frac{1}{c_{01}}\Big(n + \frac{n}{1-\eta} + \phi_1(x_0) + \frac{1}{4}\phi_1^2(x_0)\Big)z_1 \\ :=\ & -\alpha_1(x_0)z_1 \end{aligned} \tag{12}$$

leads to

$$\begin{aligned} \dot{V}_1 \le\ & -n(x_0^2 + z_1^2) - (n-1)z_1^2(t-d(t)) \\ & + d_1 z_1(z_2^{p_1} - z_2^{*p_1}) \end{aligned} \tag{13}$$

Step i ($i = 2, \cdots, n$)**.** Suppose at step $i-1$, there is a positive-definite and proper Lyapunov functional V_{i-1}, and a set of virtual controllers $z_1^*, \cdots, z_i^*$ defined by defined by (14) shown at the top of the next page, with $\alpha_1(x_0) > 0$, $\cdots$, $\alpha_{i-1}(x_0) > 0$, being $\mathcal{F}$ function, such that

$$\begin{aligned} \dot{V}_{i-1} \le\ & -(n-i+2)\sum_{k=0}^{i-1}\xi_k^2 \\ & -(n-i+1)\sum_{k=1}^{i-1}\xi_k^2(t-d(t)) \\ & + d_{i-1}\xi_{i-1}^{2-\frac{1}{p_1\cdots p_{i-2}}}(z_i^{p_{i-1}} - z_i^{*p_{i-1}}) \end{aligned} \tag{15}$$

here we let $\xi_0 = x_0$ for the simplicity of expression.

We intend to establish a similar property for $(z_1, \cdots, z_i)$-subsystem. Consider the following Lyapunov-Krasoviskii functional candidate

$$V_i = V_{i-1} + W_i + \frac{n-i+1}{1-\eta}\int_{t-d(t)}^{t}\xi_i^2(s)ds \tag{16}$$

where

$$\begin{aligned} & W_i(z_1, \cdots, z_i) \\ & = \int_{z_i^*}^{z_i}(s^{p_1\cdots p_{i-1}} - z_i^{*p_1\cdots p_{i-1}})^{(2-\frac{1}{p_1\cdots p_{i-1}})}ds \end{aligned} \tag{17}$$

For W_i, some useful properties are given by the following proposition whose proof can be found in [25] and hence omitted here.

Proposition 1. $W_i(z_1, \cdots, z_i)$ is C^1. Moreover

$$\begin{aligned} \frac{\partial W_i}{\partial z_i} &= \xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}} \\ \frac{\partial W_i}{\partial y} &= -\Big(2 - \frac{1}{p_1\cdots p_{i-1}}\Big)\frac{\partial z_i^{*p_1\cdots p_{i-1}}}{\partial y} \\ & \times \int_{z_i^*}^{z_i}(s^{p_1\cdots p_{i-1}} - z_i^{*p_1\cdots p_{i-1}})^{(1-\frac{1}{p_1\cdots p_{i-1}})}ds \end{aligned} \tag{18}$$

where y is an argument of W_i except z_i.

Proposition 2. There is a positive constant m such that

$$\Big|\frac{\partial W_i}{\partial y}\Big| \le m|\xi_i|\Big|\frac{\partial z_i^{*p_1\cdots p_{i-1}}}{\partial y}\Big| \tag{19}$$

Using Proposition 1, it is deduced from (16) that

$$\begin{array}{lllll} z_1^* & = & 0 & \xi_1 & = & z_1 - z_1^* \\ z_2^{*p_1} & = & -\alpha_1(x_0)\xi_1 & \xi_2 & = & z_2^{p_1} - z_2^{*^{p_1}} \\ & \vdots & & & \vdots & \\ z_i^{*p_1\cdots p_{i-1}} & = & -\alpha_{i-1}(x_0)\xi_{i-1} & \xi_i & = & z_i^{p_1\cdots p_{i-1}} - z_i^{*p_1\cdots p_{i-1}} \end{array} \tag{14}$$

