Reduction of Sixth-Order Ordinary Differential Equations to Laguerre Form by Fiber Preserving Transformations

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Abstract— **This paper is devoted to the study on linearization of sixth-order ordinary differential equations by fiber preserving transformations. The necessary and sufficient conditions for linearization are obtained. The procedure for obtaining the linearizing transformations and the coefficients of linear equation are provided in explicit form. Examples demonstrating the procedure of using the linearization theorems are presented.**

Keywords: Linearization problem, point transformation, fiber preserving transformation, nonlinear ordinary differential equation

1 Introduction

Nonlinear problems are considered to be a core part in many branches of sciences. The exact solutions to these problems are most required. However, it is evidently hard to analyze such problems directily. Precisely, in the nonlinear regime, many of the most basic questions remain unanswered: existence and uniqueness of solutions are not guaranteed; explicit formulae are difficult to come by; linear superposition is no longer available; numerical approximations are not always sufficiently accurate; linear superposition is no longer available; etc. One method to avoid these difficulty is transforming them into the linear differential equations, which is called the "linearization".

The linearization, that is, mapping a nonlinear differential equation into a linear differential equation, is an important tool in the theory of differential equations. The problem of linearization of ordinary differential equations attracted attention of mathematicians such as S. Lie and E. Cartan. It was admitted that Lie [1] is the first person who solved linearization problem for ordinary differential equations in 1883. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by point transformations. He found the conditions for linearization. The linearization criterion is written through relative invariants of the equivalence group. Liouville [2] and Tresse [3] attacked the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. There are other approaches for solving the linearization problem of a second-order ordinary differential equation. For example, one was developed by Cartan [4], the idea of his approach was to associate with geometric structure.

For the third-order ordinary differential equations, Bocharov, Sokolov and Svinolupov [5] considered the linearization problem with respect to point transformations. Grebot [6] studied the linearization by means of a restricted class of point transformations, namely fiber preserving transformation. However, the problem was not completely solved. Complete criteria for linearization by means of point transformations were obtained by Ibragimov and Meleshko [7].

The linearization problem of third-order ordinary differential equations under point transformations were solved by Ibragimov, Meleshko and Suksern [8]. They found the necessary and sufficient conditions for a complete linearization problem.

Later, Suksern and Pinyo [9] solved the linearization problem of fifth-order ordinary differential equations by means of fiber preserving transformations. It turns out that not every differential equation of this order can be transformed in that way. Giving a new transformation still be needed and useful.

One of main problem in linearization is the complicate calculations. Because of this task, there is no one try to solve the linearization problems for sixth and higher order yet.

By the helps of computer algebra system Reduce, we can offer the necessary and sufficient conditions of this type of linearization. Some examples are provided to illustrate the condition. Linearizing transformation and coefficients of linear equation are obtained. A program used

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for checking the linearity is also produced.

2 Point Transformations

Definition 2.1 *A transformation*

$$\begin{aligned} t = &\ \varphi(x,y), \\ u = &\ \psi(x,y) \end{aligned} \tag{1}$$

where φ and ψ are sufficiently smooth functions is called a *point transformation.* If $\varphi_y = 0$, a transformation (1) is called a *fiber preserving transformation.*

Let us explain how a point transformation maps one function into another. Assume that $y_0(x)$ is a given function, then the transformed function $u_0(t)$ is defined by the following two steps. On the first step one has to solve with respect to x the equation

$$t = \varphi(x, y_0(x)).$$

Using the Inverse Function Theorem we find $x = \alpha(t)$ is a solution of this equation. The transformed function is determined by the formula

$$u_0(t) = \psi(\alpha(t), y_0(\alpha(t))).$$

Conversely, if one has the function $u_0(t)$, then for finding the function $y_0(x)$ one has to solve the ordinary differential equation

$$u_0\left(\varphi(x, y_0(x)\right) = \psi(x, y_0(x)).$$

3 Necessary Conditions

We begin with investigating the necessary conditions for linearization. Recall that according to the Laguerre theorem, a linear sixth-order ordinary differential equation has the form

$$u^{(6)} + \alpha(t)u''' + \beta(t)u'' + \gamma(t)u' + \omega(t)u = 0. \tag{2}$$

Here we consider the sixth-order ordinary differential equations

$$y^{(6)} = f\left(x, y, y', y'', y''', y^{(4)}, y^{(5)}\right), \tag{3}$$

which can be transformed to the linear equation (2) by the fiber preserving transformation

$$\begin{aligned} t = &\ \varphi\left(x\right), \\ u = &\ \psi\left(x,y\right). \end{aligned} \tag{4}$$

So we arrive at the following theorem.

Theorem 3.1 Any sixth-order ordinary differential equations linearizable by a fiber preserving transformation has to be in the form

$$\begin{aligned} y^{(6)} &+ (A_1y' + A_0)y^{(5)} + (B_3y'' + B_2y'^2 + B_1y' \\ &+ B_0)y^{(4)} + C_0y'''^2 + ((D_5y' + D_4)y'' \\ &+ D_3y'^3 + D_2y'^2 + D_1y' + D_0)y''' + E_0y''^3 \\ &+ (F_2y'^2 + F_1y' + F_0)y''^2 + (G_4y'^4 + G_3y'^3 \\ &+ G_2y'^2 + G_1y' + G_0)y'' + H_6y'^6 + H_5y'^5 \\ &+ H_4y'^4 + H_3y'^3 + H_2y'^2 + H_1y' + H_0 = 0, \end{aligned} \tag{5}$$

