Normality Criteria of a Class of Meromorphic Functions Concerning Shared Fixed-points

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Abstract—**In this paper, by using Zalcman Lemma, we obtain some normal criterions of meromorphic functions concerning shared fixed-points, which improves some earlier related results.**

Index Terms—**meromorphic function; fixed-points; shared value; normal criterion.**

I. Introduction And Main Results

LET D be a domain in $\mathbf{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in the domain D. Then $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically, locally uniformly in D to a meromorphic function or ∞. $\mathcal{F}$ is said to be normal at a point z_0 if there exists a neighborhood of z_0 in which $\mathcal{F}$ is normal(see [1-2]). Clearly, $\mathcal{F}$ is normal in D if and only if it is normal at every point of D.

Suppose $f(z)$ is a meromorphic function in a domain D, and $z_0 \in D$, if $f(z_0) = z_0$, we say z_0 is the fixed-point of $f(z)$. Let $f(z)$ and $g(z)$ denote two meromorphic functions in D, if $f(z)-z$ and $g(z)-z$ have the same zeros (ignoring multiplicity), then we say $f(z)$ and $g(z)$ share the fix-points.

In this paper, we assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory(see [3-5]).

It is also more interesting to find normality criteria from the point of shared values. In this area, Schwick [6] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values have emerged, for instance (see [7-10]). In recent years, this subject has attracted the attention of many researchers.

In 2009, Y. T. Li and Y. X. Gu[11] gave the following result:

Theorem 1.1 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let $k, n \geq k+2$ be positive integers and $a \neq 0$ be a finite complex number. For each pair of $(f,g) \in \mathcal{F}$, if $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D, then $\mathcal{F}$ is normal in D.*

In 2009, many authors studied the functions of the form $f(f^{(k)})^n$. And D. W. Meng and P. C. Hu[12] proved:

Theorem 1.2 *Take positive integers n and k with $n, k \geq 2$ and take a non-zero complex number a. Let $\mathcal{F}$ be a family of meromorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. For each pair of $(f,g) \in \mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share a in D, then $\mathcal{F}$ is normal in D.*

Lately, Q Yang[13] extended Theorem 1.2 as follows:

Theorem 1.3 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let $n, k \geq 2$ be two positive integers. For every $f \in \mathcal{F}$, all of whose zeros have multiplicity at least $\frac{nk+2}{n-1}$. For each pair of $(f,g)\epsilon\mathcal{F}$, if $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share z in D, then $\mathcal{F}$ is normal in D.*

A natural question is: What's the result if the function $f(f^{(k)})^n$ in Theorem 1.3 is replaced by the function $f^d(f^{(k)})^n$? In this paper, we study the problem and obtain the following theorems.

Theorem 1.4 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let k, n, d be three positive integers. If for every $f \in \mathcal{F}$ with $f \neq 0$, and for each pair of $(f,g) \in \mathcal{F}$, $f^d(f^{(k)})^n$ and $g^d(g^{(k)})^n$ share z in D, then $\mathcal{F}$ is normal in D.*

Theorem 1.5 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Let $n, k \geq 2, d$ be three positive integers. For every $f \in \mathcal{F}$, all of whose zeros have multiplicity at least $\max\{\frac{nk+2}{n+d-2}, k\}$. For each pair of $(f,g) \in \mathcal{F}$, if $f^d(f^{(k)})^n$ and $g^d(g^{(k)})^n$ share z in D, then $\mathcal{F}$ is normal in D.*

II. Preliminary Lemmas

In order to obtain our theorems, we require the following Lemmas.

Lemma 2.1[7] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and k be a positive integer, such that each function $f \in \mathcal{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$, there exists a sequence of points $z_n \in D, z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0^+$, and a subsequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \varsigma)}{\rho_n^\alpha} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \leq g^\sharp(0) = kA+1$. Moreover, g has order at most 2.

Here as usual, $g^\sharp(\zeta) = \frac{|g'(\zeta)|}{1+|g(\zeta)|^2}$ is the spherical derivative of g.

