An Upwind-mixed Finite Element Method with Moving Grids for Quasi-nonlinear Sobolev Equations

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***Abstract*—An upwind-mixed finite element method with moving grids is presented to simulate quasi-nonlinear Sobolev equations. This method is constructed by two methods. The upwind method is used to approximate the the convection term of Sobolev equations, meanwhile an expanded mixed finite element method is applied to discretize the diffusion term. The scalar unknown function and the adjoint vector function can be approximated simultaneously by this method. Optimal error estimates in L^2-norm are obtained for both the scalar unknown function and the adjoint vector function. Finally, numerical experiments are presented to illustrate the efficiency of this method.**

***Index Terms*—Upwind method, mixed finite element method, moving grids, quasi-nonlinear Sobolev equations**

I. Introduction

We consider the following quasi-nonlinear Sobolev equations

$$\begin{cases} u_t - \nabla\cdot\big(a(x,t)\nabla u_t + b(x,t,u)\nabla u\big) + \boldsymbol{c}(x,t,u)\cdot\nabla u \\ \qquad\qquad = f(x,t,u), \quad x\in\Omega,\ t\in(0,T], \\ u(x,t)=0, \qquad\qquad\qquad x\in\partial\Omega,\ t\in[0,T], \\ u(x,0)=u_0(x), \qquad\qquad x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded subset of $\mathcal{R}^2$ with smooth boundary $\partial\Omega$, $u_0(x)$ and $f(x,t,u)$ are known functions, the coefficients $a(x,t)$, $b(x,t,u)$, $\boldsymbol{c}(x,t,u) = (c_1(x,t,u), c_2(x,t,u))^{\mathrm{T}}$ satisfy the following condition:

$$\begin{cases} 0<a_0\le a(x,t)\le a_1,\ |\dfrac{\partial a(x,t)}{\partial t}|\le a_2, \\ 0<b_0\le b(x,t,u)\le b_1, \\ |\boldsymbol{c}(x,t,u)| = \sqrt{c_1^2(x,t,u)+c_2^2(x,t,u)} \le c_0, \\ |\dfrac{\partial \boldsymbol{c}(x,t,u)}{\partial u}| \le K_1, \\ b(x,t,u), \boldsymbol{c}(x,t,u), f(x,t,u) \text{ and } \dfrac{\partial \boldsymbol{c}(x,t,u)}{\partial u} \\ \text{ are Lipschitz continuous with respect to } u. \end{cases} \tag{2}$$

where $a_0, a_1, a_2, b_0, b_1, c_0, K_1$ are positive constants. We assume that $u(x,t)$ satisfy the smooth condition needed in the following analysis.

For time-changing localized phenomena, such as sharp fronts and layers, the finite element method with moving grids [1]-[3] is advantageous over fixed finite element method. The reason is that the former treats the problem with the finite element method on space domain by using different meshes and different basic functions at different time level so that it has the capability of self-adaptive local grid modification (refinement or unrefinement) to efficiently capture propagating fronts or moving layers. The paper [4] had combined this method with mixed finite element method to study parabolic problems.

Sobolev equations have important applications in many mathematical and physical problems, such as the percolation theory of the fluid flowing through the cracks [5], the transfer problem of the moisture in the soil [6], and the heat conduction problem in different materials [7]. Hence, there exists great and actual significance to discuss Sobolev equations in depth. Many papers had researched on numerical methods for Sobolev equations. More attentions were paid for treating the damping term $\nabla\cdot(a\nabla u_t)$, which is a distinct character of Sobolev equations different from parabolic e-quation. For example, time-stepping Galerkin methods were presented for nonlinear Sobolev equations in [8], [9]. In [10], [11], nonlinear Sobolev equations with convection term were researched by using finite difference streamline-diffusion method and discontinuous Galerkin method, respectively. Two new least-squares mixed finite element procedures were formulated for solving convection-dominated Sobolev equations in [12]. In [13], two-grid methods for characteristic finite volume element approximations were considered for semi-linear Sobolev equations.