$$\begin{aligned} \dot{V}_i \le& -(n-i+2)\sum_{k=0}^{i-1}\xi_k^2 \\ &-(n-i+1)\sum_{k=1}^{i-1}\xi_k^2(t-d(t)) \\ &+d_{i-1}\xi_{i-1}^{2-\frac{1}{p_1\cdots p_{i-2}}}(z_i^{p_{i-1}} - z_i^{*p_{i-1}}) \\ &+d_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}(z_{i+1}^{p_i} - z_{i+1}^{*p_i}) \\ &+d_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}z_{i+1}^{*p_i} + \xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}g_i \\ &+\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial z_j}(d_j z_{j+1}^{p_j} + g_j) \\ &+\frac{\partial W_i}{\partial x_0}(d_0u_0^{p_0} + f_0) + \frac{n-i+1}{1-\eta}\xi_i^2 \\ &-\frac{(n-i+1)(1-\dot{d}(t))}{1-\eta}\xi_i^2(t-d(t)) \end{aligned} \tag{20}$$

Similarly, to give the explicit form of z_{i+1}^*, we should estimate each term on the righthand side of (20). First, from (14), Assumption 1 and Lemma 2, it follows that

$$\begin{aligned} &\Big|d_{i-1}\xi_{i-1}^{2-\frac{1}{p_1\cdots p_{i-2}}}(z_i^{p_{i-1}} - z_i^{*^{p_{i-1}}})\Big| \\ &\le c_{i-1,2}\Big|\xi_{i-1}\Big|^{2-\frac{1}{p_1\cdots p_{i-2}}}2^{1-\frac{1}{p_1\cdots p_{i-2}}} \\ &\quad\times\Big|z_i^{p_1\cdots p_{i-1}} - z_i^{*p_1\cdots p_{i-1}}\Big|^{\frac{1}{p_1\cdots p_{i-2}}} \\ &= 2^{1-\frac{1}{p_1\cdots p_{i-2}}}c_{i-1,2}\Big|\xi_{i-1}\Big|^{2-\frac{1}{p_1\cdots p_{i-2}}}\Big|\xi_i\Big|^{\frac{1}{p_1\cdots p_{i-2}}} \\ &\le \frac{1}{4}\xi_{i-1}^2 + l_{i1}\xi_i^2 \end{aligned} \tag{21}$$

where l_{i1} is a positive constant.

From (9) and (14), there are $\mathcal{F}$ functions $\rho_k(x_0)$, $k = 1,\cdots,i$ such that

$$\begin{aligned} |g_k| \le& \phi_k\Big(|z_k(t)| + |z_k(t-d(t))|\Big) \\ &+\phi_k\sum_{j=1}^{k-1}\Big(|z_j(t)|^{\frac{1}{p_j\cdots p_{k-1}}} \\ &\qquad+|z_j(t-d(t))|^{\frac{1}{p_j\cdots p_{k-1}}}\Big) \\ \le& \phi_k\Big(|\xi_k(t)|^{\frac{1}{p_1\cdots p_{k-1}}} + |\xi_k(t-d(t))|^{\frac{1}{p_1\cdots p_{k-1}}}\Big) \\ &+\phi_k\sum_{j=1}^{k-1}\Big(1 + a_j^{\frac{1}{p_1\cdots p_{k-1}}}\Big)|\xi_j(t)|^{\frac{1}{p_1\cdots p_{k-1}}} \\ &+\phi_k\sum_{j=1}^{k-1}\Big(1 + a_j^{\frac{1}{p_1\cdots p_{k-1}}}\Big)|\xi_j(t-d(t))|^{\frac{1}{p_1\cdots p_{k-1}}} \\ \le& \rho_k\sum_{j=1}^{k}\Big(|\xi_j(t)|^{\frac{1}{p_1\cdots p_{k-1}}} \\ &\qquad+|\xi_j(t-d(t))|^{\frac{1}{p_1\cdots p_{k-1}}}\Big) \end{aligned} \tag{22}$$