where

$$A_1 = (6\psi_{yy})/\psi_y, \tag{6}$$
$$A_0 = -3(5\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy})/(\varphi_x\psi_y), \tag{7}$$
$$B_3 = (15\psi_{yy})/\psi_y, \tag{8}$$
$$B_2 = (15\psi_{yyy})/\psi_y, \tag{9}$$
$$B_1 = -15(5\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy})/(\varphi_x\psi_y), \tag{10}$$
$$\begin{aligned} B_0 = &5(3(7\varphi_{xx}^2\psi_y - 5\varphi_{xx}\varphi_x\psi_{xy} + \varphi_x^2\psi_{xxy}) \\ &- 4\varphi_{xxx}\varphi_x\psi_y)/(\varphi_x^2\psi_y), \end{aligned} \tag{11}$$
$$C_0 = (10\psi_{yy})/\psi_y, \tag{12}$$
$$D_5 = (60\psi_{yyy})/\psi_y, \tag{13}$$
$$D_4 = -30(5\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy})/(\varphi_x\psi_y), \tag{14}$$
$$D_3 = (20\psi_{yyyy})/\psi_y, \tag{15}$$
$$D_2 = -30(5\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy})/(\varphi_x\psi_y), \tag{16}$$
$$\begin{aligned} D_1 = &20(3(7\varphi_{xx}^2\psi_{yy} - 5\varphi_{xx}\varphi_x\psi_{xyy} + \varphi_x^2\psi_{xxyy}) \\ &- 4\varphi_{xxx}\varphi_x\psi_{yy})/(\varphi_x^2\psi_y), \end{aligned} \tag{17}$$
$$\begin{aligned} D_0 = &-(30(14\varphi_{xx}^2\psi_y - 14\varphi_{xx}\varphi_x\psi_{xy} \\ &+ 5\varphi_x^2\psi_{xxy})\varphi_{xx} - (\varphi_x^3\psi_y\alpha + 20\psi_{xxxy})\varphi_x^3 \\ &- 10(21\varphi_{xx}\psi_y - 8\varphi_x\psi_{xy})\varphi_{xxx}\varphi_x \\ &+ 15\varphi_{xxxx}\varphi_x^2\psi_y)/(\varphi_x^3\psi_y), \end{aligned} \tag{18}$$
$$E_0 = (15\psi_{yyy})/\psi_y, \tag{19}$$
$$F_2 = (45\psi_{yyyy})/\psi_y, \tag{20}$$
$$F_1 = -45(5\varphi_{xx}\psi_{yyy} - 2\varphi_x\psi_{xyyy})/(\varphi_x\psi_y), \tag{21}$$
$$\begin{aligned} F_0 = &15(3(7\varphi_{xx}^2\psi_{yy} - 5\varphi_{xx}\varphi_x\psi_{xyy} + \varphi_x^2\psi_{xxyy}) \\ &- 4\varphi_{xxx}\varphi_x\psi_{yy})/(\varphi_x^2\psi_y), \end{aligned} \tag{22}$$
$$G_4 = (15\psi_{yyyyy})/\psi_y, \tag{23}$$
$$G_3 = -30(5\varphi_{xx}\psi_{yyyy} - 2\varphi_x\psi_{xyyyy})/(\varphi_x\psi_y), \tag{24}$$
$$\begin{aligned} G_2 = &-30(4\varphi_{xxx}\varphi_x\psi_{yyy} - 21\varphi_{xx}^2\psi_{yyy} \\ &+ 15\varphi_{xx}\varphi_x\psi_{xyyy} - 3\varphi_x^2\psi_{xxyyy})/(\varphi_x^2\psi_y), \end{aligned} \tag{25}$$
$$\begin{aligned} G_1 = &-3(30(14\varphi_{xx}^2\psi_{yy} - 14\varphi_{xx}\varphi_x\psi_{xyy} \\ &+ 5\varphi_x^2\psi_{xxyy})\varphi_{xx} - (\varphi_x^3\psi_{yy}\alpha + 20\psi_{xxxyy})\varphi_x^3 \\ &+ 15\varphi_{xxxx}\varphi_x^2\psi_{yy} - 10(21\varphi_{xx}\psi_{yy} \\ &- 8\varphi_x\psi_{xyy})\varphi_{xxx}\varphi_x)/(\varphi_x^3\psi_y), \end{aligned} \tag{26}$$

$$\begin{aligned}G_0 =&-(6\varphi_{xxxxx}\varphi_x^3\psi_y-105\varphi_{xxxx}\varphi_{xx}\varphi_x^2\psi_y\\&+45\varphi_{xxxx}\varphi_x^3\psi_{xy}-70\varphi_{xxx}^2\varphi_x^2\psi_y\\&+840\varphi_{xxx}\varphi_{xx}^2\varphi_x\psi_y-630\varphi_{xxx}\varphi_{xx}\varphi_x^2\psi_{xy}\\&+120\varphi_{xxx}\varphi_x^3\psi_{xxy}-945\varphi_{xx}^4\psi_y\\&+1260\varphi_{xx}^3\varphi_x\psi_{xy}-630\varphi_{xx}^2\varphi_x^2\psi_{xxy}\\&+3\varphi_{xx}\varphi_x^6\psi_y\alpha+150\varphi_{xx}\varphi_x^3\psi_{xxxy}\\&-\varphi_x^8\psi_y\beta-3\varphi_x^7\psi_{xy}\alpha\\&-15\varphi_x^4\psi_{xxxxy})/(\varphi_x^4\psi_y),\end{aligned} \tag{27}$$

$$H_6=(\psi_{yyyyyy})/\psi_y, \tag{28}$$

$$H_5=-3(5\varphi_{xx}\psi_{yyyyy}-2\varphi_x\psi_{xyyyyy})/(\varphi_x\psi_y), \tag{29}$$

$$\begin{aligned}H_4=&-5(4\varphi_{xxx}\varphi_x\psi_{yyyy}-21\varphi_{xx}^2\psi_{yyyy}\\&+15\varphi_{xx}\varphi_x\psi_{xyyyy}-3\varphi_x^2\psi_{xxyyyy})/(\varphi_x^2\psi_y),\end{aligned} \tag{30}$$

$$\begin{aligned}H_3=&-(15\varphi_{xxxx}\varphi_x^2\psi_{yyy}-210\varphi_{xxx}\varphi_{xx}\varphi_x\psi_{yyy}\\&+80\varphi_{xxx}\varphi_x^2\psi_{xyyy}+420\varphi_{xx}^3\psi_{yyy}\\&-420\varphi_{xx}^2\varphi_x\psi_{xyyy}+150\varphi_{xx}\varphi_x^2\psi_{xxyyy}\\&-\varphi_x^6\psi_{yyy}\alpha-20\varphi_x^3\psi_{xxxyyy})/(\varphi_x^3\psi_y),\end{aligned} \tag{31}$$

$$\begin{aligned}H_2=&-(3((420\varphi_{xx}^3-\varphi_x^6\alpha)\psi_{xyy}-5\varphi_x^3\psi_{xxxxyy})\\&-70\varphi_{xxx}^2\varphi_x\psi_{yy}+3(\varphi_x^3\psi_{yy}\alpha\\&+50\psi_{xxxyy})\varphi_{xx}\varphi_x^2+30(28\varphi_{xx}^2\psi_{yy}\\&-21\varphi_{xx}\varphi_x\psi_{xyy}+4\varphi_x^2\psi_{xxyy})\varphi_{xxx})\varphi_x\\&-((945\varphi_{xx}^4+\varphi_x^8\beta)\psi_{yy}+630\varphi_{xx}^2\varphi_x^2\psi_{xxyy}\\&-6\varphi_{xxxxx}\varphi_x^3\psi_{yy}+15(7\varphi_{xx}\psi_{yy}\\&-3\varphi_x\psi_{xyy})\varphi_{xxxx}\varphi_x^2)/(\varphi_x^4\psi_y),\end{aligned} \tag{32}$$

$$\begin{aligned}H_1=&-(315(3\varphi_{xx}^2\psi_y-6\varphi_{xx}\varphi_x\psi_{xy}\\&+4\varphi_x^2\psi_{xxy})\varphi_{xx}^3-(\varphi_x^5\psi_y\gamma+2\varphi_x^4\psi_{xy}\beta\\&+3\varphi_x^3\psi_{xxy}\alpha+6\psi_{xxxxxy})\varphi_x^5\\&+\varphi_{xxxxxx}\varphi_x^4\psi_y-3(\varphi_x^3\psi_y\alpha\\&+140\psi_{xxxy})\varphi_{xx}^2\varphi_x^3+(\varphi_x^4\psi_y\beta+6\varphi_x^3\psi_{xy}\alpha\\&+75\psi_{xxxxy})\varphi_{xx}\varphi_x^4+140(2\varphi_{xx}\psi_y\\&-\varphi_x\psi_{xy})\varphi_{xxx}^2\varphi_x^2-3(7\varphi_{xx}\psi_y\\&-4\varphi_x\psi_{xy})\varphi_{xxxxx}\varphi_x^3-5(7\varphi_{xxx}\varphi_x\psi_y\\&-42\varphi_{xx}^2\psi_y+42\varphi_{xx}\varphi_x\psi_{xy}\\&-9\varphi_x^2\psi_{xxy})\varphi_{xxxx}\varphi_x^2-(210(6\varphi_{xx}^2\psi_y\\&-8\varphi_{xx}\varphi_x\psi_{xy}+3\varphi_x^2\psi_{xxy})\varphi_{xx}\\&-(\varphi_x^3\psi_y\alpha+80\psi_{xxxy})\varphi_x^3)\varphi_{xxx})\varphi_x))/(\varphi_x^5\psi_y),\end{aligned} \tag{33}$$

