Discrete-time Geo/Geo/1 Queue with Negative Customers and Working Breakdowns

Tao Li and Liyuan Zhang

Abstract—**This paper considers a discrete-time Geo/Geo/1 queue with server breakdowns and repairs. If the server is busy in a normal state, a breakdown is represented by the arrival of a negative customer, and the failed server still works at a lower service rate rather than stopping the service completely. Applying the matrix-analytic method, we derive the necessary and sufficient condition for the system to be stable. Using the probability generating function, we deal with the joint distribution of the server state and the number of customers in the system. The relation between our discrete-time model and its continuous-time counterpart is also investigated. Finally, some numerical examples are presented to show the effect of various system parameters on the queueing characteristics. Furthermore, an operating cost function is formulated and the optimum service rate in the working breakdown period is obtained.**

Index Terms—**Geo/Geo/1, negative customer, working breakdown, repair.**

I. Introduction

Queues with negative arrivals, called G-queues, were first introduced by Gelenbe [1]. A negative customer will vanish if it arrives to an empty queue, and negative customers cannot accumulate in a queue and do not receive services. In some cases, an arrival negative customer makes the server break down when the system is in a normal state. Queues with server breakdowns can be applied in manufacturing systems, production systems, telecommunication systems, inventory systems and computer systems. Wang and Zhang [2] investigated a Geo/G/1 retrial queue with negative customers, where the server is subject to failure due to the negative arrivals. Wu and Lian [3] analyzed an M/G/1 retrial G-queue with unreliable server under Bernoulli vacation schedule. Using the matrix geometric method, Rakhee et al. [4] studied a Geo/Geo/1 queue with unreliable server, where the breakdown of the server is represented by the arrival of a negative customer. For more queueing models with server breakdowns and repairs, readers can refer to Do [5], Yang et al. [6], Gao and Wang [7] and Tsai et al. [8].

In the study of queueing systems, it is usually assumed that the server stops the service completely in a breakdown period. However, in many practical cases, the failed server still can serve a customer at a lower service rate. This type of breakdown is called as working breakdown introduced by Kalidass and Kasturi [9], and the M/M/1 queue they analyzed can be applied to studying the behavior of communication system or machine replacement problem. Liu and Song [10] extended the model in [9] to an M^X/M/1 queue, where the customers arrive in batches. Kim and Lee [11] discussed an M/G/1 queue with disasters and working breakdowns, where all present customers leave the system when a disaster happens. Recently, Jiang and Liu [12] investigated a GI/M/1 queue with disasters and working breakdowns in a multi-phase service environment. Using the matrix geometric method, Ma et al. [13] computed the steady-state distribution of an M/M/1 queue with multiple vacations and working breakdowns. Readers can also refer to Liou [14], [15] and Yen et al. [16]. Note that working breakdown is different from working vacation introduced by Servi and Finn [17]. A working vacation is taken only when the system becomes empty, while a working breakdown can occur at any time point. Some authors like Zhang and Liu [18] and Rajadurai et al. [19], [20] have studied working vacation queues with unreliable server, where the server is subject to breakdown due to the negative arrivals.

The concept of working breakdown introduced by Kalidass and Kasturi [9] does make sense in real life. For example, when a computer is infected by a virus, it may still be able to perform but in a slower service rate. Parallel to continuous-time queues, discrete-time models are more suitable to analyze computer systems or communication systems, the reason is that these systems operate on a discrete-time basis where the events (arrival of packets and their forward transmissions) only take place at regularly spaced epochs. A detailed discussion and application of discrete-time queues can be found in Woodward [21] and Ramasamy et al. [22]. Up to now, a few authors have discussed continuous-time queues with working breakdowns, but their discrete-time counterparts seem to receive very little attention in the literature. Inspired by the natural and reasonable applications of discrete-time queues, we deal with a Geo/Geo/1 queue with working breakdowns in this paper, where a breakdown occurs due to a negative arrival. In order to make a comparison with the continuous-time system [9], the killing discipline is not considered in this model.

This paper is organized as follows. Section 2 gives a brief description of the model. Using the matrix-analytic method, the stable condition is obtained. In Section 3, we deal with the steady state joint distribution of the server state and the number of customers in the system. Section 4 gives the relation between our discrete-time queue and its corresponding continuous-time model. In Section 5, some numerical examples and cost optimization analysis are presented. Finally, Section 6 concludes the paper.

II. Model description and stability condition

In this paper, for any real number $x \in [0, 1]$, we denote $\bar{x} = 1 - x$. We consider a early arrival system, and the

Manuscript received January 24, 2017; revised September 24, 2017. This work was supported by the National Natural Science Foundation of China 11301306.

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Geo/Geo/1 queue with two types of customers and working breakdowns is given as follows:

(1) Assume a potential departure (due to the completion of a service) occurs in (n^-, n), and the distribution of service time S_b in a normal period is

$$P\{S_b = k\} = \mu\bar{\mu}^{k-1}, \quad k \geq 1,\ 0 < \mu < 1.$$

A potential positive customer arrival takes place in (n, n^+), and inter-arrival times of positive customers follow the geometric distribution:

$$P\{A = k\} = p\bar{p}^{k-1}, \quad k \geq 1,\ 0 < p < 1.$$

(2) There are at least two possible choices for the location of a negative arrival: (i) in (n, n^+) and immediately after a potential positive arrival; (ii) in (n, n^+) and immediately before a potential positive arrival. In this paper, we choose (i) as Atencia and Moreno [23] did. Inter-arrival times of negative customers follow the geometric distribution:

$$P\{B = k\} = \delta\bar{\delta}^{k-1}, \quad k \geq 1,\ 0 < \delta < 1.$$

(3) In a normal period, the arrival of a negative customer makes the server break down if the server is busy (i.e., there are customers in the system) at that moment. When the server breaks down, the service rate decreases, and the distribution of service time S_w in a breakdown period is

$$P\{S_w = k\} = \eta\bar{\eta}^{k-1}, \quad k \geq 1,\ 0 < \eta < 1,$$

where $\eta < \mu$.