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Lemma 2.2 *Let k, n, d be three positive integers, and f be a non-constant rational meromorphic function such that $f \neq 0$, then $f^d(z)(f^{(k)}(z))^n - z$ has at least two distinct zeros.*
Proof: Since $f \neq 0$, let

$$f(z) = \frac{A}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i}}, \qquad (1)$$

where A is a non-zero constant and $n_j (j = 1, 2, \cdots, t)$ are positive integers.

For the sake of simplicity, we denote $N := n_1 + n_2 + \cdots + n_t$. Obviously, $N \geq t$.

From (1), we have

$$f^{(k)} = \frac{P(z)}{\prod_{i=1}^{t}(z-\beta_i)^{n_i+k}}, \qquad (2)$$

and

$$f^d(f^{(k)})^n = \frac{A^d P^n(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk}}. \qquad (3)$$

Where $P(z) \not\equiv 0$ is a polynomial, and $\deg(P) = k(t-1)$. Differentiating (3), we get

$$[f^d(f^{(k)})^n]'' = \frac{P_1(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+2}}. \qquad (4)$$

Where $P_1(z) \not\equiv 0$ is a polynomial, and $\deg(P_1) = (nk+2)(t-1)$.

If $f^d(f^{(k)})^n - z$ has at most one zero. We distinguish the following two cases:

Case 1: $f^d(f^{(k)})^n - z$ has exactly one zero z_0.

By (3), we obtain

$$f^d(f^{(k)})^n - z = \frac{B(z-z_0)^{(n+d)N+nkt+1}}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk}}, \qquad (5)$$

where B is a non-zero constant. Differentiating (5), we have

$$[f^d(f^{(k)})^n]'' = \frac{P_2(z)(z-z_0)^{(n+d)N+nkt-1}}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+2}}. \qquad (6)$$

Where $P_2(z) \not\equiv 0$ is a polynomial, and $\deg(P_2) \leq 2t$.

From (4) and (6), we get

$$\begin{aligned}(nk+2)(t-1) &\geq (n+d)N + nkt - 1\\ &\geq (n+d)t + nkt - 1,\end{aligned}$$

then $-nk \geq (n+d-2)t + 1 \geq 1$, this is a contradiction.

Case 2: $f^d(f^{(k)})^n - z$ has no zero.

By (5), we have $(n+d)N + nkt + 1 = 0$, which is a contradiction and Lemma 2.2 is hold.

Lemma 2.3 *Let $n, k, d \geq 2$ be three positive integers and f be a non-constant rational meromorphic function. If all zeros of f have multiplicity at least $\max\{\frac{nk+2}{n+d-2}, k\}$, then $f^d(z)(f^{(k)}(z))^n - z$ has at least two distinct zeros.*

Proof: Suppose that $f^d(z)(f^{(k)}(z))^n - z$ has at most one zero.

Case 1: f is a polynomial.

In this case, we find that all zeros of $f^d(z)(f^{(k)}(z))^n$ have multiplicity at least kd. Since all zeros of f have multiplicity at least $\max\{\frac{nk+2}{n+d-2}, k\}$, so $f^d(z)(f^{(k)}(z))^n - z$ has at least one zero and $[f^d(z)(f^{(k)}(z))^n]'$ has zeros with multiplicity at least $kd-1$. According to the assumption, we obtain $f^d(z)(f^{(k)}(z))^n - z$ has only one zero z_0, then there exists a non-zero constant A and a integer $l \geq 2$ such that

$$f^d(z)(f^{(k)}(z))^n = z + A(z-z_0)^l.$$

So,

$$[f^d(z)(f^{(k)}(z))^n]' = 1 + Al(z-z_0)^{l-1}.$$

Which implies that it has only simple zeros. This contradicts with the facts that all zeros of $f^d(z)(f^{(k)}(z))^n$ have multiplicity at least kd.

Case 2: f is a rational but not a polynomial.

In this case, we get

$$f(z) = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \qquad (7)$$

Where A is a non-zero constant and $m_j \geq \frac{nk+2}{n+d-2} (j = 1, 2, \cdots, s)$.