and hence by Lemma 2

$$\begin{aligned} &\Big|\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}g_i\Big| \\ &\le \Big|\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}\Big|\rho_j\sum_{j=1}^{i}\Big(|\xi_j(t)|^{\frac{1}{p_1\cdots p_{i-1}}} \\ &\quad+|\xi_j(t-d(t))|^{\frac{1}{p_1\cdots p_{i-1}}}\Big) \\ &\le \frac{1}{4}\sum_{j=1}^{i-1}\xi_j^2 + \frac{1}{2}\sum_{j=1}^{i}\xi_j^2(t-d(t)) + l_{i2}\xi_i^2 \end{aligned} \tag{23}$$

where $l_{i2}(x_0)$ is a $\mathcal{F}$ function.

For the seventh term, noting that

$$\begin{aligned} z_i^{*p_1\cdots p_{i-1}} &= -\alpha_{i-1}\xi_{i-1} \\ &= -\sum_{k=1}^{i-1}\Big(\prod_{h=k}^{i-1}\alpha_h\Big)z_k^{p_1\cdots p_{k-1}} \end{aligned} \tag{24}$$

By this, (14), Proposition 2 and Lemma 2, we obtain

$$\begin{aligned} &\Big|\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial z_j}(d_jz_{j+1}^{p_j} + g_j)\Big| \\ &\le \sum_{j=1}^{i-1}m|\xi_i|\Big|\frac{\partial z_i^{*p_1\cdots p_{i-1}}}{\partial z_j}\Big|(c_{j2}|z_{j+1}^{p_j}| + |g_j|) \\ &\le \sum_{j=1}^{i-1}mp_1\cdots p_{j-1}|\xi_i|\Big(\prod_{h=j}^{i-1}\alpha_j\Big)|z_j^{p_1\cdots p_{j-1}-1}| \\ &\quad\times(c_{j2}|z_{j+1}^{p_j}| + |g_j|) \\ &\le \sum_{j=1}^{i-1}mp_1\cdots p_{j-1}|\xi_i|\Big(\prod_{h=j}^{i-1}\alpha_j\Big) \\ &\quad\times|\xi_j + \alpha_{j-1}\xi_{j-1}|^{1-\frac{1}{p_1\cdots p_{j-1}}} \\ &\quad\times\Big[c_{j2}\Big|\xi_{j+1} + \alpha_j\xi_j\Big|^{\frac{1}{p_1\cdots p_{j-1}}} \\ &\quad+\rho_j\sum_{k=1}^{j}\Big(|\xi_k(t)|^{\frac{1}{p_1\cdots p_{j-1}}} \\ &\quad+|\xi_k(t-d(t))|^{\frac{1}{p_1\cdots p_{j-1}}}\Big)\Big] \\ &\le \frac{1}{4}\sum_{j=1}^{i-1}\xi_j^2 + \frac{1}{2}\sum_{j=1}^{i}\xi_j^2(t-d(t)) + l_{i3}\xi_i^2 \end{aligned} \tag{25}$$

where $l_{i3}(x_0)$ is a $\mathcal{F}$ function.

According to Proposition 2 and Lemma 2, we easily get

$$\begin{aligned} &\Big|\frac{\partial W_i}{\partial x_0}(d_0u_0^{p_0} + f_0)\Big| \\ &\le m|\xi_i|\Big|\frac{\partial z_i^{*p_1\cdots p_{i-1}}}{\partial x_0}\Big|(c_{02}\lambda_0 + c_{03})|x_0| \\ &\le x_0^2 + l_{i4}\xi_i^2 \end{aligned} \tag{26}$$

where $l_{i4}(x_0)$ is a $\mathcal{F}$ function.