(4) When the server breaks down, a repair procedure starts immediately. We assume that the beginning and ending of repair occur at the slot n, and the repair times follow the geometric distribution:

$$P\{R = k\} = \theta\bar{\theta}^{k-1}, \quad k \geq 1,\ 0 < \theta < 1.$$

(5) The arriving negative customer has nothing to do with the server when the server is free or under repairing, and negative customers cannot accumulate in a queue and do not receive services.

We assume that inter-arrival times, service times and repair times are mutually independent.

Here we give a practical application of this model. In a computer system, date packets arrive at the system according to a Bernoulli process with parameter p. When the computer system is operating in a normal state, the processing time (service time) for each data packet is geometrically distributed with parameter μ. The computer system may be subject to the invasion of a virus during the normal operation period, and the time interval until the presence of virus follows a geometric distribution with parameter δ. If the computer system is invaded by a virus, the CPU of the computer will not stop running completely and still can work at a lower speed. Under this situation, the processing time for each data packet is governed by a geometric distribution with parameter η ($\eta < \mu$). Meanwhile, the antivirus software begins to repair the system until the virus is cleared, and the repair times follow a geometric distribution with parameter θ.

Let J_n be the state of server at time n^+, and Q_n be the number of customers in the system at time n^+. There are two possible states of the single server as follows: (i) $J_n = 1$, the server is in a normal period at time n^+; (ii) $J_n = 2$, the server is defective (in a working breakdown period) at time n^+. Then, $\{J_n, Q_n\}$ is a Markov chain with state space

$$\Omega = \{(j, k), j = 1, 2, k \geq 0\}.$$

Remark 1: In order to make a comparison with the continuous-time model Kalidass and Kasturi [9], we do not consider killing discipline in this paper. If we consider removal discipline, such as RCH, i.e., the negative arrival can remove a customer being in service, the system will have a similar solution but is not pursued in the present work.

Using the lexicographical sequence for the states, the transition probability matrix can be written as

$$\widetilde{P} = \begin{pmatrix} A_{00} & A_{01} & & & \\ B_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_{00} = \begin{pmatrix} \bar{p} & 0 \\ \theta\bar{p} & \bar{\theta}\bar{p} \end{pmatrix}; \quad A_0 = \begin{pmatrix} \bar{\mu}p\bar{\delta} & \bar{\mu}p\delta \\ \bar{\eta}\theta p\bar{\delta} & \bar{\eta}\bar{\theta}p + \bar{\eta}\theta p\delta \end{pmatrix};$$

$$B_{10} = \begin{pmatrix} \mu\bar{p} & 0 \\ \eta\theta\bar{p} & \eta\bar{\theta}\bar{p} \end{pmatrix}; \quad A_{01} = \begin{pmatrix} p\bar{\delta} & p\delta \\ \theta p\bar{\delta} & \bar{\theta}p + \theta p\delta \end{pmatrix};$$

$$A_1 = \begin{pmatrix} \bar{\mu}\bar{p}\bar{\delta} + \mu p\bar{\delta} & \bar{\mu}\bar{p}\delta + \mu p\delta \\ \bar{\eta}\theta\bar{p}\bar{\delta} + \eta\theta p\bar{\delta} & \bar{\eta}\bar{\theta}\bar{p} + \eta\bar{\theta}p + \bar{\eta}\theta\bar{p}\delta + \eta\theta p\delta \end{pmatrix};$$

$$A_2 = \begin{pmatrix} \mu\bar{p}\bar{\delta} & \mu\bar{p}\delta \\ \eta\theta\bar{p}\bar{\delta} & \eta\bar{\theta}\bar{p} + \eta\theta\bar{p}\delta \end{pmatrix}.$$

Due to the block structure of transition probability matrix, $\{J_n, Q_n\}$ is called a QBD process.

Theorem 1: The QBD process $\{J_n, Q_n\}$ is positive recurrent if and only if $\theta\bar{\delta}(\mu\bar{p} - \bar{\mu}p) > \delta(\bar{\eta}p - \eta\bar{p})$.

Proof: Let

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} \bar{\delta} & \delta \\ \theta\bar{\delta} & \bar{\theta} + \theta\delta \end{pmatrix},$$

the Theorem 7.2.3 in [24] states that the QBD is positive recurrent if and only if $\pi A_2 e > \pi A_0 e$, where e is a column vector with all elements equal to one, and π is the unique solution of the system $\pi A = \pi$, $\pi e = 1$. After some algebraic manipulation, we have $\pi = \left(\frac{\theta\bar{\delta}}{\theta\bar{\delta}+\delta}, \frac{\delta}{\theta\bar{\delta}+\delta}\right)$, and the QBD process is positive recurrent if and only if $\mu\theta\bar{p}\bar{\delta} + \eta\bar{p}\delta > \bar{\mu}\theta p\bar{\delta} + \bar{\eta}p\delta \Leftrightarrow \theta\bar{\delta}(\mu\bar{p} - \bar{\mu}p) > \delta(\bar{\eta}p - \eta\bar{p})$. ■