Mixed finite element method has been proven to be a powerful tool to numerically solve the fluid problems. It has an advantage of approximating the unknown function and its adjoint function simultaneously. The theoretical analysis and actual applications of mixed finite element method were discussed well, such as [14], [15], [16]. For the convection dominated equation, the solutions of standard finite element method often suffer from spurious oscillations. A variety of numerical techniques were put forward to solve this problem well, such as characteristic finite element method [17], [18], characteristic finite volume method [19]. The papers [20] and [21] introduced an upwind mixed covolume method and an upwind cell-centered difference method for the problem with diagonal diffusion tensor, respectively. Hughes and Brooks [22] proposed the streamline upwind Petrov-Galerkin method (SUPG) by adding an artificial diffusion in the streamline direction to diminish the oscillations. Johnson [23] and Johnson et al. [24] stabilized the SUPG method by adding another artificial diffusion in the crosswind direction to avoid overshooting and undershooting around the sharp fronts. In [25], an upwind-mixed method on changing meshes was considered for two-phase miscible flow in porous media.

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This paper presents an upwind-mixed finite element method with moving grids for quasi-nonlinear Sobolev equations. In Section II, this method is constructed by two methods. The convection term of Sobolev equations is approximated by the upwind method, and the diffusion term is discretized by an expanded mixed finite element method. This method can approximate simultaneously the scalar unknown function and the adjoint vector function effectively. Optimal error estimates in L^2-norm are derived for both the scalar unknown function and the adjoint vector function in Section III. In Section IV, we present the results of numerical experiments, which confirm our theoretical results. We draw some conclusions in Section V.

Throughout the analysis, the symbol K will denote a generic constant, which is independent of mesh parameters Δt, h and not necessarily the same at different occurrences.

II. Upwind-mixed method with moving grids

At first, we introduce some notations and basic assumptions. The usual Sobolev spaces and norms are adopted on Ω. The inner product on $L^2(\Omega)$ is denoted by $(f,g)=\int_\Omega fgdx$. Define the following two spaces

$$W=L^2(\Omega)/\{\varphi\equiv\text{constant on }\Omega\},$$
$$V=\{\boldsymbol{v}\in H(\text{div};\Omega)|\ \boldsymbol{v}\cdot\boldsymbol{n}=0\,\text{on }\ \partial\Omega\},$$

where $\boldsymbol{n}$ is the unit outward vector normal to $\partial\Omega$.

Let $\Delta t^n>0\,(n=1,2,\cdots,N^*)$ denote different time-step size such that $T=\sum\limits_{n=1}^{N^*}\Delta t^n$ and $t^n=\sum\limits_{k=1}^{n}\Delta t^k$. We take $\Delta t=\max\limits_n\Delta t^n$. We assume that the time-step size Δt^n do not change too rapidly, that is, there exist positive constants λ_* and λ^* which are independent of n and Δt such that

$$\lambda_*\le\frac{\Delta t^n}{\Delta t^{n-1}}\le\lambda^*. \tag{3}$$

For a given function $g(x,t)$, we denote $g^n=g(x,t^n)$.

At each time level t^n, we construct a quasi-uniform partition $K_h^n=\{e_i^n\}$ of Ω for the mixed finite element space. And we assume h^n be the diameter of $e_i^n\in K_h^n$ and $\Delta t^n=O(h^n)$. We take $h=\max\limits_n h^n$. Let $W_h^n\times V_h^n\subset W\times V$ and $\text{div}V_h^n=W_h^n$ denote the "lowest-order" Raviart-Thomas spaces. That is to say, on each element $e_i^n\in K_h^n$, W_h^n is the space of functions which are constant and V_h^n is the space of vector valued functions whose components are continuous and linear. The degrees of freedom of a function $\boldsymbol{v}^n\in V_h^n$ correspond to the values of $\boldsymbol{v}^n\cdot\boldsymbol{\gamma}$ at the midpoints of ∂e_i^n. Here, ∂e_i^n is the side of e_i^n and $\boldsymbol{\gamma}$ is the unit outward vector normal to ∂e_i^n.

