On A New Approximation Method With Four Times Continuously Differentiable Quintic Integro-Differential Spline

Feng-Gong Lang, Xiao-Ping Xu

Abstract—**In this paper, we study a new method for approximating a function and its derivatives from the given integral values over the grid intervals and two given first order derivatives at boundary grid nodes by using four times continuously differentiable quintic integro-differential spline. The method mainly includes two steps: constructing three extra values and then solving a linear system. It is easy to implement. We analyze the approximation properties of the spline. We prove that the spline and its first four derivatives are able to approximate the original function and its first four derivatives respectively with high accuracy. To examine the theoretical results, we perform some numerical tests and also compare the method with some other methods. Numerical results show that the new method is very effective and practical.**

Index Terms—**Integro-differential spline, Quintic spline, Integral value, Boundary derivative, Numerical approximation.**

I. Introduction

Assumed that U is an unknown function defined over $[a, b]$, and

$$\Delta := \{a = x_0 < x_1 < \cdots < x_n = b\}$$

is a uniform partition of $[a, b]$.

In many practical problems, we are often required to approximate U from its integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ over grid intervals and its some function values $U(x_j)$ $(j = 0, 1, \cdots, n)$ or some derivative values $U'(x_j)$ $(j = 0, 1, \cdots, n)$ at grid nodes.

Some old methods have been studied for the problem, see [1], [2], [3], [4], [5], [6].

In [1], by using the given integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and the given derivatives $U'(x_j)$ $(j = 0, 1, \cdots, n)$, the original function U was firstly approximated by a discontinuous piecewise quadratic polynomial and a discontinuous piecewise trigonometric function. The two methods both require $2n + 1$ values and can only provide discontinuous approximation functions.

And then, also in [1], by only using I_j $(j = 0, 1, \cdots, n-1)$, $U'(x_0)$ and $U'(x_n)$, U was approximated by a continuously differentiable piecewise quadratic polynomial (quadratic spline) and a continuously differentiable piecewise trigonometric function. The two methods are better because they only require $n + 2$ values and can provide continuously differentiable approximation functions.

The original function U was approximated by two continuous integro-differential splines in [2] by using $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and $U(x_j)$ $(j = 0, 1, \cdots, n)$. The methods also require $2n + 1$ values and can only provide continuous approximation functions.

Similarly, two discontinuous approximations and two continuous approximations were constructed in [3] for the original function U by only using I_j $(j = 0, 1, \cdots, n-1)$. The methods only require n values but can not provide continuously differentiable approximation functions.

In [4], by using the given integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and the given function values $U(x_j)$ $(j = 0, 1, \cdots, n)$, U was firstly approximated by a continuous piecewise quartic polynomial and then approximated by some other continuous non-polynomial functions. Similarly to the methods in [2], these methods also require $2n+1$ values and also can not provide continuously differentiable approximation functions.

A continuously differentiable quartic integro-differential spline method and a twice continuously differentiable quartic integro-differential spline method were studied in [5]. However, the first method in [5] requires more data than the methods in [4] do. In fact, the first method in [5] requires the integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, the function values $U(x_j)$ $(j = 0, 1, \cdots, n)$ and the derivatives $U'(x_j)$ $(j = 0, 1, \cdots, n)$. In other words, it requires $3n + 2$ values.

In addition, the mean square approximation to U with the same splines in [5] were studied in [6].

Obviously, these old methods have the following drawbacks.

(a) Most of these methods need too many values of U. It makes the methods are not convenient to be applied.

(b) The smoothness orders of the obtained approximation functions are lower. It makes the methods can not provide approximations to the higher order derivatives.

(c) The approximation abilities of the obtained approximation functions are not higher. In fact, many of the obtained functions can not provide highly accurate approximation to the original function.

Very naturally, an important problem arises. Can we get more times continuously differentiable integro-differential spline with higher approximation order by only using fewer data of the original function U?

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In this paper, we will give a solution to the important problem. A new approximation method will be presented. The new method has the following remarkable advantages.

1) The method only needs $n+2$ values of U, that is the integral values $I_j = \int_{x_j}^{x_{j+1}} U(t)dt$ $(j = 0, 1, \cdots, n-1)$, and two first order boundary derivatives $U'(x_0)$ and $U'(x_n)$. It does not need the function values $U(x_j)$ $(j = 0, 1, \cdots, n)$, the derivatives $U'(x_j)$ $(j = 1, 2, \cdots, n-1)$ and some other values of U.
2) The method constructs a quintic integro-differential spline S that is four times continuously differentiable and is able to provide highly accurate approximations not only to U and U' but also to U'', U''' and $U^{(4)}$.
3) The method is easy to implement.

Actually, the method has very satisfactory approximation efficiency. It can be widely applied for the integro-differential interpolation problems. It is a very good progress in the integro-differential spline approximation methods.

The remainder of this paper is organized as follows. In Section II, we introduce an integro-differential interpolation problem. In Section III, we construct three new extra values for the interpolation problem. In Section IV, we study the existence and uniqueness of the interpolation problem, and the construction method for the quintic integro-differential spline is also presented at the same time. Then, in Section V, we give the concise computational procedure for the interpolation problem. The approximation properties of the obtained quintic integro-differential spline are investigated in Section VI and some numerical tests are performed in Section VII. Finally, we conclude this paper in Section VIII.

II. The integro-differential interpolation problem

In this paper, we study the following integro-differential interpolation problem.

Assume that the integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, and two boundary derivatives $U'(x_0)$ and $U'(x_n)$ are given for an unknown function $U \in C^6[a,b]$, we construct a quintic integro-differential spline S that is four times continuously differentiable to satisfy

$$\begin{aligned} \int_{x_j}^{x_{j+1}} S(x)dx &= I_j = \int_{x_j}^{x_{j+1}} U(x)dx, \quad (1) \\ & \quad (j = 0, 1, \ldots, n-1) \\ S'(x_0) &= U'(x_0), \quad (2) \\ S'(x_n) &= U'(x_n). \quad (3) \end{aligned}$$

The required data are the same as that of the continuously differentiable quadratic integro-differential spline constructed in [1].

III. Constructions of extra data

In [1], $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, $U'(x_0)$ and $U'(x_n)$ are sufficient to get a unique quadratic integro-differential spline. However, these data are not sufficient to get a unique four times continuously differentiable quintic integro-differential spline because the dimension of $\mathbf{S}_5^4$ is $n+5$, where $\mathbf{S}_5^4$ is the quintic spline space of smoothness order four over $[a,b]$ with respect to the partition Δ.

To get a unique four times continuously differentiable quintic integro-differential spline S, we will construct three new extra values in this section by using the method in [7].