III. Steady state analysis

If $\theta\bar{\delta}(\mu\bar{p} - \bar{\mu}p) > \delta(\bar{\eta}p - \eta\bar{p})$, let (J, Q) be the stationary limit of the process (J_n, Q_n), and denote

$$\begin{aligned} P_{j,k} &= P\{J = j, Q = k\} \\ &= \lim_{n\to\infty} P\{J_n = j, Q_n = k\}, (j, k) \in \Omega. \end{aligned}$$

Using the transition probability matrix $\widetilde{P}$, the balance equations governing the system are given by

$$P_{1,0} = \bar{p}P_{1,0} + \mu\bar{p}P_{1,1} + \theta\bar{p}P_{2,0} + \eta\theta\bar{p}P_{2,1}, \quad (1)$$

$$\begin{aligned} P_{1,1} &= p\bar{\delta}P_{1,0} + (\bar{\mu}\bar{p}\bar{\delta} + \mu p\bar{\delta})P_{1,1} + \mu\bar{p}\bar{\delta}P_{1,2} \\ &\quad + \theta p\bar{\delta}P_{2,0} + (\bar{\eta}\theta\bar{p}\bar{\delta} + \eta\theta p\bar{\delta})P_{2,1} + \eta\theta\bar{p}\bar{\delta}P_{2,2}, \end{aligned} \quad (2)$$

$$\begin{aligned} P_{1,k} &= \bar{\mu}p\bar{\delta}P_{1,k-1} + (\bar{\mu}\bar{p}\bar{\delta} + \mu p\bar{\delta})P_{1,k} + \mu\bar{p}\bar{\delta}P_{1,k+1} \\ &\quad + \bar{\eta}\theta p\bar{\delta}P_{2,k-1} + (\bar{\eta}\theta\bar{p}\bar{\delta} + \eta\theta p\bar{\delta})P_{2,k} \\ &\quad + \eta\theta\bar{p}\bar{\delta}P_{2,k+1}, \quad k \geq 2, \end{aligned} \quad (3)$$

$$P_{2,0} = \bar{\theta}\bar{p}P_{2,0} + \eta\bar{\theta}\bar{p}P_{2,1}, \quad (4)$$

$$\begin{aligned} P_{2,1} &= p\delta P_{1,0} + (\bar{\mu}\bar{p}\delta + \mu p\delta)P_{1,1} + \mu\bar{p}\delta P_{1,2} \\ &\quad + (\bar{\theta}p + \theta p\delta)P_{2,0} + (\bar{\eta}\bar{\theta}\bar{p} + \eta\bar{\theta}p + \bar{\eta}\theta\bar{p}\delta + \eta\theta p\delta)P_{2,1} \\ &\quad + (\eta\bar{\theta}\bar{p} + \eta\theta\bar{p}\delta)P_{2,2}, \end{aligned} \quad (5)$$

$$\begin{aligned} P_{2,k} &= \bar{\mu}p\delta P_{1,k-1} + (\bar{\mu}\bar{p}\delta + \mu p\delta)P_{1,k} \\ &\quad + \mu\bar{p}\delta P_{1,k+1} + (\bar{\eta}\bar{\theta}p + \bar{\eta}\theta p\delta)P_{2,k-1} \\ &\quad + (\bar{\eta}\bar{\theta}\bar{p} + \eta\bar{\theta}p + \bar{\eta}\theta\bar{p}\delta + \eta\theta p\delta)P_{2,k} \\ &\quad + (\eta\bar{\theta}\bar{p} + \eta\theta\bar{p}\delta)P_{2,k+1}, \quad k \geq 2. \end{aligned} \quad (6)$$

Define partial probability generating functions $P_1(z) = \sum_{k=1}^{\infty} P_{1,k}z^k$, $P_2(z) = \sum_{k=0}^{\infty} P_{2,k}z^k$. Multiplying (2) and (3) by appropriate powers of z, and summing over $k \geq 1$, we can obtain

$$\begin{aligned} &\Big(-\bar{\mu}p\bar{\delta}z^2 + (1 - \bar{\mu}\bar{p}\bar{\delta} - \mu p\bar{\delta})z - \mu\bar{p}\bar{\delta}\Big)P_1(z) \\ &- \theta\bar{\delta}\Big(\bar{\eta}pz^2 + (\bar{\eta}\bar{p} + \eta p)z + \eta\bar{p}\Big)P_2(z) \\ &= p\bar{\delta}z(z-1)P_{1,0} + \eta\theta\bar{\delta}(pz + \bar{p})(z-1)P_{2,0}. \end{aligned} \quad (7)$$

Multiplying (4), (5) and (6) by appropriate powers of z, and summing over $k \geq 0$, we can get

$$\begin{aligned} &-\Big(\bar{\mu}p\delta z^2 + (\bar{\mu}\bar{p}\delta + \mu p\delta)z + \mu\bar{p}\delta\Big)P_1(z) \\ &+ \Big(-\bar{\eta}\bar{\theta}pz^2 + (1 - \bar{\eta}\bar{\theta}\bar{p} - \eta\bar{\theta}p)z - \eta\bar{\theta}\bar{p}\Big)P_2(z) \\ &- \theta\delta\Big(\bar{\eta}pz^2 + (\bar{\eta}\bar{p} + \eta p)z + \eta\bar{p}\Big)P_2(z) \\ &= p\delta z(z-1)P_{1,0} + \eta(\bar{\theta} + \theta\delta)(pz + \bar{p})(z-1)P_{2,0}. \end{aligned} \quad (8)$$

Solving Eqs. (7) and (8), we have

$$\begin{aligned} P_1(z) &= \Big[p\bar{\delta}z\Big(\theta z + \bar{\theta}(\eta\bar{p} - \bar{\eta}pz)(z-1)\Big)P_{1,0} \\ &\quad + \eta\theta\bar{\delta}z(pz + \bar{p})P_{2,0}\Big]\Big/\phi(z), \end{aligned} \quad (9)$$

$$\begin{aligned} P_2(z) &= \Big[p\delta z^2 P_{1,0} + \eta\Big(\bar{\theta}\bar{\delta}(\mu\bar{p} - \bar{\mu}pz)(z-1) + \delta z\Big) \\ &\quad \times (pz + \bar{p})P_{2,0}\Big]\Big/\phi(z), \end{aligned} \quad (10)$$

where $\phi(z) = \bar{\delta}(\mu\bar{p} - \bar{\mu}pz)(\eta\bar{p} - \bar{\eta}pz)(z-1) + \delta z(\eta\bar{p} - \bar{\eta}pz) + \theta\bar{\delta}(\mu\bar{p} - \bar{\mu}pz)(\bar{\eta}z + \eta)(pz + \bar{p})$.