$$\begin{aligned}H_0=&-(\varphi_{xxxxxx}\varphi_x^4\psi_x-21\varphi_{xxxxx}\varphi_{xx}\varphi_x^3\psi_x\\&+6\varphi_{xxxxx}\varphi_x^4\psi_{xx}-35\varphi_{xxxx}\varphi_{xxx}\varphi_x^3\psi_x\\&+210\varphi_{xxxx}\varphi_{xx}^2\varphi_x^2\psi_x-105\varphi_{xxxx}\varphi_{xx}\varphi_x^3\psi_{xx}\\&+15\varphi_{xxxx}\varphi_x^4\psi_{xxx}+280\varphi_{xxx}^2\varphi_{xx}\varphi_x^2\psi_x\\&-70\varphi_{xxx}^2\varphi_x^3\psi_{xx}-1260\varphi_{xxx}\varphi_{xx}^3\varphi_x\psi_x\\&+840\varphi_{xxx}\varphi_{xx}^2\varphi_x^2\psi_{xx}-210\varphi_{xxx}\varphi_{xx}\varphi_x^3\psi_{xxx}\\&+\varphi_{xxx}\varphi_x^7\psi_x\alpha+20\varphi_{xxx}\varphi_x^4\psi_{xxxx}\\&+945\varphi_{xx}^5\psi_x-945\varphi_{xx}^4\varphi_x\psi_{xx}\\&+420\varphi_{xx}^3\varphi_x^2\psi_{xxx}-3\varphi_{xx}^2\varphi_x^6\psi_x\alpha\\&-105\varphi_{xx}^2\varphi_x^3\psi_{xxxx}+\varphi_{xx}\varphi_x^8\psi_x\beta\\&+3\varphi_{xx}\varphi_x^7\psi_{xx}\alpha+15\varphi_{xx}\varphi_x^4\psi_{xxxxx}\\&-\varphi_x^{11}\omega\psi-\varphi_x^{10}\psi_x\gamma-\varphi_x^9\psi_{xx}\beta\\&-\varphi_x^8\psi_{xxx}\alpha-\varphi_x^5.\end{aligned} \tag{34}$$

Proof. Applying a fiber preserving transformation (4), one obtains the following transformation of derivatives

$$\begin{aligned}u'(t) &= \frac{D_x\psi}{D_x\varphi}=\frac{\psi_x+y'\psi_y}{\varphi_x}=P(x,y,y'),\\
u''(t) &= \frac{D_xP}{D_x\varphi}=\frac{P_x+y'P_y+y''P_{y'}}{\varphi_x}\\
&= \frac{1}{\varphi_x^3}[(\varphi_x\psi_y)y''+(\varphi_x\psi_{yy})y'^2+(2\varphi_x\psi_{xy}\\
&\quad -\varphi_{xx}\psi_y)y'-\varphi_{xx}\psi_x+\varphi_x\psi_{xx}]\\
&= Q(x,y,y',y''),\\
u'''(t) &= \frac{D_xQ}{D_x\varphi}=\frac{Q_x+y'Q_y+y''Q_{y'}+y'''Q_{y''}}{\varphi_x}\\
&= \frac{1}{\varphi_x^5}[(\varphi_x^2\psi_y)y'''+(3\varphi_x^2\psi_{yy})y'y''\\
&\quad +3\varphi_x(\varphi_x\psi_{xy}-\varphi_{xx}\psi_y)y''+...]\\
&= R(x,y,y',y'',y'''),\\
u^{(4)}(t) &= \frac{D_xR}{D_x\varphi}\\
&= \frac{R_x+y'R_y+y''R_{y'}+y'''R_{y''}+y^{(4)}R_{y'''}}{\varphi_x}\\
&= \frac{1}{\varphi_x^7}[(\varphi_x^3\psi_y)y^{(4)}+(4\varphi_x^3\psi_{yy})y'y'''\\
&\quad +2\varphi_x^2(2\varphi_x\psi_{xy}-3\varphi_{xx}\psi_y)y'''+...]\\
&= S(x,y,y',y'',y''',y^{(4)}),\\
u^{(5)}(t) &= \frac{D_xS}{D_x\varphi}\\
&= (S_x+y'S_y+y''S_{y'}+y'''S_{y''}S_{y'''}+y^{(4)}S_{y'''}\\
&\quad +y^{(5)}S_{y^{(4)}})/\varphi_x\\
&= \frac{1}{\varphi_x^9}[(\varphi_x^4\psi_y)y^{(5)}+(5\varphi_x^4\psi_{yy})y'y^{(4)}\\
&\quad +5\varphi_x^3(\varphi_x\psi_{xy}-2\varphi_{xx}\psi_y)y^{(4)}+...]\\
&= V(x,y,y',y'',y''',y^{(4)},y^{(5)}),\\
u^{(6)}(t) &= \frac{D_xV}{D_x\varphi}\\
&= (V_x+y'V_y+y''V_{y'}+y'''V_{y''}+y^{(4)}V_{y'''}\\
&\quad +y^{(5)}V_{y^{(4)}}+y^{(6)}V_{y^{(5)}})/\varphi_x\\
&= \frac{1}{\varphi_x^{11}}[(\varphi_x^5\psi_y)y^{(6)}+3(2\varphi_x^5\psi_{yy})y'y^{(5)}\\
&\quad +3\varphi_x^4(2\varphi_x\psi_{xy}-5\varphi_{xx}\psi_y)y^{(5)}+...],\end{aligned}$$

where

$$
\begin{aligned}
y^{(6)} &+ ((\tfrac{6\psi_{yy}}{\psi_y})y' - (\tfrac{3(5\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy})}{(\varphi_x\psi_y)}))y^{(5)} \\
&+((\tfrac{15\psi_{yy}}{\psi_y})y'' + (\tfrac{15\psi_{yyy}}{\psi_y})y'^2 - (\tfrac{15(5\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy})}{(\varphi_x\psi_y)})y' \\
&+...)y^{(4)} + (\tfrac{10\psi_{yy}}{\psi_y})y'''^2 + ((((\tfrac{(60\psi_{yyy})}{\psi_y})y' \\
&+\tfrac{(-30(5\varphi_{xx}\psi_{yy} - 2\varphi_x\psi_{xyy}))}{(\varphi_x\psi_y)})y'' + \tfrac{(20\psi_{yyyy})}{\psi_y}y'^3 + ...)y''' \\
&+(\tfrac{15\psi_{yyy}}{\psi_y})y''^3 + ((\tfrac{45\psi_{yyyy}}{\psi_y})y'^2 + (\tfrac{-45(5\varphi_{xx}\psi_{yyy}+...)}{(\varphi_x\psi_y)})y' \\
&+\tfrac{15(3(7\varphi_{xx}^2\psi_{yy} - 5\varphi_{xx}\varphi_x\psi_{xyy} + \varphi_x^2\psi_{xxyy}) - 4\varphi_{xxx}\varphi_x\psi_{yy})}{(\varphi_x^2\psi_y)})y''^2 \\
&+((\tfrac{15\psi_{yyyyy}}{\psi_y})y'^4 + (\tfrac{-30(5\varphi_{xx}\psi_{yyyy} - 2\varphi_x\psi_{xyyyy})}{(\varphi_x\psi_y)})y'^3 \\
&+(\tfrac{-30(4\varphi_{xxx}\varphi_x\psi_{yyy} - 21\varphi_{xx}^2\psi_{yyy}+...)}{(\varphi_x^2\psi_y)})y'^2 + ...)y'' \\
&+(\tfrac{\psi_{yyyyyy}}{\psi_y})y'^6 + (\tfrac{-3(5\varphi_{xx}\psi_{yyyyy} - 2\varphi_x\psi_{xyyyyy})}{(\varphi_x\psi_y)})y'^5 \\
&+(\tfrac{-5(4\varphi_{xxx}\varphi_x\psi_{yyyy} - 21\varphi_{xx}^2\psi_{yyyy}+...)}{(\varphi_x^2\psi_y)})y'^4 + ... = 0.
\end{aligned}
$$