A. An approximation $\widehat{U_0}$ to $U(x_0)$

Expand U at $x = x_0$ and compute the integral on $[x_0, x_m]$ $(m = 1, 2, \cdots, 5)$, we have

$$\begin{aligned} & \sum_{l=0}^{m-1} I_l = \int_{x_0}^{x_m} U(x)dx \\ = & \ U(x_0)(mh) + \frac{U'(x_0)}{2!}(mh)^2 + \frac{U''(x_0)}{3!}(mh)^3 \\ & + \frac{U'''(x_0)}{4!}(mh)^4 + \frac{U^{(4)}(x_0)}{5!}(mh)^5 \\ & + \frac{U^{(5)}(x_0)}{6!}(mh)^6 + \frac{U^{(6)}(\xi_m)}{7!}(mh)^7, \quad (4) \end{aligned}$$

where $\xi_m \in [x_0, x_m]$.

We properly choose a set of parameters λ_l $(l = 1, 2, \cdots, 5)$ such that

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2^3 & 3^3 & 4^3 & 5^3 \\ 1 & 2^4 & 3^4 & 4^4 & 5^4 \\ 1 & 2^5 & 3^5 & 4^5 & 5^5 \\ 1 & 2^6 & 3^6 & 4^6 & 5^6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we get

$$\begin{aligned} \lambda_1 &= \frac{300}{137}, \quad \lambda_2 = -\frac{150}{137}, \quad \lambda_3 = \frac{200}{411}, \\ \lambda_4 &= -\frac{75}{548}, \quad \lambda_5 = \frac{12}{685}. \quad (5) \end{aligned}$$

By using the equations in (4) and the numbers in (5), we get

$$\begin{aligned} & \text{The left-hand side} \\ = & \sum_{m=1}^{5}(\lambda_m \sum_{l=0}^{m-1} I_l) \\ = & \frac{300}{137}I_0 + (-\frac{150}{137})(I_0 + I_1) + \frac{200}{411}(I_0 + I_1 + I_2) \\ & + (-\frac{75}{548})(I_0 + I_1 + I_2 + I_3) \\ & + \frac{12}{685}(I_0 + I_1 + I_2 + I_3 + I_4) \\ = & \frac{1}{8220}(12019I_0 - 5981I_1 + 3019I_2 - 981I_3 + 144I_4), \end{aligned}$$

and

$$\begin{aligned} & \text{The right-hand side} \\ = & \sum_{k=1}^{6} \frac{1}{k!}(\sum_{m=1}^{5} \lambda_m m^k) U^{(k-1)}(x_0) h^k \\ & + \frac{1}{7!}(\sum_{m=1}^{5} \lambda_m m^7) U^{(6)}(\xi_m) h^7 \\ = & \ U(x_0)h + \frac{30}{137}U'(x_0)h^2 \\ & + \frac{h^7}{7! \cdot 137}(333600U^{(6)}(\xi_6) - 326400U^{(6)}(\xi_7)), \end{aligned}$$

where $\xi_6, \xi_7 \in [x_0, x_5]$.

So we have

$$\begin{aligned} &\frac{1}{8220}(12019I_0 - 5981I_1 + 3019I_2 - 981I_3 + 144I_4) \\ = \quad & U(x_0)h + \frac{30}{137}U'(x_0)h^2 \\ & + \frac{h^7}{7! \cdot 137}(333600U^{(6)}(\xi_6) - 326400U^{(6)}(\xi_7)). \quad (6) \end{aligned}$$

By (6), an approximation $\widehat{U_0}$ to $U(x_0)$ can be obtained, see (7).

Theorem 3.1: Assume that U is a function of class $C^6[a,b]$ and $M = \max_{a \le x \le b} |U^{(6)}(x)|$. Let

$$\begin{aligned} \widehat{U_0} \quad = \quad & \frac{1}{8220h}(12019I_0 - 5981I_1 + 3019I_2 \\ & -981I_3 + 144I_4) - \frac{30}{137}U'(x_0)h, \quad (7) \end{aligned}$$

then we have

$$|e_0| = |\widehat{U_0} - U(x_0)| \le k_0 M h^6, \quad (8)$$

where $k_0 = \frac{333600+326400}{7! \cdot 137} \approx 0.956$.

Proof: Obviously, (7) and (8) can be obtained from (6). ■

B. An approximation $\widehat{U_1'}$ to $U'(x_1)$ and an approximation $\widehat{U_{n-1}'}$ to $U'(x_{n-1})$

Similarly, expand U at $x = x_1$, we have

$$\begin{aligned} I_0 \quad = \quad & U(x_1)h - \frac{1}{2}U'(x_1)h^2 + \frac{1}{3!}U''(x_1)h^3 \\ & -\frac{1}{4!}U'''(x_1)h^4 + \frac{1}{5!}U^{(4)}(x_1)h^5 \\ & -\frac{1}{6!}U^{(5)}(x_1)h^6 + \frac{1}{7!}U^{(6)}(\zeta_1)h^7, \quad (9) \end{aligned}$$

$$\begin{aligned} U'(x_0)h^2 \quad = \quad & U'(x_1)h^2 - U''(x_1)h^3 + \frac{1}{2}U'''(x_1)h^4 \\ & -\frac{1}{3!}U^{(4)}(x_1)h^5 + \frac{1}{4!}U^{(5)}(x_1)h^6 \\ & -\frac{1}{5!}U^{(6)}(\zeta_2)h^7, \quad (10) \end{aligned}$$

and

$$\begin{aligned} & \sum_{l=1}^{m} I_l = \int_{x_1}^{x_{m+1}} U(x)dx \\ = \quad & U(x_1)(mh) + \frac{U'(x_1)}{2!}(mh)^2 + \frac{U''(x_1)}{3!}(mh)^3 \\ & + \frac{U'''(x_1)}{4!}(mh)^4 + \frac{U^{(4)}(x_1)}{5!}(mh)^5 \\ & + \frac{U^{(5)}(x_1)}{6!}(mh)^6 + \frac{U^{(6)}(\zeta_{m+2})}{7!}(mh)^7, \quad (11) \end{aligned}$$

where $\zeta_1, \zeta_2 \in [x_0, x_1]$, $\zeta_{m+2} \in [x_1, x_{m+1}]$, $m = 1, 2, 3, 4$.

We properly choose a set of parameters μ_l $(l = 1, 2, \cdots, 6)$ such that

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 3 & 4 \\ -1 & 2 & 1 & 2^2 & 3^2 & 4^2 \\ 1 & -6 & 1 & 2^3 & 3^3 & 4^3 \\ -1 & 12 & 1 & 2^4 & 3^4 & 4^4 \\ 1 & -20 & 1 & 2^5 & 3^5 & 4^5 \\ -1 & 30 & 1 & 2^6 & 3^6 & 4^6 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we get

$$\begin{aligned} & \mu_1 = -\frac{1955}{3288}, \ \mu_2 = -\frac{13}{274}, \ \mu_3 = \frac{1117}{1644}, \\ & \mu_4 = -\frac{55}{1644}, \ \mu_5 = -\frac{29}{3288}, \ \mu_6 = \frac{7}{3288}. \quad (12) \end{aligned}$$

Similarly to (6), by using the equations in (9), (10), (11) and the numbers in (12), we get

$$\begin{aligned} & \mu_1 I_0 + \mu_2 U'(x_0)h^2 + \sum_{m=1}^{4}(\mu_{m+2}\sum_{l=1}^{m} I_l) \\ = \quad & -\frac{1955}{3288}I_0 + (-\frac{13}{274})U'(x_0)h^2 \\ & + \frac{1117}{1644}I_1 + (-\frac{55}{1644})(I_1 + I_2) \\ & + (-\frac{29}{3288})(I_1 + I_2 + I_3) \\ & + \frac{7}{3288}(I_1 + I_2 + I_3 + I_4) \\ = \quad & \frac{1}{3288}(-1955I_0 + 2102I_1 - 132I_2 - 22I_3 + 7I_4) \\ & -\frac{13}{274}U'(x_0)h^2 \\ = \quad & \frac{1}{2}U'(x_1)h^2 + \frac{h^7}{7! \cdot 3288}(123474U^{(6)}(\zeta_7) \\ & -79458U^{(6)}(\zeta_8)), \quad (13) \end{aligned}$$

where $\zeta_7, \zeta_8 \in [x_0, x_5]$. By (13), an approximation $\widehat{U_1'}$ to $U'(x_1)$ can be obtained, see (14).