Lemma 1: If $\theta\bar{\delta}(\mu\bar{p} - \bar{\mu}p) > \delta(\bar{\eta}p - \eta\bar{p})$, the equation $\phi(z) = 0$ has a unique root $z = r_1$ in the interval (0,1).

Proof: We can easily obtain

$$\phi(0) = -\mu\eta\bar{\theta}\bar{p}^2\bar{\delta} < 0,$$

$$\phi(1) = \delta(\eta\bar{p} - \bar{\eta}p) + \theta\bar{\delta}(\mu\bar{p} - \bar{\mu}p) > 0,$$

$$\phi\Big(\frac{\mu\bar{p}}{\bar{\mu}p}\Big) = \frac{\mu\bar{p}^2\delta}{\bar{\mu}^2 p}(\eta - \mu) < 0,$$

$$\phi(+\infty) = +\infty \ (the\ coefficient\ of\ z^3\ is\ \bar{\mu}\bar{\eta}\bar{\theta}p^2\bar{\delta} > 0).$$

Thus, the three roots of $\phi(z)$ lie in (0,1), $(1, \frac{\mu\bar{p}}{\bar{\mu}p})$ and $(\frac{\mu\bar{p}}{\bar{\mu}p}, +\infty)$, and $\phi(z) = 0$ has a unique root $z = r_1$ in the interval (0,1). ■

From Lemma 1, the numerator of $P_1(z)$ must vanish at $z = r_1$, we have

$$\begin{aligned} &p\bar{\delta}r_1\Big(\theta r_1 + \bar{\theta}(\eta\bar{p} - \bar{\eta}pr_1)(r_1 - 1)\Big)P_{1,0} \\ &\quad + \eta\theta\bar{\delta}r_1(pr_1 + \bar{p})P_{2,0} = 0, \end{aligned}$$

which means

$$\begin{aligned} P_{2,0} &= \frac{\bar{\theta}p(\eta\bar{p} - \bar{\eta}pr_1)(1 - r_1) - \theta pr_1}{\eta\theta(pr_1 + \bar{p})}P_{1,0} \\ &\triangleq L(r_1)P_{1,0}. \end{aligned} \quad (11)$$

Therefore, using (11), we have the following theorem.

Theorem 2:

$$\begin{aligned} P_1(z) &= \Big[p\bar{\delta}z\Big(\theta z + \bar{\theta}(\eta\bar{p} - \bar{\eta}pz)(z-1)\Big) \\ &\quad + \eta\theta\bar{\delta}z(pz + \bar{p})L(r_1)\Big]P_{1,0}\Big/\phi(z), \\ P_2(z) &= \Big[p\delta z^2 + \eta\Big(\bar{\theta}\bar{\delta}(\mu\bar{p} - \bar{\mu}pz)(z-1) + \delta z\Big) \\ &\quad \times (pz + \bar{p})L(r_1)\Big]P_{1,0}\Big/\phi(z), \end{aligned}$$

where $P_{1,0}$ is determined by the normalization condition

$$P_{1,0} + P_1(1) + P_2(1) = 1,$$

which leads to

$$P_{1,0} = \frac{\phi(1)}{(\delta + \theta\bar{\delta})(p + \eta L(r_1)) + \phi(1)}. \quad (12)$$

The probability generating function of the system size is given by

$$\begin{aligned} P(z) &= P_{1,0} + P_1(z) + P_2(z) \\ &= \Big[\phi(z) + pz\Big((\delta + \theta\bar{\delta})z + \bar{\theta}\bar{\delta}(\eta\bar{p} - \bar{\eta}pz)(z-1)\Big) \\ &\quad + \eta\Big((\delta + \theta\bar{\delta})z + \bar{\theta}\bar{\delta}(\mu\bar{p} - \bar{\mu}pz)(z-1)\Big) \\ &\quad \times (pz + \bar{p})L(r_1)\Big]P_{1,0}\Big/\phi(z). \end{aligned} \quad (13)$$

Let W denote the sojourn time of a customer in the system, and $W^*(s)$ is the Laplace Steiljes transform of W. Then, W and Q have the following classical relationship (see [25])

$$P(z) = W^*(pz + \bar{p}).$$

Using (13), we can get

$$\begin{aligned} W^*(s) &= \Big[p\phi(\frac{s - \bar{p}}{p}) + (s - \bar{p})\Big((\delta + \theta\bar{\delta})(s - \bar{p}) \\ &\quad + \bar{\theta}\bar{\delta}(\bar{p} - \bar{\eta}s)(s-1)\Big) + \eta s\Big((\delta + \theta\bar{\delta})(s - \bar{p}) \\ &\quad + \bar{\theta}\bar{\delta}(\bar{p} - \bar{\mu}s)(s-1)\Big)L(r_1)\Big]P_{1,0}\Big/\Big[p\phi(\frac{s - \bar{p}}{p})\Big], \end{aligned}$$

where

$$\begin{aligned} p\phi\Big(\frac{s - \bar{p}}{p}\Big) &= \bar{\delta}(\bar{p} - \bar{\mu}s)(\bar{p} - \bar{\eta}s)(s-1) \\ &\quad + \delta(s - \bar{p})(\bar{p} - \bar{\eta}s) + \theta\bar{\delta}s(\bar{p} - \bar{\mu}s)(\bar{\eta}(s - \bar{p}) + \eta p). \end{aligned}$$