By introducing variables $\tilde{\boldsymbol{z}}=-\nabla u,\boldsymbol{z}=b\tilde{\boldsymbol{z}}+a\tilde{\boldsymbol{z}}_t,\boldsymbol{g}=\boldsymbol{c}u=(c_1u,c_2u)^{\mathrm{T}}=(g_1,g_2)^{\mathrm{T}}$ and $\bar{\boldsymbol{c}}(x,t,u)=(\frac{\partial c_1}{\partial u},\frac{\partial c_2}{\partial u})^{\mathrm{T}}$, we can rewrite the first equation in (1) as

$$u_t+\nabla\cdot\boldsymbol{z}+\nabla\cdot\boldsymbol{g}+u\bar{\boldsymbol{c}}(u)\cdot\tilde{z}=f(u). \tag{4}$$

Here, we utilize the so-called "expanded" mixed finite element method, proposed by Arbogast, Wheeler and Yotov[26], which gives a gradient approximation $\tilde{\boldsymbol{z}}$ and an approximation $\boldsymbol{z}$ to the diffusion term.

Then, the weak formula of (4) is

$$\begin{cases}(u_t,w)+(\nabla\cdot\boldsymbol{z},w)+(\nabla\cdot\boldsymbol{g},w)\\ \qquad+(u\bar{\boldsymbol{c}}(u)\cdot\tilde{\boldsymbol{z}},w)=(f(u),w), & \forall\ w\in W,\\ (\tilde{\boldsymbol{z}},\boldsymbol{v})=(u,\nabla\cdot\boldsymbol{v}), & \forall\ \boldsymbol{v}\in V,\\ (\boldsymbol{z},\boldsymbol{v})=(b(u)\tilde{\boldsymbol{z}}+a\tilde{\boldsymbol{z}}_t,\boldsymbol{v}), & \forall\ \boldsymbol{v}\in V.\end{cases} \tag{5}$$

The upwind-mixed finite element method with moving grids is presented as follows: at each time level n, $\forall\,w^n\in W_h^n$, $\boldsymbol{v^n}\in V_h^n$, find $U^n\in W_h^n,\boldsymbol{Z}^n\in V_h^n$ such that

$$\begin{cases}(\dfrac{U^n-R^nU^{n-1}}{\Delta t^n},w^n)+(\nabla\cdot\boldsymbol{Z}^n,w^n)+(\nabla\cdot\boldsymbol{G}^n,w^n)\\ \qquad+(U^n\bar{\boldsymbol{c}}(U^n)\cdot\tilde{\boldsymbol{Z}}^{\boldsymbol{n}},w^n)=(f(U^n),w^n),\\ (R^nU^{n-1}-U^{n-1},w^n)=0,\\ (\tilde{\boldsymbol{Z}}^n,\boldsymbol{v^n})=(U^n,\nabla\cdot\boldsymbol{v^n}),\\ (\boldsymbol{Z}^n,\boldsymbol{v^n})=(b(U^n)\tilde{\boldsymbol{Z}}^n+a^n\dfrac{\tilde{\boldsymbol{Z}}^n-R^n\tilde{\boldsymbol{Z}}^{n-1}}{\Delta t^n},\boldsymbol{v^n}),\\ (a^n(R^n\tilde{\boldsymbol{Z}}^{n-1}-\tilde{\boldsymbol{Z}}^{n-1}),\boldsymbol{v^n})=0.\end{cases} \tag{6}$$

When different finite element spaces are used at time level t^n and t^{n-1}, the second and fifth equations of (6) give the L^2-projection $\{R^nU^{n-1},R^n\tilde{\boldsymbol{Z}}^{n-1}\}$ of the previous approximate solution $\{U^{n-1},\tilde{\boldsymbol{Z}}^{n-1}\}$ into the current finite element space $W_h^n\times V_h^n$. Then, this projection is used as initial value to calculate $\{U^n,\ \tilde{\boldsymbol{Z}}^n\}$ in the first and third equations of (6). If the finite element spaces are same at time level t^n and t^{n-1}, we know that $R^nU^{n-1}=U^{n-1},R^n\tilde{\boldsymbol{Z}}^{n-1}=\tilde{\boldsymbol{Z}}^{n-1}$.