Moreover, an approximation $\widehat{U_{n-1}'}$ to $U'(x_{n-1})$ also can be obtained by using similar manner.

Theorem 3.2: Assume that U is a function of class $C^6[a,b]$ and $M = \max_{a \le x \le b} |U^{(6)}(x)|$.

1) Let

$$\begin{aligned} \widehat{U_1'} \quad = \quad & \frac{1}{1644h^2}(-1955I_0 + 2102I_1 - 132I_2 \\ & -22I_3 + 7I_4) - \frac{13}{137}U'(x_0), \quad (14) \end{aligned}$$

then we have

$$|e_1'| = |\widehat{U_1'} - U(x_1)| \le k_1 M h^5, \quad (15)$$

where $k_1 = \frac{123474+79458}{7! \cdot 1644} \approx 0.025$.

2) Let

$$\begin{aligned} \widehat{U_{n-1}'} \quad = \quad & \frac{1}{1644h^2}(1955I_{n-1} - 2102I_{n-2} \\ & +132I_{n-3} + 22I_{n-4} - 7I_{n-5}) \\ & -\frac{13}{137}U'(x_n), \quad (16) \end{aligned}$$

then we have

$$|e_{n-1}'| = |\widehat{U_{n-1}'} - U(x_{n-1})| \le k_1 M h^5, \quad (17)$$

where $k_1 \approx 0.025$.

Proof: (14) and (15) can be obtained from (13). The others can be obtained similarly. ■

IV. THE EXISTENCE AND UNIQUENESS

In the following sections, $\widehat{U_0}$, $\widehat{U_1'}$ and $\widehat{U_{n-1}'}$ will be used as extra conditions to determine an unique quintic integro-differential spline S that satisfies (1), (2), (3) and three extra conditions

$$\begin{aligned} S(x_0) &= \widehat{U_0}, &(18)\\ S'(x_1) &= \widehat{U_1'}, &(19)\\ S'(x_{n-1}) &= \widehat{U_{n-1}'}. &(20)\end{aligned}$$

Since S is a four times continuously differentiable piece-wise quintic polynomial, it can be expressed as

$$S(x) = \sum_{i=-2}^{n+2} c_i B_i(x),$$

where

$$B_i(x) = \frac{1}{120h^5}\begin{cases} (x-x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}] \\ (x-x_{i-3})^5 \\ -6(x-x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}] \\ (x-x_{i-3})^5 \\ -6(x-x_{i-2})^5 \\ +15(x-x_{i-1})^5, & x \in [x_{i-1}, x_i] \\ (x_{i+3}-x)^5 \\ -6(x_{i+2}-x)^5 \\ +15(x_{i+1}-x)^5, & x \in [x_i, x_{i+1}] \\ (x_{i+3}-x)^5 \\ -6(x_{i+2}-x)^5, & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3}-x)^5, & x \in [x_{i+2}, x_{i+3}] \\ 0, & \text{else} \end{cases} \quad (21)$$

is a typical quintic B-spline and $h = \frac{b-a}{n}$. (Refer to [8], [9], [10], [11], [12], [13], [14], [15], [16] for more details of quintic B-splines.)

Moreover, by using (21), we have

$$\begin{aligned} \int_{x_{i-3}}^{x_{i-2}} B_i(x)dx &= \int_{x_{i+2}}^{x_{i+3}} B_i(x)dx = \frac{1}{720}h, &(22)\\ \int_{x_{i-2}}^{x_{i-1}} B_i(x)dx &= \int_{x_{i+1}}^{x_{i+2}} B_i(x)dx = \frac{57}{720}h, &(23)\\ \int_{x_{i-1}}^{x_{i}} B_i(x)dx &= \int_{x_{i}}^{x_{i+1}} B_i(x)dx = \frac{302}{720}h, &(24)\\ \int_{x_{j}}^{x_{j+1}} B_i(x)dx &= 0, (j \geq i+3 \ or \ j \leq i-4). &(25)\end{aligned}$$

For $j = 0, 1, \cdots, n-1$, by using (1) and (22), (23), (24), (25), we have

$$\begin{aligned} &\int_{x_j}^{x_{j+1}} S(x)dx \\ =& \int_{x_j}^{x_{j+1}} \sum_{i=-2}^{n+2} c_i B_i(x)dx \\ =& \int_{x_j}^{x_{j+1}} \sum_{i=j-2}^{j+3} c_i B_i(x)dx \\ =& \sum_{i=j-2}^{j+3} c_i \int_{x_j}^{x_{j+1}} B_i(x)dx \\ =& \frac{h}{720}(c_{j-2} + 57c_{j-1} + 302c_j + 302c_{j+1} \\ & +57c_{j+2} + c_{j+3}) = I_j. \qquad (26)\end{aligned}$$

By using (2) and (21), we get

$$\begin{aligned} &\sum_{i=-2}^{n+2} c_i B_i'(x_0) \\ =& c_{-2}B_{-2}'(x_0) + c_{-1}B_{-1}'(x_0) \\ & +c_0 B_0'(x_0) + c_1 B_1'(x_0) + c_2 B_2'(x_0) \\ =& U'(x_0).\end{aligned}$$

It is

$$-c_{-2} - 10c_{-1} + 10c_1 + c_2 = 24hU'(x_0). \qquad (27)$$

Similarly, by using (3) and (21), we get

$$-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2} = 24hU'(x_n). \qquad (28)$$