The probability that the server is busy in the normal state is given by

$$P_N = P_1(1) = \frac{\theta p\bar{\delta} + \eta\theta\bar{\delta}L(r_1)}{\phi(1)}P_{1,0}.$$

The probability that the server is busy in the defective state (working breakdown period) is given by

$$P_W = P_2(1) - P_{2,0} = \frac{p\delta + (\eta\delta - \phi(1))L(r_1)}{\phi(1)} P_{1,0}.$$

The probability that the server is free in the normal state and in the defective state are given by $P_{1,0}$ and $P_{2,0}$, respectively.

Let $E[L_j]$ denote the average number of customers in the system when the server's state is $j, j = 1, 2$. From Theorem 2, after some calculations we can get

$$\begin{aligned} E[L_1] &= \lim_{z\to 1} P_1'(z) = \Big[\Big(\theta\bar{\delta}(p + \eta L(r_1)) \\ &\quad + \theta p\bar{\delta}(1 + \eta L(r_1)) + \bar{\theta}p\bar{\delta}(\eta\bar{p} - \bar{\eta}p)\Big)\phi(1) \\ &\quad - \theta\bar{\delta}(p + \eta L(r_1))\phi'(1)\Big] P_{1,0}\Big/\phi^2(1), \\ E[L_2] &= \lim_{z\to 1} P_2'(z) = \Big[\Big(\delta(p + \eta L(r_1)) \\ &\quad + p\delta(1 + \eta L(r_1)) + \eta\bar{\theta}\bar{\delta}(\mu\bar{p} - \bar{\mu}p)L(r_1\Big)\phi(1) \\ &\quad - \delta(p + \eta L(r_1))\phi'(1)\Big] P_{1,0}\Big/\phi^2(1), \end{aligned}$$

where

$$\begin{aligned} \phi'(1) &= \bar{\delta}(\mu\bar{p} - \bar{\mu}p)(\eta\bar{p} - \bar{\eta}p) + \delta(\eta\bar{p} - 2\bar{\eta}p) \\ &\quad + \theta\bar{\delta}\big((\mu\bar{p} - \bar{\mu}p)(\bar{\eta} + p) - \bar{\mu}p\big). \end{aligned}$$

The expected number of customers in the system is given by

$$E[L] = \lim_{z\to 1} P'(z) = E[L_1] + E[L_2].$$

Let $E[W]$ be the expected sojourn time of a customer in the system, using Little's formula, $E[W] = \frac{E[L]}{p}$.

IV. Relation to the continuous-time queueing system

This section discusses the relation between our discrete-time queue and its corresponding continuous-time system [9]. If it is assumed that the time is slotted into sufficiently small intervals of equal length ϵ, the continuous-time queue [9] can be approximated by our discrete-time queueing system for which

$$p = \widetilde{\lambda}\epsilon, \quad \delta = \widetilde{\alpha}\epsilon, \quad \mu = \widetilde{\mu}\epsilon, \quad \eta = \widetilde{\mu}_1\varepsilon, \quad \theta = \widetilde{\beta}\epsilon.$$

From Eqs. (9) and (10), we can obtain

$$\begin{aligned} &\widetilde{P}_1(z) \triangleq \lim_{\epsilon\to 0} P_1(z) \\ &= \Big[\widetilde{\lambda}z\Big(\widetilde{\beta}z + (\widetilde{\mu}_1 - \widetilde{\lambda}z)(z-1)\Big)\widetilde{P}_{1,0} + \widetilde{\mu}_1\widetilde{\beta}z\widetilde{P}_{2,0}\Big]\Big/\widetilde{\phi}(z), \\ &\widetilde{P}_2(z) \triangleq \lim_{\epsilon\to 0} P_2(z) \\ &= \Big[\widetilde{\lambda}\widetilde{\alpha}z^2\widetilde{P}_{1,0} + \widetilde{\mu}_1\Big((\widetilde{\mu} - \widetilde{\lambda}z)(z-1) + \widetilde{\alpha}z\Big)\widetilde{P}_{2,0}\Big]\Big/\widetilde{\phi}(z), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\phi}(z) &= (\widetilde{\mu} - \widetilde{\lambda}z)(\widetilde{\mu}_1 - \widetilde{\lambda}z)(z-1) \\ &\quad + \widetilde{\alpha}z(\widetilde{\mu}_1 - \widetilde{\lambda}z) + \widetilde{\beta}z(\widetilde{\mu} - \widetilde{\lambda}z). \end{aligned}$$

From Eqs. (11) and (12), we can have

$$\begin{aligned} \widetilde{P}_{2,0} &\triangleq \lim_{\epsilon\to 0} P_{2,0} = \frac{\widetilde{\lambda}(\widetilde{\mu}_1 - \widetilde{\lambda}\widetilde{r}_1)(1 - \widetilde{r}_1) - \widetilde{\beta}\widetilde{\lambda}\widetilde{r}_1}{\widetilde{\mu}_1\widetilde{\beta}}\widetilde{P}_{1,0} \\ &\triangleq L(\widetilde{r}_1)\widetilde{P}_{1,0}, \\ \widetilde{P}_{1,0} &\triangleq \lim_{\epsilon\to 0} P_{1,0} = \frac{\widetilde{\phi}(1)}{(\widetilde{\alpha} + \widetilde{\beta})(\widetilde{\lambda} + \widetilde{\mu}_1 L(\widetilde{r}_1)) + \widetilde{\phi}(1)}, \end{aligned}$$