Substituting (21), (23), (25)and (26) into (20) yields

$$\begin{aligned}\dot{V}_i \le & -(n-i+1)\sum_{k=0}^{i-1}\xi_k^2 - (n-i)\sum_{k=1}^{i}\xi_k^2(t-d(t)) \\ & + d_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}} z_{i+1}^{*p_i} + \Big(\sum_{j=1}^{4} l_{ij} + \frac{n-i+1}{1-\eta}\Big)\xi_i^2 \\ & + d_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}(z_{i+1}^{p_i} - z_{i+1}^{*p_i})\end{aligned} \tag{27}$$

Now, it easy to see that the virtual controller

$$\begin{aligned} z_{i+1}^{*p_1\cdots p_i} &= -\Big(n-i+1+\sum_{j=1}^{4} l_{ij} + \frac{n-i+1}{1-\eta}\Big)\xi_i \\ &:= -\alpha_i(x_0)\xi_i \end{aligned} \tag{28}$$

renders

$$\begin{aligned}\dot{V}_i \le & -(n-i+1)\sum_{k=0}^{i}\xi_k^2 - (n-i)\sum_{k=1}^{i}\xi_k^2(t-d(t)) \\ & + d_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}(z_{i+1}^{p_i} - z_{i+1}^{*p_i})\end{aligned} \tag{29}$$

As $i = n$, the last step, we can construct explicitly a change of coordinates $(\xi_1,\cdots,\xi_n)$, a positive-definite and proper Lyapunov-Krasoviskii functional $V_n(\xi_1,\cdots,\xi_n)$ and a state feedback controller z_{n+1}^* of form (28) such that

$$\dot{V}_n \le -\sum_{k=0}^{n}\xi_k^2 + d_n\xi_n^{2-\frac{1}{p_1\cdots p_{n-1}}}(u_1^{p_n} - z_{n+1}^{*p_n}) \tag{30}$$

Therefore, by choosing the actual control u_1 as

$$u_1 = z_{n+1}^* = -\Big(\alpha_n(x_0)\xi_n\Big)^{\frac{1}{p_1\cdots p_n}} \tag{31}$$

we get

$$\dot{V}_n \le -\sum_{k=0}^{n}\xi_k^2 \tag{32}$$

Thus far the controller design procedure for $x_0(t_0) \neq 0$ has been completed.

IV. Switching Controller and Main Result

In the preceding section, we have given controller design for $x_0(t_0) \neq 0$. Now, we discuss how to select the control laws u_0 and u_1 when the initial $x_0(t_0) = 0$. Without loss of generality, we can assume that $t_0 = 0$. In the absence of the disturbances, most of the commonly used control strategies use constant control $u_0 = u_0^* \neq 0$ in time interval $[0, t_s)$. In this paper, we also use this method when $x_0(0) = 0$, with u_0 chosen as follows:

$$u_0 = u_0^*, \;\; u_0^* > 0. \tag{33}$$

Since $f_0(t, x_0(t))$ in this paper satisfies the linear growth condition, the x_0-state does not escape and $x_0(t_s) \neq 0$, for any given finite time $t_s > 0$. Thus, input-state-scaling for the control design can be carried out.

During the time period $[0, t_s)$, using u_0 defined in (33), new control law $u_1 = u_1^*(x_0, x)$ can be obtained by the control procedure described above to the original $x-$subsystem in (1). Then we can conclude that the $x-$state of (1) cannot blow up during the time period $[0, t_s)$. Since $x(t_s) \neq 0$ at t_s, we can switch the control inputs u_0 and u_1 to (3) and (31), respectively.

We are now ready to state the main theorem of this paper.

Theorem 1. Under Assumptions 1-2, if the proposed control design procedure together with the above switching control strategy is applied to system (1), then, for any initial conditions in the state space $(x_0, x) \in R^{n+1}$, the closed-loop system is globally asymptotically regulated at origin .

Proof. According to the above analysis, it suffices to prove the statement in the case where $x_0(0) \neq 0$.