Furthermore, by using (18) and (21), we get

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = 120\widehat{U_0}; \qquad (29)$$

by using (19) and (21), we get

$$-c_{-1} - 10c_0 + 10c_2 + c_3 = 24h\widehat{U_1'}; \qquad (30)$$

by using (20) and (21), we get

$$-c_{n-3} - 10c_{n-2} + 10c_n + c_{n+1} = 24h\widehat{U_{n-1}'}. \qquad (31)$$

Take (29), (27), (30), (26), (31) and (28) together, we get a linear system

$$AC = Y, \qquad (32)$$

where

$$A = \begin{pmatrix} 1 & 26 & 66 & 26 & 1 & 0 & & & & & \\ -1 & -10 & 0 & 10 & 1 & 0 & & & & & \\ 0 & -1 & -10 & 0 & 10 & 1 & & & & & \\ 1 & 57 & 302 & 302 & 57 & 1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & & & 1 & 57 & 302 & 302 & 57 & 1 \\ & & & & & -1 & -10 & 0 & 10 & 1 & 0 \\ & & & & & 0 & -1 & -10 & 0 & 10 & 1 \end{pmatrix}, \qquad (33)$$

and

$$C = (c_{-2}, c_{-1}, c_0, c_1, \cdots, c_{n+1}, c_{n+2})^T, \qquad (34)$$

$$\begin{aligned} Y =& (120\widehat{U_0}, 24hU'(x_0), 24h\widehat{U_1'}, \frac{720}{h}I_0, \\ & \cdots, \frac{720}{h}I_{n-1}, 24h\widehat{U_{n-1}'}, 24hU'(x_n))^T. \qquad (35)\end{aligned}$$

Theorem 4.1: ([17]) If A is a strictly diagonally dominant square matrix, then A is nonsingular and $\|A^{-1}\|_\infty$ is bounded.

Proof: Refer to [17]. ∎

In the following theorem, we prove the existence and uniqueness of the quintic integro-differential spline S.

Theorem 4.2: The quintic integro-differential spline S defined by (1), (2), (3) and three extra conditions (18), (19), (20) exists uniquely.

Proof: It only needs to prove that the coefficient matrix A (33) is nonsingular. Let

$$P = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & 1 & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \end{pmatrix}_{(n+5)\times(n+5)},$$

P is nonsingular obviously. Furthermore, we have $A = BP$, where

$$B = \begin{pmatrix} 16 & -15 & 41 & 25 & 1 & 0 & & \\ 0 & -1 & -9 & 9 & 1 & 0 & & \\ 0 & 0 & -1 & -9 & 9 & 1 & & \\ 0 & 1 & 56 & 246 & 56 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 56 & 246 & 56 & 1 \\ & & 0 & -1 & -9 & 9 & 1 & 0 \\ & & 0 & 0 & -1 & -9 & 9 & 1 \end{pmatrix}.$$

Let $R(i)$ denotes the i-th row of B. We continue to perform ten elementary row-addition transformations to B in the following order, where $R(i) := R(i) + kR(j)$ means adding k times $R(j)$ to $R(i)$.

Step 1:

$$\begin{aligned} R(4) &:= R(4) + R(2), \\ R(n+3) &:= R(n+3) - R(n+5); \end{aligned}$$

Step 2:

$$\begin{aligned} R(4) &:= R(4) + 47R(3), \\ R(n+3) &:= R(n+3) - 47R(n+4); \end{aligned}$$

Step 3:

$$\begin{aligned} R(5) &:= R(5) + R(3), \\ R(n+2) &:= R(n+2) - R(n+4); \end{aligned}$$

Step 4:

$$\begin{aligned} R(5) &:= R(5) + \frac{47}{168}R(4), \\ R(n+2) &:= R(n+2) + \frac{47}{168}R(n+3); \end{aligned}$$

Step 5:

$$\begin{aligned} R(6) &:= R(6) + \frac{1}{168}R(4), \\ R(n+1) &:= R(n+1) + \frac{1}{168}R(n+3), \end{aligned}$$

then B is transformed to the following matrix

$$C = \begin{pmatrix} C_{11} & C_{12} & \mathbf{0}_{4\times 3} \\ \mathbf{0}_{(n-2)\times 4} & C_{22} & \mathbf{0}_{(n-2)\times 3} \\ \mathbf{0}_{3\times 4} & C_{32} & C_{33} \end{pmatrix},$$

where

$$C_{11} = \begin{pmatrix} 16 & -15 & 41 & 25 \\ 0 & -1 & -9 & 9 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 0 & -168 \end{pmatrix}_{4\times 4},$$

$$C_{12} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 9 & 1 & 0 & \cdots & 0 \\ 480 & 48 & 0 & \cdots & 0 \end{pmatrix}_{4\times (n-2)},$$

$$C_{22} = \begin{pmatrix} \frac{2725}{7} & \frac{493}{7} & 1 & & & & \\ \frac{412}{7} & \frac{1724}{7} & 56 & 1 & & & \\ 1 & 56 & 246 & 56 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 56 & 246 & 56 & 1 \\ & & & 1 & 56 & \frac{1724}{7} & \frac{412}{7} \\ & & & & 1 & \frac{493}{7} & \frac{2725}{7} \end{pmatrix},$$

$$C_{32} = \begin{pmatrix} 0 & \cdots & 0 & 48 & 480 \\ 0 & \cdots & 0 & -1 & -9 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}_{3\times (n-2)},$$

and

$$C_{33} = \begin{pmatrix} -168 & 0 & 0 \\ 9 & 1 & 0 \\ -9 & 9 & 1 \end{pmatrix}_{3\times 3}.$$

It is easy to observe that $|C_{11}| \neq 0$ and $|C_{33}| \neq 0$. At the same time, $|C_{22}| \neq 0$ because C_{22} is an $(n-2)\times(n-2)$ strictly diagonally dominant square matrix (Theorem 4.1, [17]). Hence

$$|C| = |C_{11}| \cdot |C_{22}| \cdot |C_{33}| \neq 0.$$

It implies that $|B| \neq 0$ and $|A| \neq 0$. Hence, the coefficient matrix A (33) is nonsingular and the theorem is proved. ■

V. The computational procedure

Under the assumption that the integral values $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, and two boundary derivatives $U'(x_0)$ and $U'(x_n)$ are given for an unknown function $U \in C^6[a,b]$, we can construct a four times continuously differentiable quintic integro-differential spline S that satisfies (1), (2), (3), (18), (19) and (20) by using the following computational procedure.

Step 1:Construct three extra values.

- Compute $\widehat{U_0}$ by (7);
- Compute $\widehat{U_1'}$ by (14);
- Compute $\widehat{U_{n-1}'}$ by (16).

Step 2:Solve the linear system $AC = Y$ (see (32), (33), (34) and (35)) to get the unknowns c_i $(i = -2, -1, 0, \cdots, n+2)$.

Step 3:Approximate $U^{(k)}$ by $S^{(k)} = \sum_{i=-2}^{n+2} c_i B_i^{(k)}(x)$.

Obviously, the computational procedure is very concise and easy to implement.

VI. Approximation properties

A. *The local approximation properties at the knots*

We first study the local approximation behaviors of $S^{(k)}$ $(k = 0, 1, 2, 3, 4)$ at the knots x_j $(j = 0, 1, \cdots, n)$.