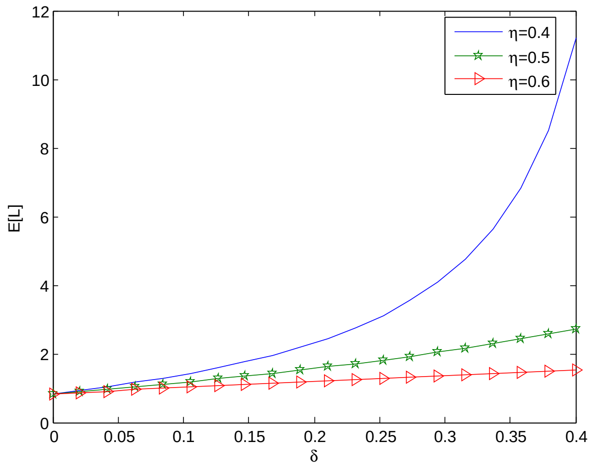

Fig. 1. The expected queue length versus δ.

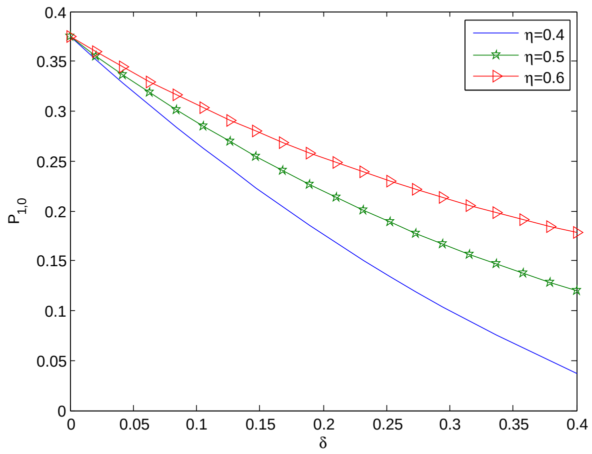

Fig. 2. The probability of server being free in the normal state versus δ.

where $\widetilde{\phi}(1) = \widetilde{\alpha}(\widetilde{\mu}_1 - \widetilde{\lambda}) + \widetilde{\beta}(\widetilde{\mu} - \widetilde{\lambda})$ and $\widetilde{r}_1$ is the unique root of the equation $\widetilde{\phi}(z) = 0$ in the interval (0,1).

After some calculations, we can find the expressions of $\widetilde{P}_1(z)$, $\widetilde{P}_2(z)$, $\widetilde{P}_{2,0}$ and $\widetilde{P}_{1,0}$ are agreement with the expressions obtained in Kalidass and Kasturi [9].

V. Numerical results

In this section, we present some numerical examples to illustrate the effect of varying parameters on some crucial performance measures of our model. Moreover, a cost minimization problem is also considered. Under the stable condition, all the computations are done by developing program in Matlab software and the values of parameters are arbitrarily chosen as μ=0.8, η=0.4, θ=0.3, p=0.5 and δ=0.2, unless they are considered as variables in the respective figures.

A. Sensitivity analysis

The effect of δ on the expected queue length $E[L]$ and the probability of server being free in the normal state $P_{1,0}$ are presented in Fig.1 and Fig.2, respectively. We can find that $E[L]$ increases with increasing values of δ, while $P_{1,0}$ decreases as δ increases. The reason is that as δ increases, the system is more likely to break down and the service rate will decrease, which in turn increases the system queue length. Moreover, the effect of δ on $E[L]$ is not obvious when the

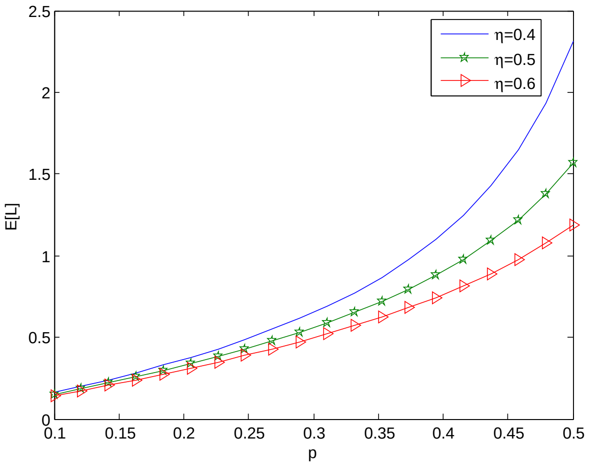

Fig. 3. The expected queue length versus p.

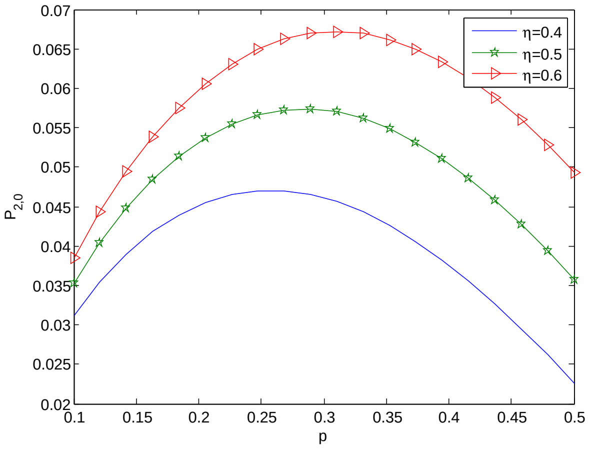

Fig. 4. The probability of server being free in the defective state versus p.

value of η is large. An especial case is $\delta \to 0$, i.e., the server can not break down, and the system will be in normal state with probability 1, we can see that η has no effect on $E[L]$ and $P_{1,0}$.