For simplicity, we denote

$$M = m_1 + m_2 + \cdots + m_s \geq \frac{(nk+2)s}{n+d-2}, \qquad (8)$$

$$N = n_1 + n_2 + \cdots + n_t \geq t. \qquad (9)$$

From (7), we have

$$f^{(k)} = \frac{\prod_{j=1}^{s}(z-\alpha_j)^{m_j-k}g(z)}{\prod_{i=1}^{t}(z-\beta_i)^{n_i+k}}, \qquad (10)$$

and

$$f^d(f^{(k)})^n = \frac{\prod_{j=1}^{s}(z-\alpha_j)^{(n+d)m_j-nk}g^n(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk}} = \frac{P(z)}{Q(z)}. \qquad (11)$$

Where $g(z) \not\equiv 0$ is a polynomial and $\deg(g) \leq k(s+t-1)$.

Differentiating (11), we have

$$[f^d(f^{(k)})^n]' = \frac{\prod_{j=1}^{s}(z-\alpha_j)^{(n+d)m_j-nk-1}g_1(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+1}}, \qquad (12)$$

and

$$[f^d(f^{(k)})^n]'' = \frac{\prod_{j=1}^{s}(z-\alpha_j)^{(n+d)m_j-nk-2}g_2(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+2}}. \qquad (13)$$

Where $g_1(z) \not\equiv 0, g_2(z) \not\equiv 0$ are polynomials, and $\deg(g_1) \leq (nk+1)(s+t-1), \deg(g_2) \leq (nk+2)(s+t-1)$.

Now, we distinguish the two subcases:

Subcase 2.1: $f^d(z)(f^{(k)}(z))^n - z$ has exactly one zero z_0.

From (11), we obtain

$$f^d(f^{(k)})^n = z + \frac{B(z-z_0)^l}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk}} = \frac{P(z)}{Q(z)}. \quad (14)$$

Differentiating (14), we have

$$[f^d(f^{(k)})^n]' = 1 + \frac{(z-z_0)^{l-1}g_1^*(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+1}}, \quad (15)$$

and

$$[f^d(f^{(k)})^n]'' = \frac{(z-z_0)^{l-2}g_2^*(z)}{\prod_{i=1}^{t}(z-\beta_i)^{(n+d)n_i+nk+2}}. \quad (16)$$

Where $g_1^*(z) \not\equiv 0, g_2^*(z) \not\equiv 0$ are polynomials, $\deg(g_1^*) \leq t, \deg(g_2^*) \leq 2t$. By (12) and (15), we have $z_0 \neq \alpha_j (j = 1, 2, \cdots, s)$.

Further, we consider the following two subcases.

Subcase 2.1.1: $l \neq (n+d)N + nkt + 1$.

From (14), it is easily obtained that $\deg(P) \geq \deg(Q)$. Thus (11) implies

$$\begin{aligned}\deg(Q) &= (n+d)N + nkt \leq \deg(P) \\ &= (n+d)M - nks + n\deg(g) \\ &\leq (n+d)M - nks + nk(s+t-1) \\ &= (n+d)M + nkt - nk.\end{aligned}$$

So, $M \geq N + \frac{nk}{n+d}$, that is $M > N$. From (11) and (14), noting that $z_0 \neq \alpha_j (j = 1, 2, \cdots, s)$, we have

$$(n+d)M - (nk+2)s \leq \deg(g_2^*) \leq 2t.$$

It follows that $(n+d)M \leq (nk+2)s + 2t$, by (8) and (9), we obtain

$$\begin{aligned}(n+d)M &\leq (nk+2)s + 2t \\ &\leq (nk+2)\tfrac{(n+d-2)M}{nk+2} + 2N \\ &< (n+d)M\end{aligned}$$

which is impossible.

Subcase 2.1.2: $l = (n+d)N + nkt + 1$.

If $M > N$, the similar to the Subcase 2.1.1. It follows that $(n+d)M < (n+d)N$, which is impossible.