Since we have already proven that x_0 can be globally exponentially regulated to zero as $t \to \infty$ in Section 3.1, we just need to show that $\lim_{t\to\infty} x(t) = 0$. In this case, choose the Lyapunov functional

$$\begin{aligned} V = V_n = & \frac{n}{2}x_0^2 + \frac{1}{2}\sum_{k=1}^{n}\xi_i^2 \\ & + \sum_{k=1}^{n}\frac{n-i+1}{1-\eta}\int_{t-d(t)}^{t}\xi_i^2(s)ds \end{aligned} \tag{34}$$

from (32), we obtain

$$\dot{V} \le -(x_0^2 + \xi_1^2 + \cdots + \xi_n^2) \tag{35}$$

Then by Lyapunov-Krasovskii stability theorem [20], we have $\lim_{t\to\infty}\xi(t) = 0$. This together with the definitions of z_i^*'s and the input-state-scaling transformation(6) directly concludes that $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 1.

Remark 4. From the above design procedure, we can see that the upper bound of the change rate of time delays have important impact on the control effort. To keep the control effort within the certain range, the upper bound of the change rate of time delays cannot be arbitrarily close to 1, which should be considered in practical engineering design.

Remark 5. It should be mentioned that the control law u_1 may exhibit extremely large value when $x_0(t_0) \neq 0$ is sufficiently small. This is unacceptable from a practical point of view. It is therefore recommended to apply (33) in order to enlarge the initial value of x_0 before we appeal to the converging controllers (3) and (31).

V. Simulation Example

To verify the proposed controller, we consider the following low-dimensional system

$$\begin{cases} \dot{x}_0(t) = (1.5 + 0.5\cos t)u_0(t) \\ \dot{x}_1(t) = x_2^3(t)u_0(t) + \sin x_0(t)x_1(t-d(t)) \\ \dot{x}_2(t) = u_1^3(t) \end{cases} \tag{36}$$

where $d(t) = \frac{1}{2}(1+\sin t)$. It is very easy to verify that Assumptions 1-2 holds. Hence the controller proposed in this paper is applicable.

If $x(0) = 0$, controls u_0 and u_1 are set as in Section 4 in interval $[0, t_s)$, such that $x(t_s) \neq 0$, then we can adopt the controls developed below. Therefore, without loss of generality, we assume that $x(0) \neq 0$. Noting that $\dot{d}(t) = \frac{1}{2}\cos t \le \frac{1}{2} < 1$, we define the control law $u_0(t) = -\lambda_0 x_0(t)$ and introduce the state scaling transformation

$$z_1(t) = \begin{cases} \dfrac{x_1(t)}{u_0(t)}, & t \ge 0 \\ x_1(t), & -1 \le t < 0 \end{cases}, \; z_2(t) = x_2(t) \tag{37}$$

In new $z-$coordinates, the $(x_1, x_2)-$subsystem of (36) is rewritten as

$$\begin{cases} \dot{z}_i(t) = z_{i+1}^3(t) \\ \qquad +g_1(t, x_0(t), x(t), x(t-d(t))) \\ \dot{z}_2(t) = u_1^3(t) \\ \qquad +g_2(t, x_0(t), x(t), x(t-d(t))) \end{cases} \tag{38}$$

where

$$\begin{aligned} & g_i(t, x_0(t), x(t), x(t-d(t))) \\ & \quad = \frac{f_i(t, x_0(t), x(t), x(t-d(t)))}{u_0^{r_i}(t)} \\ & \quad +r_i z_i(t) \frac{u_0^{p_0}(t) + f_0(t, x_0(t))}{p_0 x_0(t)} \end{aligned} \tag{39}$$

Similar to (10), it is very easy to verify that Lemma 4 is satisfied with $\phi_1 = 1 + e^{\lambda_0}$ and $\phi_2 = 1$

Obviously, the $(z_1, z_2)-$ subsystem of (48) with nonlinear parameter ε. Define $\gamma_1 = -1.5g_0$, $\Theta = 1+\varepsilon^2$, this subsystem satisfies Lemma 4, i.e. $|g_0(1-0.5\varepsilon^2)z_1| \leq |z_1|\gamma_1\Theta$. Now consider $V_1 = x_0^2 + \frac{1}{2}z_1^2 + 4\int_{t-d(t)}^t z_1^2(s)ds$. A simple calculation yields

$$\begin{aligned} \dot{V}_1 \leq & -2x_0^2 + z_1 z_2^3 \\ & +z_1^2\Big(4 + \phi_1 + \frac{1}{4}\phi_1^2\Big) - z_1^2(t-d(t)) \end{aligned} \tag{40}$$