Fig.3 illustrates that $E[L]$ increases as p increases, which agrees with the intuitive expectation. When p is small, which means the number of customers in the system is small, it can be found that the effect of η on $E[L]$ is not obvious. In Fig.4, we see that as p increases, the probability of server being free in the defective state $P_{2,0}$ first increases and then decreases. However, the effect of p on $P_{2,0}$ is not obvious. For example, if we choose η=0.4, with the change of p, the values of $P_{2,0}$ only varies from 0.023 to 0.047. The main reason is that the failure rate can be regarded as δ=0.2, the mean repair time is $1/\theta$, and the server can not break down if the system is empty.

Fig.5 and Fig.6 provide the effect of η on the expected queue length $E[L]$ and the probability of server being busy in the defective state P_W, respectively. As expected, $E[L]$ and P_W both decrease with increasing values of η. The effect of η is more obvious when θ is smaller, this is due to the fact that the expected repair time is $1/\theta$, and the defective system will be repaired in a longer time. In Fig.5, an especial case is $\eta \to \mu$, i.e., the lower service rate equals to the normal service rate, it can be observed that θ has no effect on $E[L]$.

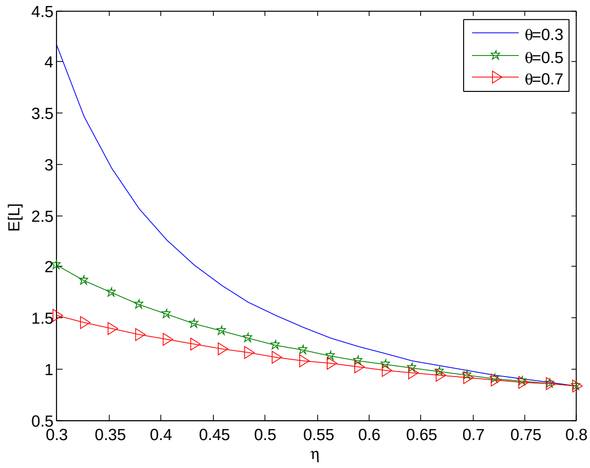

Fig. 5. The expected queue length versus η.

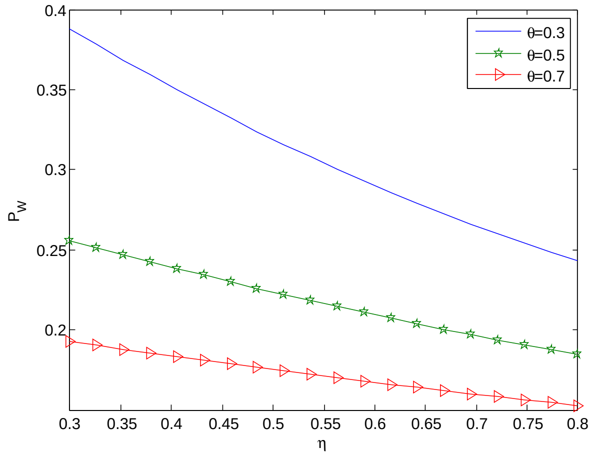

Fig. 6. The probability of server being busy in the defective state versus η.

As shown in Fig.7, since $\eta < \mu$, it is obvious that $E[L]$ decreases as θ increases, and the effect of θ on $E[L]$ is not obvious when θ is large. The reason is that the mean repair time is becoming shorter with the increasing repair rate θ, and therefore the waiting customers have a greater chance to be served by normal service rete, which can reduce the

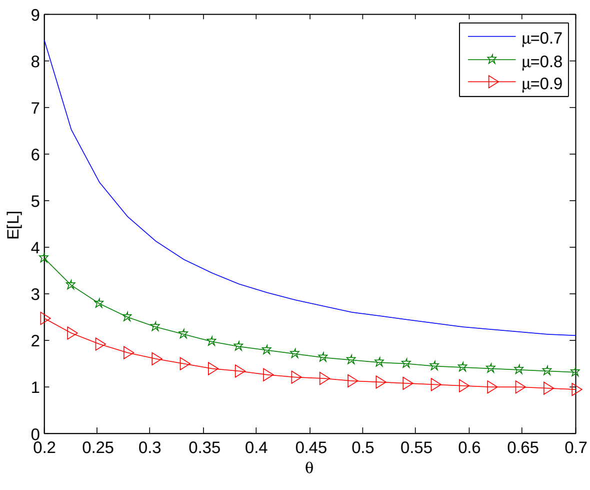

Fig. 7. The expected queue length versus θ.

system queue length. Fig.8 indicates that the probability of server being busy in the normal state P_N increases with the increment of θ, this is also because that the duration of server in the defective state is shorter for larger value of θ. Further, as intuitively expected, for a fixed repair rate θ, $E[L]$ and P_N both decrease as μ increases.

B. Cost analysis

In practice, from the perspective of economic profit, queueing managers are always interested in minimizing operating cost of unit time. Therefore, in this subsection, we establish a cost function to search for the optimal service rate η, so as to minimize the expected operating cost function per unit time.

Define the following cost elements:

C_L=cost per unit time for each customer present in the system;

C_μ=cost per customer served by the normal service rate μ;

C_η=cost per customer served by the lower service rate η;

C_θ=fixed cost per unit time when the server is in a repair (working breakdown) period.