From (21), we get

$$\begin{aligned} B_i(x_{i-2}) &= B_i(x_{i+2}) = \frac{1}{120}, \\ B_i(x_{i-1}) &= B_i(x_{i+1}) = \frac{26}{120}, \\ B_i(x_i) &= \frac{66}{120}, \\ B_i(x_k) &= 0, \quad k \neq i, i\pm 1, i\pm 2; \end{aligned}$$

$$\begin{aligned} B_i'(x_{i-2}) &= -B_i'(x_{i+2}) = \frac{1}{24h}, \\ B_i'(x_{i-1}) &= -B_i'(x_{i+1}) = \frac{10}{24h}, \\ B_i'(x_k) &= 0, \quad k \neq i\pm 1, i\pm 2; \end{aligned}$$

$$\begin{aligned} B_i''(x_{i-2}) &= B_i''(x_{i+2}) = \frac{1}{6h^2}, \\ B_i''(x_{i-1}) &= B_i''(x_{i+1}) = \frac{2}{6h^2}, \\ B_i''(x_i) &= -\frac{6}{6h^2}, \\ B_i''(x_k) &= 0, \quad k \neq i, i\pm 1, i\pm 2; \end{aligned}$$

$$\begin{aligned} B_i'''(x_{i-2}) &= -B_i'''(x_{i+2}) = \frac{1}{2h^3}, \\ B_i'''(x_{i-1}) &= -B_i'''(x_{i+1}) = -\frac{2}{2h^3}, \\ B_i'''(x_k) &= 0, \quad k \neq i\pm 1, i\pm 2; \end{aligned}$$

and

$$\begin{aligned} B_i''''(x_{i-2}) &= B_i''''(x_{i+2}) = \frac{1}{h^4}, \\ B_i''''(x_{i-1}) &= B_i''''(x_{i+1}) = -\frac{4}{h^4}, \\ B_i''''(x_i) &= \frac{6}{h^4}, \\ B_i''''(x_k) &= 0, \quad k \neq i, i\pm 1, i\pm 2. \end{aligned}$$

For $j = 0, 1, \cdots, n$, since $S = \sum_{i=-2}^{n+2} c_i B_i(x)$, by using the above results, we have the following formulae

$$S(x_j) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}), \quad (36)$$

$$S'(x_j) = \frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}), \quad (37)$$

$$S''(x_j) = \frac{1}{6h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}), \quad (38)$$

$$S'''(x_j) = \frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}), \quad (39)$$

$$S''''(x_j) = \frac{1}{h^4}(c_{j-2} - 4c_{j-1} + 6c_j - 4c_{j+1} + c_{j+2}). \quad (40)$$

For the sake of simplicity, let $s_j := S(x_j)$, $m_j := S'(x_j)$, $M_j := S''(x_j)$, $T_j := S'''(x_j)$ and $F_j := S''''(x_j)$. We also use U_j to denote $U(x_j)$, and U_j', U_j'', U_j''', U_j'''' $(j = 0, 1, \cdots, n)$ are similar abbreviations. At the same time, we define $e_j := s_j - U_j$, $e_j' := m_j' - U_j'$, $e_j'' := M_j - U_j''$, $e_j''' := T_j - U_j'''$ and $e_j'''' := F_j - U_j''''$, $j = 0, 1, \cdots, n$.

We first present some relations between s_j, m_j, M_j, T_j, F_j and I_j of the quintic integro-differential spline S.

Theorem 6.1:

1) For $j = 2, 3, \cdots, n-2$, we have

$$m_{j-2} + 56m_{j-1} + 246m_j + 56m_{j+1} + m_{j+2} = \frac{30}{h^2}(-I_{j-2} - 9I_{j-1} + 9I_j + I_{j+1}). \quad (41)$$

2) For $j = 0, 1, \cdots, n-3$, we have

$$T_j = \frac{2}{3h^2}(28m_j + 245m_{j+1} + 56m_{j+2} + m_{j+3}) + \frac{20}{h^4}(10I_j - 9I_{j+1} - I_{j+2}), \quad (42)$$

$$T_{j+3} = \frac{2}{3h^2}(m_j + 56m_{j+1} + 245m_{j+2} + 28m_{j+3}) - \frac{20}{h^4}(10I_{j+2} - 9I_{j+1} - I_j). \quad (43)$$

3) For $j = 0, 1, \cdots, n-1$, we have

$$s_{j+1} = -s_j + \frac{2}{h}I_j - \frac{h}{6}(m_j - m_{j+1}) + \frac{h^3}{360}(T_j - T_{j+1}). \quad (44)$$

4) For $j = 0, 1, \cdots, n-1$, we have

$$M_j = \frac{10}{h^3}I_j - \frac{10}{h^2}s_j - \frac{13}{3h}m_j - \frac{2}{3h}m_{j+1} - \frac{h}{9}T_j + \frac{h}{36}T_{j+1}, \quad (45)$$

$$M_{j+1} = -\frac{10}{h^3}I_j + \frac{10}{h^2}s_j + \frac{7}{3h}m_j + \frac{8}{3h}m_{j+1} - \frac{h}{18}T_j + \frac{5h}{36}T_{j+1}. \quad (46)$$

5) For $j = 0, 1, \cdots, n-1$, we have

$$F_j = -\frac{120}{h^5}I_j + \frac{120}{h^4}s_j + \frac{40}{h^3}m_j + \frac{20}{h^3}m_{j+1} - \frac{11}{3h}T_j - \frac{4}{3h}T_{j+1}, \quad (47)$$

$$F_{j+1} = \frac{120}{h^5}I_j - \frac{120}{h^4}s_j - \frac{40}{h^3}m_j - \frac{20}{h^3}m_{j+1} + \frac{5}{3h}T_j + \frac{10}{3h}T_{j+1}. \quad (48)$$

Proof: These equations can be proved by using (26), (36), (37), (38), (39), (40) and comparing the coefficients of c_j $(j = -2, -1, \cdots, n+2)$.

For example, (48) can be proved as follows

$$\begin{aligned} & \text{The right-hand side of (48)} \\ = \; & \frac{120}{h^5}\cdot\frac{h}{720}(c_{j-2} + 57c_{j-1} + 302c_j + 302c_{j+1} + 57c_{j+2} + c_{j+3}) \\ & -\frac{120}{h^4}\cdot\frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}) \\ & -\frac{40}{h^3}\cdot\frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}) \\ & -\frac{20}{h^3}\cdot\frac{1}{24h}(-c_{j-1} - 10c_j + 10c_{j+2} + c_{j+3}) \\ & +\frac{5}{3h}\cdot\frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}) \\ & +\frac{10}{3h}\cdot\frac{1}{2h^3}(-c_{j-1} + 2c_j - 2c_{j+2} + c_{j+3}) \\ = \; & \frac{1}{h^4}(c_{j-1} - 4c_j + 6c_{j+1} - 4c_{j+2} + c_{j+3}) \\ = \; & F_{j+1}. \end{aligned}$$

■

Theorem 6.2: Let S be the four times continuously differentiable quintic integro-differential spline that satisfies (1), (2), (3), (18), (19) and (20), where $U \in C^6[a, b]$. For $j = 0, 1, \cdots, n$, we have

$$s_j = U_j + O(h^6), \quad (49)$$

$$m_j = U_j' + O(h^5), \quad (50)$$

$$M_j = U_j'' + O(h^4), \quad (51)$$

$$T_j = U_j''' + O(h^3), \quad (52)$$

$$F_j = U_j'''' + O(h^2). \quad (53)$$

Proof:

1) We first prove (50). For $j = 2, 3, \cdots, n-2$, by using (41) and the Taylor formula, we have

$$\begin{aligned} & e'_{j-2} + 56e'_{j-1} + 246e'_j + 56e'_{j+1} + e'_{j+2} \\ = & \frac{30}{h^2}(-I_{j-2} - 9I_{j-1} + 9I_j + I_{j+1}) \\ & -(U'_{j-2} + 56U'_{j-1} + 246U'_j + 56U'_{j+1} + U'_{j+2}) \\ = & \frac{30}{h^2}(12h^2U'_j + 2h^4U'''_j + \frac{1}{5}h^6U^{(5)}_j + O(h^7)) \\ & -(360U'_j + 60h^2U'''_j + 6h^4U^{(5)}_j + O(h^5)) \\ = & O(h^5). \end{aligned}$$

Besides, by using (2) and (3), we have $e'_0 = 0$ and $e'_n = 0$; by using (14), (15) and (16), (17), we have $e'_1 = O(h^5)$ and $e'_{n-1} = O(h^5)$. Hence, we get

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ 1 & 56 & 246 & 56 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 56 & 246 & 56 & 1 \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} e'_0 \\ e'_1 \\ e'_2 \\ \vdots \\ e'_{n-2} \\ e'_{n-1} \\ e'_n \end{pmatrix} = \begin{pmatrix} 0 \\ O(h^5) \\ O(h^5) \\ \vdots \\ O(h^5) \\ O(h^5) \\ 0 \end{pmatrix}.$$

Its coefficient matrix is strictly diagonally dominant. The infinity norm of its inverse is bounded (Theorem 4.1, [17]). Hence, we get

$$e'_j = O(h^5), \quad j = 0, 1, \cdots, n.$$

So (50) is proved.

2) By using (42) and (50), we have

$$\begin{aligned} e'''_j = & \frac{2}{3h^2}(28U'_j + 245U'_{j+1} + 56U'_{j+2} + U'_{j+3}) \\ & +\frac{20}{h^4}(10I_j - 9I_{j+1} - I_{j+2}) \\ & +\frac{2}{3h^2}(28e'_j + 245e'_{j+1} + 56e'_{j+2} + e'_{j+3}) \\ & -U'''_j \\ & \text{(continue to expand it at } x_j) \\ = & O(h^3), \quad j = 0, 1, \cdots, n-3. \end{aligned}$$

Similarly, by using (43) and (50), we get

$$e'''_{j+3} = O(h^3), \quad j = 0, 1, \cdots, n-3.$$

Hence, (52) holds.

3) The formulae (7) and (8) imply that (49) holds for $j = 0$. Moreover, when $j = 0$, by using (44), (50) and (52), we have

$$\begin{aligned} e_1 = & -e_0 + \frac{2}{h}I_0 - \frac{h}{6}(e'_0 - e'_1) + \frac{h^3}{360}(e'''_0 - e'''_1) \\ & -U_0 - \frac{h}{6}(U'_0 - U'_1) + \frac{h^3}{360}(U'''_0 - U'''_1) - U_1 \\ & \text{(continue to expand it at } x_0) \\ = & O(h^6). \end{aligned} \quad (54)$$

The formula (54) shows that (49) holds for $j = 1$. By similar deduction, we can get (49) holds for $j = 2, 3, \cdots, n$.

4) We use (45), (46), (49), (50) and (52) to prove (51). By using (45), (49), (50) and (52), we get

$$\begin{aligned} e''_j = & \frac{10}{h^3}I_j - \frac{10}{h^2}e_j - \frac{13}{3h}e'_j - \frac{2}{3h}e'_{j+1} \\ & -\frac{h}{9}e'''_j + \frac{h}{36}e'''_{j+1} - \frac{10}{h^2}U_j - \frac{13}{3h}U'_j \\ & -\frac{2}{3h}U'_{j+1} - \frac{h}{9}U'''_j + \frac{h}{36}U'''_{j+1} - U''_j \\ & \text{(continue to expand it at } x_j) \\ = & O(h^4), \quad j = 0, 1, \cdots, n-1. \end{aligned}$$

By using (46), (49), (50) and (52), we get

$$e''_{j+1} = O(h^4), \quad j = 0, 1, \cdots, n-1.$$

Hence, (51) holds.

5) Similarly, (53) can be proved by using (47), (48), (49), (50) and (52).

■

B. The global approximation properties

We give the global approximation properties of S in the next theorem.

Theorem 6.3: Let S be the four times continuously differentiable quintic integro-differential spline that satisfies (1), (2), (3), (18), (19) and (20), where $U \in C^6[a, b]$, we have

$$\|S^{(k)} - U^{(k)}\|_\infty = O(h^{6-k}), \quad k = 0, 1, 2, 3, 4, \quad (55)$$

where $\|y\|_\infty = \max_{a \le x \le b} |y(x)|$.

Proof:

1) $S^{(4)}$ is a piecewise continuous linear polynomial. For $j = 1, 2, \cdots, n$, let $S^{(4)}_j$ denotes the restriction of $S^{(4)}(x)$ over $[x_{j-1}, x_j]$, we have

$$S^{(4)}_j = F_{j-1} + \frac{F_j - F_{j-1}}{h}(x - x_{j-1}). \quad (56)$$

By using (53) and (56), we have

$$\begin{aligned} S^{(4)}_j = & (U''''_{j-1} + O(h^2)) \\ & +(\frac{U''''_j - U''''_{j-1}}{h} + O(h))(x - x_{j-1}) \\ = & U''''_{j-1} + U^{(5)}_{j-1}(x - x_{j-1}) \\ & +O(h)(x - x_{j-1}) + O(h^2). \end{aligned} \quad (57)$$

Besides, on $[x_{j-1}, x_j]$, we have

$$\begin{aligned} U^{(4)} = & U''''_{j-1} + U^{(5)}_{j-1}(x - x_{j-1}) \\ & +\frac{1}{2}U^{(6)}_{j-1}(x - x_{j-1})^2 + O(h^2). \end{aligned} \quad (58)$$

By using (57), (58) and take the infinite norm on $[x_{j-1}, x_j]$, for $j = 1, 2, \cdots, n$, we have

$$\|S^{(4)}_j - U^{(4)}\|_\infty = \max_{x_{j-1} \le x \le x_j} |S^{(4)}_j - U^{(4)}| = O(h^2). \quad (59)$$

Hence, (55) holds for $k = 4$.