Hence, the virtual controller

$$\begin{aligned} z_2^{*3} &= -\Big(6 + \phi_1 + \frac{1}{4}\phi_1^2\Big)z_1 \\ &:= -\alpha_1 z_1 \end{aligned} \tag{41}$$

renders

$$\dot{V}_1 \leq -(x_0^2 + z_1^2) - z_1^2(t-d(t)) + z_1(z_2^3 - z_2^{*3}) \tag{42}$$

Next, define $\xi_2 = z_2^3 - z_2^{*3}$ and construct Lypunov Lyapunov-Krasoviskii functional

$$V_2 = V_2 + W_2 + 2\int_{t-d(t)}^t \xi_i^2(s)ds \tag{43}$$

where

$$W_2 = \int_{z_2^*}^{z_2} (s^3 - z_2^{*3})^{\frac{5}{3}} ds \tag{44}$$

Clearly

$$\begin{aligned} \dot{V}_2 \leq & -(x_0^2 + z_1^2) - z_1^2(t-d(t)) \\ & +z_1(z_2^3 - z_2^{*3}) + \xi_2^{\frac{5}{3}} u_1^3 + \xi_2^{\frac{5}{3}} g_2 \\ & +\frac{\partial W_2}{\partial z_1}(z_2^3 + g_1) + \frac{\partial W_2}{\partial x_0} d_0 u_0 \\ & +2\xi_1^2 - \xi_1^2(t-d(t)) \end{aligned} \tag{45}$$

By Lemma 2, we have

$$z_1(z_2^3 - z_2^{*3}) \leq \frac{1}{4}z_1^2 + l_{21}\xi_2^2 \tag{46}$$

$$\xi_2^{\frac{5}{3}} g_2 \leq \frac{1}{4}z_1^2 + l_{22}\xi_2^2 \tag{47}$$

$$\frac{\partial W_2}{\partial z_1}(z_2^3 + g_1) \leq \frac{1}{4}z_1^2 + \xi_1^2(t-d(t)) + l_{23}\xi_2^2 \tag{48}$$

$$\frac{\partial W_2}{\partial x_0} d_0 u_0 \leq_1^2 + l_{24}\xi_2^2 \tag{49}$$

where l_{2j}, $j = 1, 2, 3, 4$ are known $\mathcal{F}$ functions

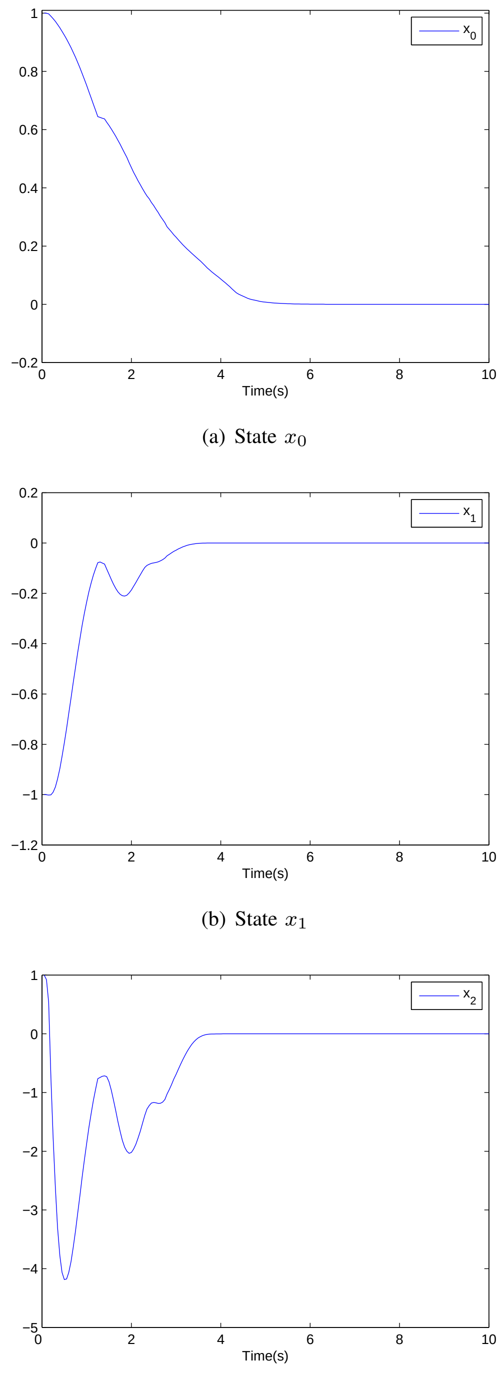
It is easy to verify that the controller