Based on the definitions of each cost element listed above, the expected operating cost function per unit time is given by

$$\min_{\eta} : f(\eta) = C_L E[L] + C_\mu \mu + C_\eta \eta + C_\theta \theta. \qquad (14)$$

Because the expected operating cost function per unit time is highly non-linear and complex, we can use the parabolic method to find the optimum value of η, say η^*. The essence of the parabolic method is to generate a quadratic function through the evaluated points in each iteration, and the objective function $f(x)$ is approximated by the quadratic function in generating an estimate of the optimum value. According to the polynomial approximation theory, the unique optimum of the quadratic function agreeing with $f(x)$ at 3-point pattern $\{x_0, x_1, x_2\}$ occurs at

$$\bar{x} = \frac{1}{2}\frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{f(x_0)(x_1 - x_2) + f(x_1)(x_2 - x_0) + f(x_2)(x_0 - x_1)}.$$

The steps of the parabolic method are given as follows [26]:

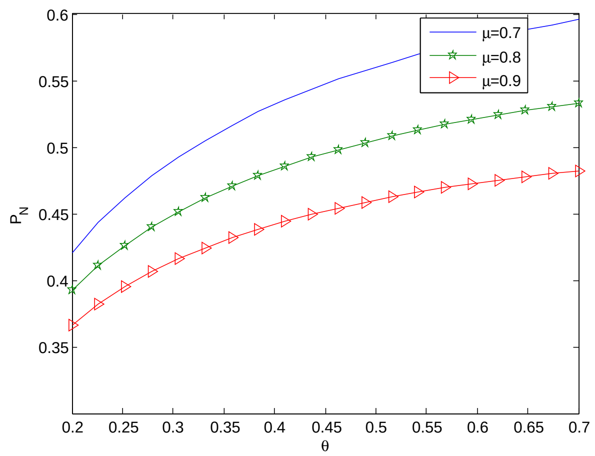

Fig. 8. The probability of server being busy in the normal state versus θ.

Step 1: Choose a starting 3-point pattern $\{x_0, x_1, x_2\}$ along with a stopping tolerance ε, and initialize the iteration counter i=0.

Step 2: Compute $\bar{x}$ according to the above equation. If $|\bar{x} - x_1| \le \varepsilon$, stop and report approximate optimum solution $\bar{x}$.

Step 3: If $\bar{x} \le x_1$, go to Step 4. If $\bar{x} > x_1$, go to Step 5.

Step 4: If $f(x_1)$ is less than $f(\bar{x})$, update $\bar{x} \to x_0$. Otherwise, replace $x_1 \to x_2, \bar{x} \to x_1$. Either way, advance $i = i + 1$, and return to Step 2.

Step 5: If $f(x_1)$ is less than $f(\bar{x})$, update $\bar{x} \to x_2$. Otherwise, replace $x_1 \to x_0, \bar{x} \to x_1$. Either way, advance $i = i + 1$, and return to Step 2.

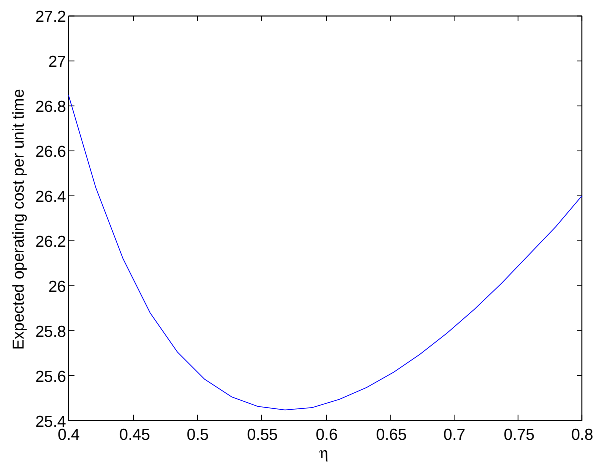

Fig. 9. The effect of η on the expected operating cost per unit time.

Assume C_L=3, C_μ=18, C_η=10 and C_θ=5, Fig.9 shows that there is an optimal service rate η to make the cost minimize. Using the parabolic method and the error is controlled by ε=10^{-5}. After three iterations, Table 1 shows that the minimum expected operating cost per unit time converges to the solution η^*=0.568600 with a value $f(\eta^*)$=25.444881.

VI. Conclusion

This paper generalizes the model of Kalidass and Kasturi [9] to a Geo/Geo/1 queue. During the breakdown period, the service still continues at a lower rate. Using the matrix-analytic method, we obtain the condition of stability. The probability generating function of the number of customers in the system is also discussed. Moreover, various system performance measures are developed, and the effect of some parameters are examined numerically. The novelty of this investigation is the first time to consider working breakdowns in a discrete-time queue. For future study, one can analyze a similar system with retrial customers or extend this model to a Geo/G/1 queue.

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TABLE I
THE PARABOLIC METHOD IN SEARCHING FOR THE OPTIMAL SERVICE RATE η.

iterations	η_0	η_1	η_2	$f(\eta_0)$	$f(\eta_1)$	$f(\eta_2)$	$\bar{\eta}$	$f(\bar{\eta})$	tolerance
0	0.550000	0.600000	0.650000	25.455291	25.471059	25.601916	0.568149	25.444886	0.031851
1	0.550000	0.568149	0.600000	25.455291	25.444886	25.471059	0.569348	25.444897	0.001199
2	0.550000	0.568149	0.569348	25.455291	25.444886	25.444897	0.568609	25.444881	4.599073×10^{-4}
3	0.568149	0.568609	0.569348	25.444886	25.444881	25.444897	0.568600	25.444881	9.262377×10^{-6}

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