Denoting $A_i, B_i, C_i, D_i, E_i, F_i, G_i$ and H_i as equations (6)-(34), so we obtain the necessary form (5). These prove the theorem.

4 Sufficient Conditions and Linearizing Transformations

We have shown the previous section that every linearizable sixth-order odinary differential equation belong to the class of equation (5). In this section, we formulate the main theorems containing sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

Theorem 4.1 Sufficient conditions for equation (5) to be linearizable via a fiber preserving tranformation are as follows:

$$A_{0y} = A_{1x}, \tag{35}$$
$$B_3 = (5A_1)/2, \tag{36}$$
$$A_{1y} = (-5A_1^2 + 12B_2)/30, \tag{37}$$
$$A_{1x} = (-5A_0A_1 + 6B_1)/30, \tag{38}$$
$$C_0 = (5A_1)/3, \tag{39}$$
$$D_5 = 4B_2, \tag{40}$$
$$D_4 = 2B_1, \tag{41}$$
$$B_{2y} = (-2A_1B_2 + 9D_3)/12, \tag{42}$$
$$B_{2x} = (-2A_0B_2 + 3D_2)/12, \tag{43}$$
$$B_{0y} = (-2A_1B_0 + 3D_1)/12, \tag{44}$$
$$E_0 = B_2, \tag{45}$$
$$D_3 = (4F_2)/9, \tag{46}$$
$$F_1 = (3D_2)/2, \tag{47}$$
$$F_0 = (3D_1)/4, \tag{48}$$
$$F_{2y} = (-A_1F_2 + 18G_4)/6, \tag{49}$$
$$F_{2x} = (-2A_0F_2 + 9G_3)/12, \tag{50}$$
$$D_{1y} = (-A_1D_1 + 4G_2)/6, \tag{51}$$

$$
\begin{aligned}
D_{1x} =&(-150A_{0x}A_0A_1 + 180A_{0x}B_1 + 180B_{0x}A_1 \\
&- 25A_0^3A_1 + 30A_0^2B_1 + 90A_0A_1B_0 \\
&- 45A_0D_1 - 135A_1D_0 - 72B_0B_1 \\
&+ 270G_1)/270,
\end{aligned} \tag{52}
$$
$$G_{4y} = (-A_1G_4 + 90H_6)/6, \tag{53}$$
$$G_{4x} = (-A_0G_4 + 15H_5)/6, \tag{54}$$
$$G_{2y} = (-A_1G_2 + 36H_4)/6, \tag{55}$$
$$G_{1y} = (-A_1G_1 + 18H_3)/6, \tag{56}$$
$$
\begin{aligned}
G_{1x} =&(150A_{0x}A_0^2A_1 - 180A_{0x}A_0B_1 - 180A_{0x}A_1B_0 \\
&+ 270A_{0x}D_1 - 180B_{0x}A_0A_1 + 216B_{0x}B_1 \\
&+ 270D_{0x}A_1 + 25A_0^4A_1 - 30A_0^3B_1 - 120A_0^2A_1B_0 \\
&+ 45A_0^2D_1 + 180A_0A_1D_0 + 108A_0B_0B_1 \\
&- 90A_0G_1 + 72A_1B_0^2 - 360A_1G_0 - 108B_0D_1 \\
&- 162B_1D_0 + 2160H_2)/540,
\end{aligned} \tag{57}
$$
$$\lambda_{1y} = \lambda_{2y} = \lambda_{3y} = \lambda_{4y} = \lambda_{5y} = \lambda_{6y}, \lambda_{7y} = \lambda_{8y} = 0, \tag{58}$$

where

$$\lambda_1 = -30A_{0x} - 5A_0^2 + 12B_0, \tag{59}$$
$$\lambda_2 = -72B_{0x} - 3\lambda_{1x} - 12A_0B_0 - 2A_0\lambda_1 + 54D_0, \tag{60}$$
$$\lambda_3 = -108B_{0x} - 18A_0B_0 - 3A_0\lambda_1 + 81D_0 - 5\lambda_2, \tag{61}$$
$$
\begin{aligned}
\lambda_4 =&28350D_{0x} + 210\lambda_{2x} + 210\lambda_{3x} + 4725A_0D_0 \\
&+ 875A_0\lambda_2 + 175A_0\lambda_3 + 630B_0\lambda_1 \\
&- 37800G_0 + 3\lambda_1^2,
\end{aligned} \tag{62}
$$
$$
\begin{aligned}
\lambda_5 =&396900D_{0x} + 2100\lambda_{3x} + 66150A_0D_0 \\
&+ 12250A_0\lambda_2 + 2450A_0\lambda_3 + 8820B_0\lambda_1 \\
&- 529200G_0 + 150\lambda_1^2 - 21\lambda_4,
\end{aligned} \tag{63}
$$
$$
\begin{aligned}
\lambda_6 =&113400D_{0x}A_0 - 226800G_{0x} + \lambda_{4x} + \lambda_{5x} \\
&+ 18900A_0^2D_0 + 3500A_0^2\lambda_2 + 700A_0^2\lambda_3 \\
&+ 2520A_0B_0\lambda_1 - 189000A_0G_0 - 4200B_0\lambda_2 \\
&- 840B_0\lambda_3 - 3780D_0\lambda_1 + 567000H_1 \\
&- 360\lambda_1\lambda_2 - 60\lambda_1\lambda_3,
\end{aligned} \tag{64}
$$
$$
\begin{aligned}
\lambda_7 =&283500D_{0x}A_0 - 567000G_{0x} - 15\lambda_{4x} \\
&+ 47250A_0^2D_0 + 8750A_0^2\lambda_2 + 1750A_0^2\lambda_3 \\
&+ 6300A_0B_0\lambda_1 - 472500A_0G_0 - 10500B_0\lambda_2 \\
&- 2100B_0\lambda_3 - 9450D_0\lambda_1 + 1417500H_1 \\
&- 1075\lambda_1\lambda_2 - 250\lambda_1\lambda_3 - 6\lambda_6,
\end{aligned} \tag{65}
$$
$$
\begin{aligned}
\lambda_8 =&83349000D_{0x}A_0^2 - 100018800D_{0x}B_0 \\
&- 9525600D_{0x}\lambda_1 - 166698000G_{0x}A_0 \\
&- 3000564000H_{0y} + 500094000H_{1x} \\
&+ 252\lambda_{7x} + 13891500A_0^3D_0 + 2572500A_0^3\lambda_2 \\
&+ 514500A_0^3\lambda_3 + 1852200A_0^2B_0\lambda_1 \\
&- 138915000A_0^2G_0 - 16669800A_0B_0D_0
\end{aligned}
$$

$$
\begin{aligned}
&- 6174000A_0B_0\lambda_2 - 1234800A_0B_0\lambda_3\\
&- 4365900A_0D_0\lambda_1 + 500094000A_0H_1\\
&- 294000A_0\lambda_1\lambda_2 - 58800A_0\lambda_1\lambda_3\\
&- 500094000A_1H_0 - 2222640B_0^2\lambda_1\\
&+ 133358400B_0G_0 - 211680B_0\lambda_1^2\\
&+ 4630500D_0\lambda_2 + 926100D_0\lambda_3\\
&+ 18257400G_0\lambda_1 - 1260\lambda_1^3 - 147\lambda_1\lambda_4\\
&+ 3\lambda_1\lambda_5 + 171500\lambda_2^2 + 107800\lambda_2\lambda_3\\
&+ 14700\lambda_3^2. \qquad (66)
\end{aligned}
$$