$$u_1(t) = -\Big(3 + \sum_{j=1}^4 l_{ij}\Big)^{\frac{1}{9}} \tag{50}$$

renders

$$\dot{V}_2 \leq -x_0^2 - z_1^2 - \xi_2^2 \tag{51}$$

thus achieving global stability with asymptotic state regulation.



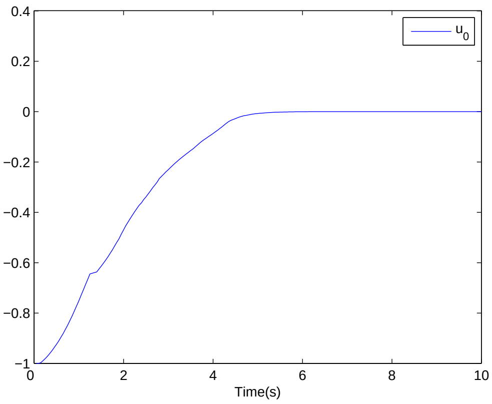
(a) State x_0

(b) State x_1

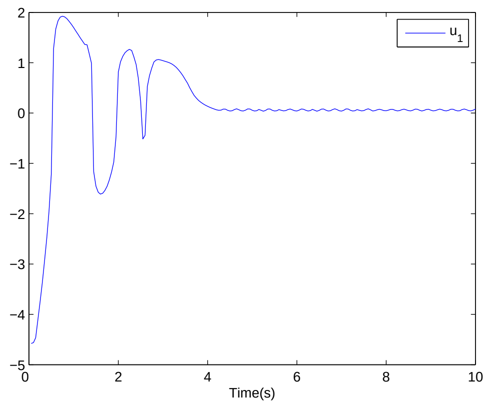
(c) State x_2

Fig. 1. The responses of system states.

In the simulation, by choosing design parameter as $\lambda_0 = 1$, the responses of the closed-loop system for initial conditions $(x_0(0), x_1(0), x_2(0)) = (1, -1, 1)$ are shown in Figs. 1 and



(a) Input u_0



(b) Input u_1

Fig. 2. The responses of control inputs.

2. From the figures, we can see that under the constructed controller, the solution process of the closed-loop system asymptotically converges to zero.

VI. Conclusion

In this paper, a state-feedback stabilization controller independent of time-delays is presented for a class of high-order nonholonomic systems with time-varying delays. It should be mentioned that the stabilization approaches in literature may fail be applied for the existence of time delays. In order to overcome the difficulty, a novel Lyapunov-Krasovskii functional is introduced to deal with time delays. The controller design is developed by using input-state-scaling and adding a power integrator techniques. Based on switching control strategy, global asymptotic regulation of the closed-loop system is achieved. It should be noted that the proposed controller can only work well when the whole state vector is measurable. Therefore, a natural and more interesting problem is how to design output feedback stabilization controller for the systems studied in the paper if only partial state vector are measurable, which are now under our further investigation.

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