We may assume that $M \leq N$, by (13) and (16), we have

$$l - 2 \leq \deg(g_2) \leq (nk+2)(s+t-1),$$

and

$$\begin{aligned}(n+d)N &= l - nkt - 1 \leq (nk+2)(s+t-1) - nkt + 1 \\ &= (nk+2)s + 2t - (nk+1) \\ &< (nk+2)\tfrac{(n+d-2)M}{nk+2} + 2N \\ &\leq (n+d)N.\end{aligned}$$

This is also a contradiction.

Subcase 2.2: $f(z)(f^{(k)}(z))^n - z$ has no zero.

Then $l = 0$ for (14). And differentiating (14), the similarly to the proof of Subcase 2.1, we also obtain a contradiction. Hence, the Lemma 2.3 is hold.

Lemma 2.4[14] *Let $f(z)$ be a transcendental meromorphic function, n, k, d be three positive integers. Then, when $k \geq 1$, $n, d \geq 2$, $f^d(f^{(k)})^n - \varphi(z)$ has infinitely many zeros, where $\varphi(z) \not\equiv 0, T(r,\varphi) = S(r,f)$.*

III. Proofs Of Theorems

Proof of Theorem 1.5. From theorem 1.3, the theorem 1.5 holds when $d = 1$. Next, we will prove the case $d \geq 2$.

Case 1: $z_0 = 0$.

Let $\mathcal{F}_1 = \{F_j : F_j(z) = \frac{f_j(z)}{z^{\frac{1}{n+d}}} | f_j \epsilon \mathcal{F}\}$. If $\mathcal{F}_1$ is not normal at 0, by Lemma 2.1, there exists a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$, a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ such that

$$g_j(\xi) = \rho_j^{-\frac{nk}{n+d}} F_j(z_j + \rho_j\xi) \to g(\xi),$$

locally uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$.

Here we distinguish two subcases.

Subcase 1.1: $\frac{z_j}{\rho_j} \to c$, where c is a finite complex number. Then

$$\begin{aligned}\phi_j(\xi) &= \frac{f_j(\rho_j\xi)}{\rho_j^{\frac{1+nk}{n+d}}} = \frac{F_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{\rho_j^{\frac{nk}{n+d}}} \frac{(\rho_j\xi)^{\frac{1}{n+d}}}{\rho_j^{\frac{1}{n+d}}} \\ &\to \xi^{\frac{1}{n+d}} g(\xi - c) = H(\xi),\end{aligned}$$

locally uniformly on compact subsets of $\mathbf{C}$ disjoint from the poles of g, where $H(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $\max\{\frac{nk+2}{n+d-2}, k\}$.

Hence,

$$\begin{aligned}\phi_j^d(\xi)(\phi_j^{(k)}(\xi))^n - \frac{\rho_j\xi}{\rho_j} &= \frac{f_j^d(\rho_j\xi)(f_j^{(k)}(\rho_j\xi))^n - (\rho_j\xi)}{\rho_j} \\ &\to H^d(\xi)(H^{(k)}(\xi))^n - \xi,\end{aligned}$$

spherically locally uniformly in $\mathbf{C}$ disjoint from the poles of g.

If $H^d(\xi)(H^{(k)}(\xi))^n \equiv \xi$, since $H(\xi)$ has zeros with multiplicity at least $\max\{\frac{nk+2}{n+d-2}, k\}$, obviously this is a contradiction. Hence, $H^d(\xi)H^{(k)}(\xi) \not\equiv \xi$. Since the multiplicity of all zeros of $H(\xi)$ is at least $\max\{\frac{nk+2}{n+d-2}, k\}$, by Lemma 2.3 and 2.4, $H^d(\xi)(H^{(k)}(\xi))^n - \xi$ has at least two distinct zeros.