Proof. For obtaining sufficient conditions, one has to solve the compatibility problem. Considering the representations of the coefficients $A_i, B_i, C_i, D_i, E_i, F_i, G_i$ and H_i through the unknown functions φ and ψ. We first rewrite the expressions (6) and (7) for A_1 and A_0 in the following forms

$$\psi_{yy} = (\psi_y A_1)/6, \qquad (67)$$

$$\psi_{xy} = (\psi_y(15\varphi_{xx} + \varphi_x A_0))/(6\varphi_x). \qquad (68)$$

The mixed derivative $(\psi_{yy})_x = (\psi_{xy})_y$ provides the condition (35). From equations (8)-(10), one gets the conditions (36)-(38). The relation $(A_{1x})_y = (A_{1y})_x$ provides the condition

$$B_{1y} = (12B_{2x} + 2A_0B_2 - A_1B_1)/6.$$

From equation (11), we have

$$\varphi_{xxx} = (-30A_{0x}\varphi_x^2+315\varphi_{xx}^2-5\varphi_x^2A_0^2+12\varphi_x^2B_0)/(210\varphi_x). \qquad (69)$$

Since $\varphi_y = 0$, then differentiating (69) with respect to y, one arrives at the condition

$$B_{1x} = (30A_{0x}A_1 + 72B_{0y} + 5A_0^2A_1 - 6A_0B_1)/36.$$

The relation $(B_{1x})_y = (B_{1y})_x$ provides the condition

$$B_{0yy} = (-B_{0y}A_1 + 3B_{2xx} + B_{2x}A_0)/3.$$

From equations (12)-(16), one gets the conditions (39)-(43). The relation $(B_{2x})_y = (B_{2y})_x$ provides the condition

$$D_{2y} = (18D_{3x} + 3A_0D_3 - A_1D_2)/6.$$

From equation (17), one gets the conditions (44). The relation $(B_{0y})_y = B_{0yy}$ provides the condition

$$
\begin{aligned}
D_{2x} =&(60A_{0x}B_2 + 90D_{1y} + 10A_0^2B_2 - 15A_0D_2 + 15A_1D_1\\
&- 24B_0B_2)/90.
\end{aligned}
$$

The relation $(D_{2x})_y = (D_{2y})_x$ provides the condition

$$D_{1yy} =(-5D_{1y}A_1 + 45D_{3xx} + 15D_{3x}A_0 + 3B_0D_3 - B_2D_1)/15.$$

One can determine α from equation (18). Since $\varphi = \varphi(x)$, then $\alpha_y = 0$, which yields the condition

$$
\begin{aligned}
D_{0y} =&(150A_{0x}A_0A_1 - 180A_{0x}B_1 - 180B_{0x}A_1\\
&+ 270D_{1x} + 25A_0^3A_1 - 30A_0^2B_1 - 90A_0A_1B_0\\
&+ 45A_0D_1 + 72B_0B_1)/810.
\end{aligned}
$$

From equations (19)-(24), one finds the conditions (45)-(50). The relation $(F_{2x})_y = (F_{2y})_x$ provides the condition

$$G_{3y} =(24G_{4x} + 4A_0G_4 - A_1G_3)/6.$$

From equation (25), one gets the conditions (51). The relation $(D_{1x})_y = (D_{1y})_x$ provides the condition

$$
\begin{aligned}
G_{2x} =&(-60A_{0x}A_0B_2 + 90A_{0x}D_2 + 72B_{0x}B_2 + 270G_{1y}\\
&- 10A_0^3B_2 + 15A_0^2D_2 + 36A_0B_0B_2 - 30A_0G_2\\
&+ 45A_1G_1 - 36B_0D_2 - 54B_2D_0)/180.
\end{aligned}
$$

From equation (26), one obtains the condition (52). One can determine β from equation (27). Since $\varphi = \varphi(x)$, then $\beta_y = 0$, which yields the conditiion

$$
\begin{aligned}
G_{0y} =& - (150A_{0x}A_0^2A_1 - 180A_{0x}A_0B_1 - 180A_{0x}A_1B_0\\
&+ 270A_{0x}D_1 - 180B_{0x}A_0A_1 + 216B_{0x}B_1\\
&+ 270D_{0x}A_1 - 540G_{1x} + 25A_0^4A_1 - 30A_0^3B_1\\
&- 120A_0^2A_1B_0 + 45A_0^2D_1 + 180A_0A_1D_0\\
&+ 108A_0B_0B_1 - 90A_0G_1 + 72A_1B_0^2 - 108B_0D_1\\
&- 162B_1D_0)/2160.
\end{aligned}
$$

Equations (28) and (29) provide the conditions (53) and (54). The relation $(G_{4x})_y = (G_{4y})_x$ provides the condition

$$H_{5y} =(36H_{6x} + 6A_0H_6 - A_1H_5)/6.$$

From equation (30), one gets the condition (55). The relation $(G_{2y})_x = (G_{2x})_y$ provides the condition

$$
\begin{aligned}
G_{1yy} =&(60A_{0x}A_0F_2 - 270A_{0x}G_3 - 72B_{0x}F_2 - 270G_{1y}A_1\\
&+ 3240H_{4x} + 10A_0^3F_2 - 45A_0^2G_3 - 36A_0B_0F_2\\
&+ 540A_0H_4 + 108B_0G_3 - 54B_2G_1 + 54D_0F_2)/810.
\end{aligned}
$$

From equation (31), one gets the condition (56). The relation $(G_{1y})_x = (G_{1x})_y$ provides the condition

$$
\begin{aligned}
H_{3y} =&(60A_{0x}A_0F_2 - 270A_{0x}G_3 - 72B_{0x}F_2 + 3240H_{4x}\\
&+ 10A_0^3F_2 - 45A_0^2G_3 - 36A_0B_0F_2 + 540A_0H_4\\
&- 405A_1H_3 + 108B_0G_3 + 54D_0F_2)/2430.
\end{aligned}
$$

From equation (32), one finds the condition (57). The relation $(G_{1x})_y = (G_{1y})_x$ provides the condition

$$
\begin{aligned}
H_{3x} =&(300A_{0x}A_0^2B_2 - 450A_{0x}A_0D_2 - 360A_{0x}B_0B_2\\
&+ 900A_{0x}G_2 - 360B_{0x}A_0B_2 + 540B_{0x}D_2\\
&+ 540D_{0x}B_2 + 10800H_{2y} + 50A_0^4B_2 - 75A_0^3D_2\\
&- 240A_0^2B_0B_2 + 150A_0^2G_2 + 270A_0B_0D_2\\
&+ 360A_0B_2D_0 - 1350A_0H_3 + 1800A_1H_2\\
&+ 144B_0^2B_2 - 360B_0G_2 - 720B_2G_0\\
&- 405D_0D_2)/8100.
\end{aligned}
$$