2) $S^{(3)}$ is a piecewise quadratic polynomial. For $j = 1, 2, \cdots, n$, let $S^{(3)}_j$ denotes the restriction of $S^{(3)}(x)$

TABLE I
THE MAXIMUM ABSOLUTE ERRORS OF THE RUNGE FUNCTION (61)

	$E_{0,n}$	$E_{1,n}$	$E_{2,n}$
$n = 40$	$4.181e-5$	$1.099e-3$	$2.431e-1$
$n = 80$	$1.661e-7$	$1.482e-5$	$7.624e-3$
$n = 160$	$1.655e-9$	$2.235e-7$	$4.023e-4$
$n = 320$	$2.304e-11$	$3.398e-9$	$2.415e-5$
$n = 640$	$4.071e-13$	$8.376e-11$	$1.527e-6$

TABLE II
THE MAXIMUM ABSOLUTE ERRORS OF THE RUNGE FUNCTION (61)

	$E_{3,n}$	$E_{4,n}$
$n = 40$	$9.501e+0$	$2.833e+3$
$n = 80$	$5.573e-1$	$6.136e+2$
$n = 160$	$3.548e-2$	$1.481e+2$
$n = 320$	$2.183e-3$	$3.672e+1$
$n = 640$	$1.468e-4$	$9.205e+0$

over $[x_{j-1}, x_j]$. For any $x \in [x_{j-1}, x_j]$, by using (52) and (59), we have

$$\begin{aligned} & S_j^{(3)}(x) - U^{(3)}(x) \\ = & \int_{x_{j-1}}^{x} (U_j^{(4)}(t) - U^{(4)}(t))dt + (T_{j-1} - U_{j-1}^{(3)}) \\ = & O(h^2)(x - x_{j-1}) + O(h^3). \end{aligned} \quad (60)$$

By using (60), we get

$$\|S_j^{(3)} - U^{(3)}\|_\infty = \max_{x_{j-1} \le x \le x_j} |S_j^{(3)} - U^{(3)}| = O(h^3),$$

$j = 1, 2, \cdots, n$. Hence, (55) holds for $k = 3$.

3) By similar manners, we also can prove that (55) holds for $k = 2, 1, 0$.

■

VII. NUMERICAL TESTS

In this section, we perform some numerical tests by Matlab to check the approximation efficiency of our new method.

A. Numerical tests with a Runge function

We first test our method by using the following Runge function

$$U = \frac{1}{1 + 25x^2}, \qquad x \in [-1, 1]. \quad (61)$$

We list our maximum absolute errors of (61) in Table I and Table II. $aeb = a \times 10^b$, for example $4.181e-5 = 4.181 \times 10^{-5}$.

The maximum absolute error $E_{k,n}$ is defined as follows

$$E_{k,n} = \max_{0 \le j \le n} |S^{(k)}(x_j) - U^{(k)}(x_j)|, \qquad k = 0, 1, 2, 3, 4,$$

where S is the four times continuously differentiable quintic integro-differential spline determined by using (1), (2), (3), (18), (19) and (20).

From Table I and Table II, we find S can provide satisfactory approximations to the Runge function U (61) and its first order to fourth order derivatives.

To check the convergence properties of S, we also list the approximate convergence orders O_k of the maximum absolute errors $E_{k,n}$ in Table III, where O_k is defined as follows

$$O_k = \frac{\log(E_{k,n_1}) - \log(E_{k,n_2})}{\log(n_2) - \log(n_1)}, \qquad k = 0, 1, 2, 3, 4.$$

TABLE III
THE APPROXIMATE CONVERGENCE ORDERS OF THE MAXIMUM ABSOLUTE ERRORS OF THE RUNGE FUNCTION (61)

$n_1 \to n_2$	O_0	O_1	O_2	O_3	O_4
$40 \to 640$	6.65	5.91	4.32	4.00	2.06

TABLE IV
THE NUMERICAL RESULTS OF THE RUNGE FUNCTION (61) IN [1]

Method	M.I.1 ([1])	M.I.2 ([1])	M.I.3,4 ([1])	Our method
N_{10}	21	21	12	12
$E_{0,10}$	$4.5e-4$	$4.9e-4$	$1.0e-1$	$4.7e-2$
N_{100}	201	201	102	102
$E_{0,100}$	$6.8e-7$	$7.2e-7$	$3.8e-5$	$3.5e-8$
C^k	C^{-1}	C^{-1}	C^1	C^4

TABLE V
THE NUMERICAL RESULTS OF THE RUNGE FUNCTION (61) IN [2] AND [3]

Method	M.II.1,2 ([2])	M.III.1,2 ([3])	M.III.3,4 ([3])
N_{20}	41	20	20
$E_{0,20}$	$4.1e-3$	$3.2e-2$	$2.1e-2$
N_{200}	401	200	200
$E_{0,20}$	/	/	$1.6e-5$
C^k	C^0	C^{-1}	C^0

From Table III, we find that O_0, O_1, O_2, O_3 are higher than the theoretical ones and O_4 is about the same with the theoretical order. Anyway, these results in Table I, Table II and Table III are in accordance with the theoretical expectations.

For the sake of comparison, we collect many other numerical results in Table IV, Table V, Table VI and Table VII. In these tables, we use N_n to denote the number of the needed data of a method. Moreover, we note a method with C^k we mean that the obtained approximation is k times continuously differentiable. Especially, C^{-1} means that the obtained approximation is discontinuous and C^0 means that the obtained approximation is merely continuous.

We compare our method with the methods of [1] in Table IV, where

- M.I.1 denotes the discontinuous piecewise quadratic polynomial method and M.I.2 denotes the discontinuous piecewise trigonometric function method in [1], the needed data of the two methods are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and $U'_j = U'(x_j)$ $(j = 0, 1, \cdots, n)$;
- M.I.3 denotes the continuously differentiable piecewise quadratic polynomial method and M.I.4 denotes the continuously differentiable piecewise trigonometric function method in [1], the needed data of the two methods are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, $U'(x_0)$ and $U'(x_n)$.

From Table IV, at fist glance, one can find the $E_{0,10}$ of M.I.1 ([1]) and M.I.2 ([1]) are better than our $E_{0,10}$. This

TABLE VI
THE NUMERICAL RESULTS OF THE RUNGE FUNCTION (61) IN [4]

Method	M.IV.1 ([4])	M.IV.2,3,4 ([4])	Our method
N_{20}	41	41	22
$E_{0,20}$	$2.25e-2$	$2.23e-2$	$2.1e-4$
N_{200}	401	401	202
$E_{0,200}$	$5.59e-9$	/	$4.1e-10$
C^k	C^0	C^0	C^4

is mainly because they are obtained based on 21 exact data of U, while our $E_{0,10}$ is obtained based on only 12 exact data of U. Beyond that, our else results are better than the corresponding ones of [1]. For example, the $E_{0,100}$ of M.I.1 ([1]) is $6.8e-7$ and it is obtained from 201 exact data of U, while our $E_{0,100}$ is $3.5e-8$ and it is obtained from only 102 exact data of U.

We compare our method with the methods of [2], [3], [4] in Table V, and Table VI, where

- M.II.1 and M.II.2 denote the polynomial integro-differential spline method and the trigonometrical integro-differential spline method in [2], the needed data of the two methods are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and $U_j = U(x_j)$ $(j = 0, 1, \cdots, n)$;
- M.III.1, M.III.2, M.III.3 and M.III.4 denote the discontinuous piecewise quadratic polynomial method, the discontinuous piecewise trigonometric spline method, the continuous quadratic polynomial method and the continuous trigonometric spline method in [3], the needed data of these methods are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$;
- M.IV.1 denotes the continuous quartic integro-differential spline method in [4], M.IV.2, M.IV.3 and M.IV.4 denote the other three non-polynomial integro-differential spline methods in [4], the needed data of these methods are also $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and $U_j = U(x_j)$ $(j = 0, 1, \cdots, n)$;
- "/ " denotes that the corresponding results were not provided in the cited papers.