Suppose that ξ_0, ξ_0^* are two distinct zeros of $H^d(\xi)(H^{(k)}(\xi))^n - \xi$. Then we can choose a positive number δ small enough such that $D_1 \bigcap D_2 = \emptyset$ and $H^d(\xi)(H^{(k)}(\xi))^n - \xi$ has no other zeros in $D_1 \bigcup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1 = \{\xi \in \mathbf{C} || \xi - \xi_0| < \delta\}, D_2 = \{\xi \in \mathbf{C} || \xi - \xi_0^*| < \delta\}.$$

By Hurwitz's Theorem, for sufficiently large j, there exist points $\xi_j \in D_1, \xi_j^* \in D_2$ such that

$$f_j^d(\rho_j\xi_j^*)(f_j^{(k)}(\rho_j\xi_j^*))^n - (\rho_j\xi_j^*) = 0,$$

$$f_j^d(\rho_j\xi_j)(f_j^{(k)}(\rho_j\xi_j))^n - (\rho_j\xi_j) = 0.$$

By the assumption in Theorem 1.5, $f_m^d(f_m^{(k)})^n$ and $f_j^d(f_j^{(k)})^n$ share z, it follows that

$$f_m^d(\rho_j\xi_j^*)(f_m^{(k)}(\rho_j\xi_j^*))^n - (\rho_j\xi_j^*) = 0,$$

$$f_m^d(\rho_j\xi_j)(f_m^{(k)}(\rho_j\xi_j))^n-(\rho_j\xi_j)=0.$$

Fix m, take $j\to\infty$, and note $\rho_j\xi_j\to 0, \rho_j\xi_j^*\to 0$, we obtain

$$f_m^d(0)(f_m^{(k)}(0))^n=0.$$

Since the zeros of $f_j^d(\xi)(f_j^{(k)}(\xi))^n-\xi$ has no accumulation point, for sufficiently large j, we have

$$\rho_j\xi_j=0, \rho_j\xi_j^*=0.$$

Therefore, when j is large enough, $\xi_0=\xi_0^*$. This contradicts with the facts $\xi_0\in D_1, \xi_0^*\in D_2, D_1\bigcap D_2=\emptyset$. Thus, $\mathcal{F}_1$ is normal at 0.

Subcase 1.2: $\frac{z_j}{\rho_j}\to\infty$.

We have

$$\begin{aligned} f_j^{(k)}(z) &= z^{\frac{1}{n+d}}F_j^{(k)}(z)+\sum_{l=1}^{k}c_k^l(z^{\frac{1}{n+d}})^{(l)}F_j^{(k-l)}(z)\\ &= z^{\frac{1}{n+d}}F_j^{(k)}(z)+\sum_{l=1}^{k}c_l z^{\frac{1}{n+d}-l}F_j^{(k-l)}(z),\end{aligned}$$

where $c_l=\frac{1}{n+d}(\frac{1}{n+d}-1)\cdots(\frac{1}{n+d}-l+1)C_k^l$. Thus, we get

$$\begin{aligned} & f_j^d(z)(f_j^{(k)}(z))^n\\ &=[z^{\frac{1}{n+d}}F_j^{(k)}(z)+\sum_{l=1}^{k}c_l z^{\frac{1}{n+d}-l}F_j^{(k-l)}(z)]^n z^{\frac{d}{n+d}}F_j^d(z),\\ & \frac{f_j^d(z)(f_j^{(k)}(z))^n}{z}\\ &=[F_j^{(k)}(z)F_j^{\frac{d}{n}}(z)+\sum_{l=1}^{k}c_l\frac{F_j^{(k-l)}(z)F_j^{\frac{d}{n}}(z)}{z^l}]^n.\end{aligned}$$

Since $F_j^{(k-l)}(z_j+\rho_j\xi)=\rho_j^{\frac{kn}{n+d}-(k-l)}g_j^{(k-l)}(\xi)$, we have

$$\begin{aligned} & \frac{f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n}{z_j+\rho_j\xi}\\ &=(g_j^{(k)}(\xi)g_j^{\frac{d}{n}}(\xi)+\sum_{l=1}^{k}c_l\frac{g_j^{(k-l)}(\xi)g_j^{\frac{d}{n}}(\xi)}{(\frac{z_j}{\rho_j}+\xi)^l})^n.\end{aligned}$$

On the other hand, for $l=1,2,\cdots,k$, we have

$$\lim_{j\to\infty}\frac{c_l}{(\frac{z_j}{\rho_j}+\xi)^l}=0,$$

Thus,

$$\frac{f_j^d(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^n}{z_j+\rho_j\xi}\to g^d(\zeta)(g^{(k)}(\xi))^n,$$

spherically locally uniformly in $\mathbf{C}$ disjoint from the poles of g.