The relation $(H_{3x})_y = (H_{3y})_x$ provides the condition

$$\begin{aligned} H_{2yy} =&(90A_{0xx}A_0F_2 - 405A_{0xx}G_3 - 30A_{0x}A_0^2F_2 \\ &+ 135A_{0x}A_0G_3 + 72A_{0x}B_0F_2 - 3240A_{0x}H_4 \\ &- 108B_{0xx}F_2 - 162B_{0x}G_3 - 1620H_{2y}A_1 \\ &+ 4860H_{4xx} + 1620H_{4x}A_0 - 10A_0^4F_2 + 45A_0^3G_3 \\ &+ 48A_0^2B_0F_2 - 540A_0^2H_4 - 162A_0B_0G_3 \\ &- 54A_0D_0F_2 - 36B_0^2F_2 + 1620B_0H_4 \\ &- 324B_2H_2 + 243D_0G_3 + 108F_2G_0)/4860. \end{aligned}$$

One can determine γ from equation (33). Since $\varphi = \varphi(x)$, then $\gamma_y = 0$, which yields the condition

$$\begin{aligned} H_{1y} =&(750A_{0x}A_0^3A_1 - 900A_{0x}A_0^2B_1 - 1800A_{0x}A_0A_1B_0 \\ &+ 1350A_{0x}A_0D_1 + 1350A_{0x}A_1D_0 + 1080A_{0x}B_0B_1 \\ &- 2700A_{0x}G_1 - 900B_{0x}A_0^2A_1 + 1080B_{0x}A_0B_1 \\ &+ 1080B_{0x}A_1B_0 - 1620B_{0x}D_1 + 1350D_{0x}A_0A_1 \\ &- 1620D_{0x}B_1 - 2700G_{0x}A_1 + 16200H_{2x} \\ &+ 125A_0^5A_1 - 150A_0^4B_1 - 750A_0^3A_1B_0 \\ &+ 225A_0^3D_1 + 1125A_0^2A_1D_0 + 720A_0^2B_0B_1 \\ &- 450A_0^2G_1 + 900A_0A_1B_0^2 - 2250A_0A_1G_0 \\ &- 810A_0B_0D_1 - 1080A_0B_1D_0 + 2700A_0H_2 \\ &- 1350A_1B_0D_0 - 432B_0^2B_1 + 1080B_0G_1 \\ &+ 2160B_1G_0 + 1215D_0D_1)/40500. \end{aligned}$$

One can determine ω from equation (34). Since $\varphi = \varphi(x)$, then $\omega_y = 0$, which yields the derivative $\psi_{xxxxxxx}$. Forming the mixed derivative $(\psi_{xxxxxxx})_y$=$(\psi_{xy})_{xxxxxx}$, one obtains the condition

$$\begin{aligned} H_{0yy} =&(1125A_{0xx}A_0^3A_1 - 1350A_{0xx}A_0^2B_1 \\ &- 2700A_{0xx}A_0A_1B_0 + 2025A_{0xx}A_0D_1 \\ &+ 2025A_{0xx}A_1D_0 + 1620A_{0xx}B_0B_1 \\ &- 4050A_{0xx}G_1 - 1125A_{0x}A_0^4A_1 \\ &+ 1350A_{0x}A_0^3B_1 + 4500A_{0x}A_0^2A_1B_0 \\ &- 2025A_{0x}A_0^2D_1 - 4725A_{0x}A_0A_1D_0 \\ &- 3780A_{0x}A_0B_0B_1 + 4050A_{0x}A_0G_1 \\ &- 2160A_{0x}A_1B_0^2 + 5400A_{0x}A_1G_0 \\ &+ 3240A_{0x}B_0D_1 + 3240A_{0x}B_1D_0 \\ &- 32400A_{0x}H_2 - 1350B_{0xx}A_0^2A_1 \\ &+ 1620B_{0xx}A_0B_1 + 1620B_{0xx}A_1B_0 \\ &- 2430B_{0xx}D_1 + 900B_{0x}A_0^3A_1 \\ &- 1080B_{0x}A_0^2B_1 - 2700B_{0x}A_0A_1B_0 \\ &+ 1620B_{0x}A_0D_1 + 2430B_{0x}A_1D_0 \\ &+ 1944B_{0x}B_0B_1 - 4860B_{0x}G_1 \\ &+ 2025D_{0xx}A_0A_1 - 2430D_{0xx}B_1 \\ &- 675D_{0x}A_0^2A_1 + 810D_{0x}A_0B_1 \\ &+ 1620D_{0x}A_1B_0 - 2430D_{0x}D_1 \\ &- 4050G_{0xx}A_1 - 1620G_{0x}B_1 \\ &- 60750H_{0y}A_1 + 24300H_{2xx} + 8100H_{2x}A_0 \\ &- 250A_0^6A_1 + 300A_0^5B_1 + 1800A_0^4A_1B_0 \\ &- 450A_0^4D_1 - 2475A_0^3A_1D_0 - 1800A_0^3B_0B_1 \\ &+ 900A_0^3G_1 - 3330A_0^2A_1B_0^2 + 4275A_0^2A_1G_0 \\ &+ 2160A_0^2B_0D_1 + 2430A_0^2B_1D_0 - 5400A_0^2H_2 \\ &+ 6345A_0A_1B_0D_0 - 10125A_0A_1H_1 + 2268A_0B_0^2B_1 \\ &- 3240A_0B_0G_1 - 4050A_0B_1G_0 - 2835A_0D_0D_1 \\ &+ 10125A_1^2H_0 + 972A_1B_0^3 - 6210A_1B_0G_0 \\ &- 2430A_1D_0^2 - 1458B_0^2D_1 - 3402B_0B_1D_0 \\ &+ 14580B_0H_2 + 12150B_1H_1 - 24300B_2H_0 \\ &+ 4860D_0G_1 + 5670D_1G_0)/364500. \end{aligned}$$

One can rewrite the expression of equation (69) as

$$\varphi_{xxx} = (315\varphi_{xx}^2 + \varphi_x^2\lambda_1)/(210\varphi_x), \tag{70}$$

where λ_1 is in the form of equation (59). Rewriting the representation of the derivative $\psi_{xxxxxxx}$ as