From Table V and Table VI, it is easy to find that our results $E_{0,20} = 2.1e-4$ and $E_{0,200} = 4.1e-10$ are better than all the corresponding results of [2], [3], [4]. Furthermore, the smoothness order of our new approximation is obviously higher than that of [2], [3], [4].

We continue to compare our method with the methods of [5] in Table VII, where

- M.V.1 denotes the continuously differentiable quartic integro-differential spline method in [5], the needed data of the method are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$, $U_j = U(x_j)$ $(j = 0, 1, \cdots, n)$ and $U'_j = U'(x_j)$ $(j = 0, 1, \cdots, n)$;
- M.V.2 denotes the twice continuously differentiable quartic integro-differential spline method in [5], the needed data of the method are $I_j = \int_{x_j}^{x_{j+1}} U(x)dx$ $(j = 0, 1, \cdots, n-1)$ and $U_j = U(x_j)$ $(j = 0, 1, \cdots, n)$.

From Table VII, we also can find our results are better than all the corresponding results of [5]. At the same time, we point that our N_n is only about $\frac{1}{3}$ of that of M.V.1 ([5]) and is only about $\frac{1}{2}$ of that of M.V.2 ([5]) for a certain n. It is also an attractive advantage in practical applications.

TABLE VII
THE NUMERICAL RESULTS OF THE RUNGE FUNCTION (61) IN [5]

Method	M.V.1 ([5])	M.V.2 ([5])	Our method
N_{20}	62	41	22
$E_{0,20}$	$2.1e-4$	$2.7e-3$	$2.1e-4$
$E_{1,20}$	$1.4e-2$	$8.9e-2$	$2.2e-3$
N_{200}	602	401	202
$E_{0,200}$	$2.3e-9$	$1.8e-7$	$4.1e-10$
$E_{1,200}$	$1.63e-6$	$6.0e-5$	$5.7e-8$
$E_{2,200}$	/	$2.1e-2$	$1.6e-4$
C^k	C^1	C^2	C^4

TABLE VIII
THE MAXIMUM ABSOLUTE ERRORS OF THE OSCILLATING FUNCTION (62)

	$E_{0,n}$	$E_{1,n}$	$E_{2,n}$
$n = 40$	$1.706e-5$	$1.839e-4$	$1.350e-1$
$n = 80$	$3.225e-7$	$6.636e-6$	$9.974e-3$
$n = 160$	$5.198e-9$	$2.121e-7$	$6.404e-4$
$n = 320$	$8.130e-11$	$6.623e-9$	$4.003e-5$
$n = 640$	$1.252e-12$	$2.130e-10$	$2.478e-6$

TABLE IX
THE MAXIMUM ABSOLUTE ERRORS OF THE OSCILLATING FUNCTION (62)

	$E_{3,n}$	$E_{4,n}$
$n = 40$	$1.140e+1$	$7.738e+2$
$n = 80$	$1.640e+0$	$2.263e+2$
$n = 160$	$2.095e-1$	$5.802e+1$
$n = 320$	$2.618e-2$	$1.451e+1$
$n = 640$	$3.280e-3$	$3.885e+0$

TABLE X
THE APPROXIMATE CONVERGENCE ORDERS OF THE MAXIMUM ABSOLUTE ERRORS OF THE OSCILLATING FUNCTION (62)

$n_1 \to n_2$	O_0	O_1	O_2	O_3	O_4
$40 \to 640$	5.92	4.93	3.93	2.94	1.91

B. Numerical tests with an oscillating function

We continue to test our method by using the following oscillating function

$$U = \sin 3x \cos 5x, \qquad x \in [-1, 1]. \tag{62}$$

The maximum absolute errors of (62) are given in Table VIII, Table IX and the results are well accepted.

The approximate convergence orders of the errors are given in Table X. Here, the approximate convergence orders are slightly lower than the theoretical ones. It is a normal numerical phenomenon when approximating an oscillating function.

Generally, these results in Table Table VIII, Table IX and Table X are also in accordance with the theoretical expectations.

The comparison results of the oscillating function (62) are given in Table XI and Table XII. (The oscillating function

TABLE XI
THE NUMERICAL RESULTS OF THE OSCILLATING FUNCTION (62) IN [1]

Method	M.I.1,2 ([1])	M.I.3,4 ([1])	Our method
N_{10}	21	12	12
$E_{0,10}$	$5.2e-2$	$2.5e-2$	$5.1e-2$
N_{100}	201	102	102
$E_{0,100}$	$8.2e-5$	$1.6e-5$	$8.6e-8$
C^k	C^{-1}	C^0	C^4

TABLE XII
THE NUMERICAL RESULTS OF THE OSCILLATING FUNCTION (62) IN [2] AND [5]

Method	M.II.1,2 ([2])	M.V.1 ([5])	M.V.2 ([5])	Our method
N_{20}	41	62	41	22
$E_{0,20}$	$1.9e-3$	$1.2e-5$	$5.1e-4$	$2.7e-4$
$E_{1,20}$	/	$8.1e-4$	$1.8e-2$	$2.4e-3$
N_{200}	201	602	401	202
$E_{0,200}$	/	$1.2e-10$	$1.1e-8$	$1.4e-9$
$E_{1,200}$	/	$8.5e-8$	$3.6e-6$	$7.0e-8$
$E_{2,200}$	/	/	$1.1e-3$	$2.6e-4$
C^k	C^0	C^1	C^2	C^4

(62) was not tested in [3] and [4].) These comparison results also show that our new method is very preferable and competitive because it has satisfactory approximation ability, has higher smoothness order and needs fewer exact data of U.

VIII. CONCLUSIONS

Function approximation is an important topic of applied mathematics and computational mathematics. It has wide applications in many fields, see the above-cited papers and refer to [18], [19], [20], [21] for more examples.

In the field of function approximation, there is an important integro interpolation approximation problem. That is to approximate a function and its derivatives from its integral values over grid intervals and some function values or some derivative values at grid nodes. The old methods for the problem have many evident drawbacks. We are in great need of a new effective method.

In this paper, we have presented a new integro-differential spline approximation method for the integro interpolation approximation problem. We have successfully obtained an integro-differential spline with higher approximation order and higher smoothness order to approximate a function and its first four derivatives by only using fewer data of the original function. In fact, our new method can easily produce a four times continuously differentiable quintic integro-differential spline by only using $I_j = \int_{x_j}^{x_{j+1}} U(t)dt$ $(j = 0, 1, \cdots, n-1)$, $U'(x_0)$ and $U'(x_n)$. The approximation efficiency of the new method is very high and has been examined by a Runge function and an oscillating function. In conclusions, the new method is very effective and practical.

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