If $g^d(\xi)(g^{(k)}(\xi))^n\equiv 1$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant Meromorphic function of order at most 2, there exist constant $c_i(i=1,2), (c_1,c_2)\neq(0,0)$, and $g(\xi)=e^{c_0+c_1\xi+c_2\xi^2}$, obviously, this contradicts the case $g^d(\xi)(g^{(k)}(\xi))^n\equiv 1$. Hence, $g^d(\xi)(g^{(k)}(\xi))^n\not\equiv 1$.

Since the multiplicity of all zeros of g is at least $\max\{\frac{nk+2}{n+d-2},k\}$. By Lemma 2.4, $g^d(\xi)(g^{(k)}(\xi))^n-1$ has at least two distinct zeros.

Suppose that ξ_1,ξ_1^* are two distinct zeros of $g^d(\xi)(g^{(k)}(\xi))^n-1$. We choose a positive number δ small enough such that $D_3\bigcap D_4=\emptyset$ and $g^d(\xi)(g^{(k)}(\xi))^n-1$ has no other zeros in $D_3\bigcup D_4$ except for ξ_1 and ξ_1^*, where

$$D_3=\{\xi\in\mathbf{C}||\xi-\xi_1|<\delta\}, D_4=\{\xi\in\mathbf{C}||\xi-\xi_1^*|<\delta\}.$$

By Hurwitz's Theorem, for sufficiently large j there exist points $\widehat{\xi_j}\in D_1, \widetilde{\xi_j}\in D_2$ such that

$$\begin{aligned} & f_j^d(z_j+\rho_j\widehat{\xi_j})(f_j^{(k)}(z_j+\rho_j\widehat{\xi_j}))^n-(z_j+\rho_j\widehat{\xi_j})=0,\\ & f_j^d(z_j+\rho_j\widetilde{\xi_j})(f_j^{(k)}(z_j+\rho_j\widetilde{\xi_j}))^n-(z_j+\rho_j\widetilde{\xi_j})=0.\end{aligned}$$

By the assumption in Theorem 1.5, $f_m^d(f_m^{(k)})^n$ and $f_j^d(f_j^{(k)})^n$ share z, it follows that

$$\begin{aligned} & f_m^d(z_j+\rho_j\widehat{\xi_j})(f_m^{(k)}(z_j+\rho_j\widehat{\xi_j}))^n-(z_j+\rho_j\widehat{\xi_j})=0,\\ & f_m^d(z_j+\rho_j\widetilde{\xi_j})(f_m^{(k)}(z_j+\rho_j\widetilde{\xi_j}))^n-(z_j+\rho_j\widetilde{\xi_j})=0.\end{aligned}$$

Similar to the proof of Subcase 1.1, Fix m, take $j\to\infty$, we also get $\xi_1=\xi_1^*$. This contradicts the facts $\xi_1\in D_3, \xi_1^*\in D_4, D_3\bigcap D_3=\emptyset$. Thus, $\mathcal{F}_1$ is normal at 0.

From Subcases 1.1 and 1.2, we know $\mathcal{F}_1$ is normal at 0, there exists $\Delta=\{z:|z|<\rho\}$ and a subsequence of F_j, we may still denote it as F_j, such that F_j converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞ in Δ. Here we distinguish the following two cases.

Case i: $f_j(0)\neq 0$, when j is large enough.

Then $F(0)=\infty$. Thus, for each $F_j(z)\in\mathcal{F}_1$, there exists $\delta>0$ such that $|F(z)|>1$ for all $z\in\Delta_\delta=\{z:|z|<\delta\}$ when $F(z)\in\mathcal{F}_1$. So, for sufficiently large j, $|F_j(z)|\geq 1$, $\frac{1}{f_j}$ is holomorphic in Δ_δ. Therefore, for all $f_j\in\mathcal{F}$, we have

$$|\frac{1}{f_j}|=|\frac{1}{F_j(z)z^{\frac{1}{n+d}}}|\leq(\frac{2}{\delta})^{\frac{1}{n+d}}.$$

when $|z|=\frac{\delta}{2}$. By maximum Principle and Montel's Theorem, $\mathcal{F}$ is normal at $z=0$.