In equation (6), $\boldsymbol{G}^n$ is constructed by the upwind method [25]. Since $\boldsymbol{g}=\boldsymbol{c}u=0$ on $\partial\Omega$, we set the integral average of $\boldsymbol{G}^n\cdot\boldsymbol{\gamma}$ equal to zero on boundary edges. Suppose that elements e_1 and e_2 share an interior edge l,x_l be the midpoint of the edge l, and $\boldsymbol{\gamma}_l$ point from e_1 to e_2. Then we adopt ([25])

$$\boldsymbol{G}^n\cdot\boldsymbol{\gamma}_l=\begin{cases}U_{e_1}^n(\boldsymbol{c}(U^{n-1})\cdot\boldsymbol{\gamma}_l)(x_l),\text{if}\,(\boldsymbol{c}(U^{n-1})\cdot\boldsymbol{\gamma}_l)(x_l)\ge0,\\ U_{e_2}^n(\boldsymbol{c}(U^{n-1})\cdot\boldsymbol{\gamma}_l)(x_l),\text{if}\,(\boldsymbol{c}(U^{n-1})\cdot\boldsymbol{\gamma}_l)(x_l)<0,\end{cases}$$

where $U_{e_1}^n$ and $U_{e_2}^n$ are the constant values of U^n on the elements e_1 and e_2, respectively.

III. Error estimates

In order to derive optimal error estimates, we need three projections. First, define $\prod u^n\in W_h^n$, $\tilde{\prod}\tilde{\boldsymbol{z}}^n\in V_h^n$ to be the L^2-projection of $u^n\in H^1(\Omega)$ and $\tilde{\boldsymbol{z}}^n\in H(\text{div},\Omega)$ respectively, which satisfy

$$\begin{cases}(u^n,w^n)=(\prod u^n,w^n), & \forall\,w^n\in W_h^n,\\ (a^n\tilde{\boldsymbol{z}}^n,\boldsymbol{v}^n)=(a^n\tilde{\prod}\tilde{\boldsymbol{z}}^n,\boldsymbol{v}^n), & \forall\,\boldsymbol{v}^n\in V_h^n.\end{cases} \tag{7}$$

Then, define $\pi\boldsymbol{z}^n\in V_h^n$ to be the π-projection of $\boldsymbol{z}^n\in H(\text{div},\Omega)$, which satisfies

$$(\nabla\cdot(\boldsymbol{z}^n-\pi\boldsymbol{z}^n),w^n)=0,\ \forall\ w^n\in W_h^n. \tag{8}$$

According to [25], [27], these projections have the following approximate properties

$$\begin{cases}\|u^n-\prod u^n\|\le Kh^n,\\ \|\tilde{\boldsymbol{z}}^n-\tilde{\prod}\tilde{\boldsymbol{z}}^n\|+\|(\tilde{\boldsymbol{z}}^n-\tilde{\prod}\tilde{\boldsymbol{z}}^n)_t\|\le Kh^n,\\ \|\boldsymbol{z}^n-\pi\boldsymbol{z}^n\|\le Kh^n.\end{cases} \tag{9}$$

At time level t^n, for $\forall\ w^n\in W_h^n,\ \boldsymbol{v^n}\in V_h^n$, we know that the exact solutions satisfy