$$\begin{aligned} \psi_{xxxxxxx} =&(67512690000\varphi_{xx}^5\psi_x \\ &- 337563450000\varphi_{xx}^4\varphi_x\psi_{xx} \\ &+ 450084600000\varphi_{xx}^3\varphi_x^2\psi_{xxx} \\ &- 2143260000\varphi_{xx}^3\varphi_x^2\psi_x\lambda_1 \\ &+ 416745000\varphi_{xx}^3\varphi_x^2\lambda_2\psi \\ &- 225042300000\varphi_{xx}^2\varphi_x^3\psi_{xxxx} \\ &+ 4286520000\varphi_{xx}^2\varphi_x^3\psi_{xx}\lambda_1 \\ &+ 238140000\varphi_{xx}^2\varphi_x^3\psi_x\lambda_3 \\ &+ 396900\varphi_{xx}^2\varphi_x^3\lambda_4\psi \\ &+ 45008460000\varphi_{xx}\varphi_x^4\psi_{xxxxx} \\ &- 2143260000\varphi_{xx}\varphi_x^4\psi_{xxx}\lambda_1 \\ &+ 59535000\varphi_{xx}\varphi_x^4\psi_{xx}(-7\lambda_2 - 4\lambda_3) \\ &+ 56700\varphi_{xx}\varphi_x^4\psi_x\lambda_5 + 13230\varphi_{xx}\varphi_x^4\lambda_6\psi \\ &+ 285768000\varphi_x^5\psi_{xxxx}\lambda_1 \\ &+ 15876000\varphi_x^5\psi_{xxx}(7\lambda_2 + 3\lambda_3) \\ &+ 11340\varphi_x^5\psi_{xx}(-60\lambda_1^2 - 7\lambda_4 - 2\lambda_5) \\ &+ 1512\varphi_x^5\psi_x\lambda_7 + 3000564000\varphi_x^5\psi_yH_0 \\ &+ \varphi_x^5\lambda_8\psi)/(3000564000\varphi_x^5), \end{aligned} \tag{71}$$

where $\lambda_2, \lambda_3, ..., \lambda_8$ are in the form of equations (60)-(66). The relations $(A_{0x})_y = (A_{0y})_x$, $(\lambda_{1x})_y = (\lambda_{1y})_x$, $(B_{0x})_y = (B_{0y})_x$, $(\lambda_{2x})_y = (\lambda_{2y})_x$, $(\lambda_{3x})_y = (\lambda_{3y})_x$, $(\lambda_{5x})_y = (\lambda_{5y})_x$, $(\lambda_{4x})_y = (\lambda_{4y})_x$ and $(\lambda_{7x})_y = (\lambda_{7y})_x$ provide the conditions (58).

Finally, the representations for α, β, γ and ω become

$$\alpha = \lambda_2/(54\varphi_x^3), \quad (72)$$
$$\beta = (-3150\varphi_{xx}\lambda_2 - \varphi_x\lambda_4)/(37800\varphi_x^5), \quad (73)$$
$$\gamma = (94500\varphi_{xx}^2\lambda_2 + 60\varphi_{xx}\varphi_x\lambda_4 + \varphi_x^2\lambda_6)/(567000\varphi_x^7), \quad (74)$$
$$\omega = (-416745000\varphi_{xx}^3\lambda_2 - 396900\varphi_{xx}^2\varphi_x\lambda_4 - 13230\varphi_{xx}\varphi_x^2\lambda_6 - \varphi_x^3\lambda_8)/(3000564000\varphi_x^9). \quad (75)$$

This completes the proof of Theorem 4.1.

Corollary 4.2 Provided that the sufficient conditions in Theorem 4.1 are satisfied, the transformation (4) mapping equation (5) to a linear equation (2) is obtained by solving the compatible system of equations (67), (68), (70) and (71) for the functions $\varphi(x)$ and $\psi(x, y)$. Finally, the coefficients α, β, γ and ω of the resulting linear equation (2) are given by equations (72), (73), (74) and (75).

5 Illustration of the Linearization Theorems

Example 5.1 Consider the nonlinear sixth-order ordinary differential equation

$$24y'^2x + 12y'y''x^2 + 240y'y''' + 120y'y^{(4)}x + 12y'y^{(5)}x^2 + 24y'y + 180y''^2 + 240y''y'''x + 30y''y^{(4)}x^2 + 24y''xy + 20y'''^2 + 4y'''x^2y + 60y^{(4)}y + 24y^{(5)}xy + x^2y(2y^{(6)} + y) = 0. \quad (76)$$

It is an equation of the form (5) in Theorem 3.1 with the coefficients

$$A_1 = \frac{6}{y},\ A_0 = \frac{12}{x},\ B_3 = \frac{15}{y},\ B_2 = 0,\ B_1 = \frac{60}{xy},$$
$$B_0 = \frac{30}{x^2},\ C_0 = \frac{10}{y},\ D_5 = 0, D_4 = \frac{120}{xy},\ D_3 = 0,$$
$$D_2 = 0,\ D_1 = \frac{120}{x^2y},\ D_0 = 0,\ E_0 = 0,\ F_2 = 0,$$
$$F_1 = 0,\ F_0 = \frac{90}{x^2y},\ G_4 = 0,\ G_3 = 0,\ G_2 = 0,$$
$$G_1 = \frac{6}{y},\ G_0 = \frac{12}{x},\ H_6 = 0,\ H_5 = 0,\ H_4 = 0,$$
$$H_3 = 0,\ H_2 = \frac{12}{xy},\ H_1 = \frac{12}{x^2},\ H_0 = \frac{y}{2},\ \lambda_1 = 0,$$
$$\lambda_2 = 180,\ \lambda_3 = -378,\ \lambda_4 = 0,\ \lambda_5 = 0,\ \lambda_6 = 0,$$
$$\lambda_7 = 0,\ \lambda_8 = 300056400.$$

One can check that these coefficients obey the conditions in Theorem 4.1. Hence an equation (76) is linearizable via a fiber preserving transformation. Applying Corollary 4.2, the linearizing transformation is found by solving the following equations

$$\psi_{yy} = \psi_y/y, \quad (77)$$
$$\psi_{xy} = \psi_y(5\varphi_{xx}x + 4\varphi_x)/(2\varphi_x x), \quad (78)$$
$$\varphi_{xxx} = (3\varphi_{xx}^2)/(2\varphi_x), \quad (79)$$
$$\psi_{xxxxxx} = (45\varphi_{xx}^5\psi_x - 225\varphi_{xx}^4\varphi_x\psi_{xx} + 300\varphi_{xx}^3\varphi_x^2\psi_{xxx} + 30\varphi_{xx}^3\varphi_x^2\psi - 150\varphi_{xx}^2\varphi_x^3\psi_{xxxx} - 60\varphi_{xx}^2\varphi_x^3\psi_x + 30\varphi_{xx}\varphi_x^4\psi_{xxxxx} + 30\varphi_{xx}\varphi_x^4\psi_{xx} - 4\varphi_x^5\psi_{xxx} + \varphi_x^5\psi_y y - 2\varphi_x^5\psi)/(2\varphi_x^5). \quad (80)$$

Consider equation (79), one can choose the particular solution

$$\varphi = x.$$

So that system of equations (77), (78) and (80) are written as

$$\psi_{yy} = \frac{\psi_y}{y}, \quad (81)$$
$$\psi_{xy} = \frac{2\psi_y}{x}, \quad (82)$$
$$\psi_{xxxxxx} = \frac{-4\psi_{xxx} + \psi_y y - 2\psi}{2}. \quad (83)$$