Case ii: There exists a subsequence of f_j, we may still denote it as f_j such that $f_j(0)=0$.

Since $f\in\mathcal{F}$, the multiplicity of all zeros of f is at least $\max\{\frac{nk+2}{n+d-2},k\}$, then $F_j(0)=0$. Thus, there exists $0<r<\rho$ such that $F_j(z)$ is holomorphic in $\Delta_r=\{z:|z|<r\}$ and has a unique zero $z=0$ in Δ_r. Then F_j converges spherically locally uniformly to a holomorphic function $F(z)$ in Δ_r, f_j converges spherically locally uniformly to a holomorphic function $F(z)z^{\frac{1}{n+d}}$ in Δ_r. Hence, $\mathcal{F}$ is normal at $z=0$.

From Case i and ii, we know that $\mathcal{F}$ is normal at $z=0$.

Case 2: $z_0\neq 0$.

Suppose that F is not normal in D. Then there exists at least one point z_0 such that F is not normal at the point z_0. By Lemma 2.1, there exist a sequence $\{z_{n_j}\}$ of complex numbers with $z_{n_j}\to z_0$, a sequence $\{\rho_n\}$ of positive numbers with $\rho_n\to 0$ such that

$$g_n(\xi)=\rho_n^{-\frac{nk}{n+d}}f_n(z_n+\rho_n\xi)\to g(\xi), \quad (17)$$

locally uniformly on compact subsets of $\mathbf{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $\max\{\frac{nk+2}{n+d-2},k\}$. Moreover, $g(\xi)$ has order at most 2.

From (17), we get

$$\begin{aligned}&g_n^d(\xi)(g_n^{(k)}(\xi))^n-(z_n+\rho_n\xi)\\&=f_n^d(z_n+\rho_n\xi)(f_n^{(k)}(z_n+\rho_n\xi))^n-(z_n+\rho_n\xi)\\&\to g^d(\xi)(g^{(k)}(\xi))^n-z_0,\end{aligned}$$

spherically locally uniformly in $\mathbf{C}$ disjoint from the poles of g.

If $g^d(g^{(k)})^n\equiv z_0$, then g has no zeros. Of course, g also has no poles. Since g is a non-constant meromorphic function of order at most 2, then there exist constant $c_i(i=1,2)\neq 0$, and $g(\xi)=e^{c_0+c_1\xi+c_2\xi^2}$. Obviously, this is contrary to the case $g^d(g^{(k)})^n\equiv z_0$. Hence $g^d(g^{(k)})^n\not\equiv z_0$.

By Lemma 2.4, we deduce that $g^d(g^{(k)})^n-z_0$ has at least two distinct zeros. Next we show that it is impossible. Let ξ_2 and ξ_2^* be two distinct zeros of $g^d(g^{(k)})^n-z_0$. We choose a positive number δ small enough such that $D_5\bigcap D_6=\emptyset$ and such that $g^d(g^{(k)})^n-z_0$ has no other zeros in $D_5\bigcup D_6$ expect for ξ_2 and ξ_2^*, where

$$D_5=\{\xi\in\mathbf{C}||\xi-\xi_2|<\delta\}, D_6=\{\xi\in\mathbf{C}||\xi-\xi_2^*|<\delta\}.$$

By Hurwitz's Theorem, similar to the proof of Case 1, we can get $\xi_2=\xi_2^*$. This is contrary to the facts $\xi_2\in D_5,\xi_2^*\in D_6, D_5\bigcap D_6=\emptyset$. Thus, $\mathcal{F}_1$ is normal at z_0. Hence, the Theorem 1.5 is hoid.

The proof of Theorem 1.4 is similar to Theorem 1.5, only replace Lemma 2.3 by Lemma 2.2 in the proof procedure, here we omit the proof.

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