$$\begin{cases} (\frac{u^n - u^{n-1}}{\Delta t^n}, w^n) + (\nabla \cdot \boldsymbol{z}^n, w^n) + (\nabla \cdot \boldsymbol{g}^n, w^n) \\ \quad +(u^n \bar{\boldsymbol{c}}(u^n) \cdot \tilde{\boldsymbol{z}}^n, w^n) = (f(u^n), w^n) - (\rho^n, w^n), \\ (\tilde{\boldsymbol{z}}^n, \boldsymbol{v^n}) = (u^n, \nabla \cdot \boldsymbol{v^n}), \\ (\boldsymbol{z}^n, \boldsymbol{v^n}) = (b(u^n)\tilde{\boldsymbol{z}}^n + a^n \frac{\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \boldsymbol{v^n}) + (\boldsymbol{r}^n, \boldsymbol{v^n}), \end{cases} \tag{10}$$

where $\rho^n = u_t^n - \frac{u^n - u^{n-1}}{\Delta t^n}$, $\boldsymbol{r}^n = a^n(\tilde{\boldsymbol{z}}_t^n - \frac{\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{z}}^{n-1}}{\Delta t^n})$.

Denote

$$\begin{aligned} &\xi_u = U - \prod u, \quad \eta_u = u - \prod u, \\ &\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}} = \tilde{\boldsymbol{Z}} - \tilde{\prod} \tilde{\boldsymbol{z}}, \quad \tilde{\boldsymbol{\eta}}_z = \tilde{\boldsymbol{z}} - \tilde{\prod} \tilde{\boldsymbol{z}}, \\ &\boldsymbol{\xi_z} = \boldsymbol{Z} - \pi \boldsymbol{z}, \quad \boldsymbol{\eta_z} = \boldsymbol{z} - \pi \boldsymbol{z}. \end{aligned}$$

Using the projections (7) and (8), we subtract (10) from (6) to get

$$\begin{cases} (\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t^n}, w^n) + (\nabla \cdot \boldsymbol{\xi}_{\boldsymbol{z}}^n, w^n) + (\nabla \cdot (\boldsymbol{G}^n - \boldsymbol{g}^n), w^n) \\ \quad +(U^n \bar{\boldsymbol{c}}(U^n) \cdot \tilde{\boldsymbol{Z}}^n, w^n) - (u^n \bar{\boldsymbol{c}}(u^n) \cdot \tilde{\boldsymbol{z}}^n, w^n) \\ = (\rho^n, w^n) + (f(U^n) - f(u^n), w^n) + (\frac{\eta_u^n - \eta_u^{n-1}}{\Delta t^n}, w^n), \\ (\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \boldsymbol{v^n}) = (\xi_u^n, \nabla \cdot \boldsymbol{v^n}), \\ (\boldsymbol{\xi}_{\boldsymbol{z}}^n, \boldsymbol{v^n}) = (b(U^n)\tilde{\boldsymbol{Z}}^n - b(u^n)\tilde{\boldsymbol{z}}^n, \boldsymbol{v^n}) - (\boldsymbol{r}^n, \boldsymbol{v^n}) \\ \qquad +(a^n \frac{\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{Z}}^{n-1}}{\Delta t^n}, \boldsymbol{v^n}) + (\boldsymbol{\eta}_{\boldsymbol{z}}^n, \boldsymbol{v^n}) \\ \qquad -(a^n \frac{\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \boldsymbol{v^n}). \end{cases} \tag{11}$$

Here, the last term of the first equation of (11) is related to the moving grids. If the grids don't change, this term is equal to zero.

Taking $w^n = \xi_u^n$, $\boldsymbol{v^n} = \boldsymbol{\xi}_{\boldsymbol{z}}^n$ and $\boldsymbol{v^n} = \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n$ sequentially in (11) and adding together, we obtain

$$\begin{aligned} &(\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t}, \xi_u^n) + (b(U^n)\tilde{\boldsymbol{Z}}^n - b(u^n)\tilde{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\quad +(a^n \frac{\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{Z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^n \frac{\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &= (f(U^n) - f(u^n), \xi_u^n) + (\rho^n, \xi_u^n) + (\nabla \cdot (\boldsymbol{g}^n - \boldsymbol{G}^n), \xi_u^n) \\ &\quad -(U^n \bar{\boldsymbol{c}}(U^n) \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) + (u^n \bar{\boldsymbol{c}}(u^n) \cdot \tilde{\boldsymbol{z}}^n, \xi_u^n) \\ &\quad +(\boldsymbol{r}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (\boldsymbol{\eta}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + (\frac{\eta_u^n - \eta_u^{n-1}}{\Delta t^n}, \xi_u^n). \end{aligned} \tag{12}$$