Consider equations (81) and (82), we have

$$\psi_y = Kx^2y, \qquad K = constant.$$

Hence,

$$\psi = \frac{Kx^2y^2}{2} + f(x).$$

Since one can use any particular solution, we set $K = 2, f(x) = 0$ and take $\psi = x^2y^2$. One can readily verify that this function solves equation (83) as well. Hence, one obtains the linearizing transformation

$$t = x, \quad u = x^2y^2. \quad (84)$$

From Corollary 4.2 the coefficients α, β, γ and ω of the resulting linear equation (2) are

$$\alpha = 0,\ \beta = 2,\ \gamma = 0,\ \omega = 1.$$

Hence, the nonlinear equation (76) can be mapped by transformation (84) into the linear equation

$$u^{(6)} + 2u''' + u = 0. \quad (85)$$

The solution of equation (85) is

$$u(t) = e^{-t}(C_0 + C_1t) + e^{\frac{t}{2}}((C_2 + C_3t)\cos(\frac{\sqrt{3}}{2}t) + (C_4 + C_5t)\sin(\frac{\sqrt{3}}{2}t)), \quad (86)$$

where C_0, C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. Substituting equation (84) into equation (86), we get the solution

$$x^2y^2 = e^{-x}(C_0 + C_1x) + e^{\frac{x}{2}}((C_2 + C_3x)\cos(\frac{\sqrt{3}}{2}x) + (C_4 + C_5x)\sin(\frac{\sqrt{3}}{2}x)).$$

Example 5.2 Consider the nonlinear sixth-order ordinary differential equation

$$\begin{aligned}&y'^6 - 15y'^5 + 15y'^4y'' + 85y'^4 - 150e^y y'^3y'' + 20y'^3y'''\\&+ y'^3(e^{3x} - 225) + 45y'^2y''^2 + 510y'^2y'' - 150y'^2y'''\\&+ 15y'^2y^{(4)} + y'^2(-3e^{3x} + 274) - 225y'y''^2 + 60y'y''y'''\\&+ 3y'y''(e^{3x} - 225) + 340y'y''' - 75y'y^{(4)} + 6y'y^{(5)}\\&+ 2y'(e^{3x} - 60) + 15y''^3 + 255y''^2 - 150y''y'''\\&+ 15y''y^{(4)} + y''(-3e^{3x} + 274) + 10y'''^2\\&+ y'''(e^{3x} - 225) + 85y^{(4)} - 15y^{(5)} + y^{(6)} = 0. \qquad (87)\end{aligned}$$

It is an equation of the form (5) in Theorem 3.1 with the coefficients

$$\begin{aligned}&A_1 = 6,\ A_0 = -15,\ B_3 = 15,\ B_2 = 15,\ B_1 = -75,\\&B_0 = 85,\ C_0 = 10,\ D_5 = 60,\ D_4 = -150,\ D_3 = 20,\\&D_2 = -150,\ D_1 = 340, D_0 = e^{3x} - 225,\ E_0 = 15,\\&F_2 = 45,\ F_1 = -225,\ F_0 = 255,\ G_4 = 15, G_3 = -150,\\&G_2 = 510,\ G_1 = 3(e^{3x} - 225),\ G_0 = -3e^{3x} + 274,\\&H_6 = 1,\ H_5 = -15,\ H_4 = 85,\ H_3 = e^{3x} - 225,\\&H_2 = -3e^{3x} + 274,\ H_1 = 2(e^{3x} - 60),\ H_0 = 0,\\&\lambda_1 = -150,\ \lambda_2 = 54e^{3x}, \lambda_3 = -189e^{3x},\\&\lambda_4 = -170100e^{3x},\ \lambda_5 = 1190700(e^{3x} + 1),\\&\lambda_6 = 5103000e^{3x},\ \lambda_7 = -18852750e^{3x},\\&\lambda_8 = -22504230000e^{3x}.\end{aligned}$$

One can check that these coefficients obey the conditions in Theorem 4.1. Hence an equation (76) is linearizable via a fiber preserving transformation. Applying Corollary 4.2, the linearizing transformation is found by solving the following equations

$$\psi_{yy} = \psi_y, \qquad (88)$$

$$\psi_{xy} = 5\psi_y(\varphi_{xx} - \varphi_x)/(2\varphi_x), \qquad (89)$$

$$\varphi_{xxx} = (3\varphi_{xx}^2 - \varphi_x^2)/(2\varphi_x), \qquad (90)$$

$$\begin{aligned}\psi_{xxxxxx} =& (45\varphi_{xx}^5\psi_x - 225\varphi_{xx}^4\varphi_x\psi_{xx} + 300\varphi_{xx}^3\varphi_x^2\psi_{xxx}\\&+ 150\varphi_{xx}^3\varphi_x^2\psi_x + 15e^{3x}\varphi_{xx}^3\varphi_x^2\psi\\&- 150\varphi_{xx}^2\varphi_x^3\psi_{xxxx} - 300\varphi_{xx}^2\varphi_x^3\psi_{xx}\\&- 30e^{3x}\varphi_{xx}^2\varphi_x^3\psi_x - 45e^{3x}\varphi_{xx}^2\varphi_x^3\psi\\&+ 30\varphi_{xx}\varphi_x^4\psi_{xxxxx} + 150\varphi_{xx}\varphi_x^4\psi_{xxx}\\&+ 15e^{3x}\varphi_{xx}\varphi_x^4\psi_{xx} + 45e^{3x}\varphi_{xx}\varphi_x^4\psi_x\\&+ 45\varphi_{xx}\varphi_x^4\psi_x + 45e^{3x}\varphi_{xx}\varphi_x^4\psi\\&- 20\varphi_x^5\psi_{xxxx} - 2e^{3x}\varphi_x^5\psi_{xxx}\\&- 9e^{3x}\varphi_x^5\psi_{xx} - 23\varphi_x^5\psi_{xx} - 19e^{3x}\varphi_x^5\psi_x\\&- 15e^{3x}\varphi_x^5\psi)/(2\varphi_x^5). \qquad (91)\end{aligned}$$

Consider equation (90), one can choose the particular solution

$$\varphi = e^x.$$

So that system of equations (88), (89) and (91) as rewritten as

$$\psi_{yy} = \psi_y, \qquad (92)$$

$$\psi_{xy} = 0, \qquad (93)$$

$$\begin{aligned}\psi_{xxxxxx} =& 15\psi_{xxxxx} - 85\psi_{xxxx} - e^{3x}\psi_{xxx}\\&+ 225\psi_{xxx} + 3e^{3x}\psi_{xx} - 274\psi_{xx}\\&- 2e^{3x}\psi_x + 120\psi_x. \qquad (94)\end{aligned}$$

Consider equation (92) and (93), we have

$$\psi_y = Ke^y, \qquad K = constant.$$

Hence,

$$\psi = Ke^y + f(x).$$

Since one can use any particular solution, we set $K = 1, f(x) = 0$ and take $\psi = Ke^y$. One can readily verify that this function solves equation (94) as well. Hence, one obtains the linearizing transformation

$$t = e^x, \quad u = e^y. \qquad (95)$$

From Corollary 4.2 the coefficients α, β, γ and ω of the resulting linear equation (2) are

$$\alpha = 1,\ \beta = \gamma = \omega = 0.$$

Hence, the nonlinear equation (87) can be mapped by transformation (95) into the linear equation

$$u^{(6)} + u''' = 0. \qquad (96)$$

The solution of equation (96) is

$$u(t) = (C_0 + C_1t + C_2t^2) + C_3e^{-t} + e^{\frac{t}{2}}(C_4\cos(\frac{\sqrt{3}}{2}t) + C_5\sin(\frac{\sqrt{3}}{2}t)), \qquad (97)$$

where C_0, C_1, C_2, C_3, C_4 and C_5 are arbitrary constants. Substituting equation (95) into equation (97), we get the solution

$$e^y = (C_0 + C_1e^x + C_2e^{x^2)} + C_3e^{-e^x} + e^{\frac{e^x}{2}}(C_4\cos(\frac{\sqrt{3}}{2}e^x) + C_5\sin(\frac{\sqrt{3}}{2}e^x)).$$

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