Furthermore, we notice that

(I)

$$\begin{aligned} &(b(U^n)\tilde{\boldsymbol{Z}}^n - b(u^n)\tilde{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &= (b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (b(U^n)\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\quad +([b(U^n) - b(u^n)]\tilde{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n), \end{aligned} \tag{13}$$

(II)

$$\begin{aligned} &(a^n \frac{\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{Z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^n \frac{\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &= (a^n \frac{\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^n \frac{\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n), \end{aligned} \tag{14}$$

(III)

$$\begin{aligned} &(U^n \bar{\boldsymbol{c}}(U^n) \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) - (u^n \bar{\boldsymbol{c}}(u^n) \cdot \tilde{\boldsymbol{z}}^n, \xi_u^n) \\ &= ([\xi_u^n - \eta_u^n]\bar{\boldsymbol{c}}(U^n) \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) + (u^n[\bar{\boldsymbol{c}}(U^n) - \bar{\boldsymbol{c}}(u^n)] \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) \\ &\quad +(u^n \bar{\boldsymbol{c}}(u^n) \cdot [\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n], \xi_u^n). \end{aligned} \tag{15}$$

The last second term in (14) is related to the moving grids. If the grids don't change, this term is equal to zero.

We substitute (13)-(15) into (12) to yield

$$\begin{aligned} &(\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t^n}, \xi_u^n) + (b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + (a^n \frac{\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &= (f(U^n) - f(u^n), \xi_u^n) + (\rho^n, \xi_u^n) + (\nabla \cdot (\boldsymbol{g}^n - \boldsymbol{G}^n), \xi_u^n) \\ &\quad +(\boldsymbol{r}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (\boldsymbol{\eta}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + ([\eta_u^n - \xi_u^n]\bar{\boldsymbol{c}}(U^n) \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) \\ &\quad -(u^n[\bar{\boldsymbol{c}}(U^n) - \bar{\boldsymbol{c}}(u^n)] \cdot \tilde{\boldsymbol{Z}}^n, \xi_u^n) + (b(U^n)\tilde{\boldsymbol{\eta}}_z^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\quad +(\frac{\eta_u^n - \eta_u^{n-1}}{\Delta t^n}, \xi_u^n) + (u^n \bar{\boldsymbol{c}}(u^n) \cdot [\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n], \xi_u^n) \\ &\quad +([b(u^n) - b(U^n)]\tilde{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + (a^n \frac{\tilde{\boldsymbol{\eta}}_z^n - \tilde{\boldsymbol{\eta}}_z^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\stackrel{Def.}{=} T_1 + T_2 + \cdots + T_{12}. \end{aligned} \tag{16}$$

Now, we turn to analyze each term in (16). First of all, for the first and third terms on the left-hand side, we have

$$\begin{aligned} &(\frac{\xi_u^n - \xi_u^{n-1}}{\Delta t^n}, \xi_u^n) \\ &= \frac{1}{2\Delta t^n}\Big\{[\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_u^n - \xi_u^{n-1}\|^2\Big\}, \end{aligned} \tag{17}$$

$$\begin{aligned} &(a^n \frac{\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\geq \frac{1}{2\Delta t^n}\Big\{[(a^n \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^{n-1}\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1})] \\ &\quad +a_0\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2\Big\} - \frac{a_2}{2}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2. \end{aligned} \tag{18}$$

Following from (16), (17) and (18), we derive

$$\begin{aligned} &\frac{1}{2\Delta t^n}\Big\{[\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_u^n - \xi_u^{n-1}\|^2\Big\} + (b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) \\ &\quad +\frac{1}{2\Delta t^n}\Big\{[(a^n \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^{n-1}\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1})] \\ &\quad +a_0\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2\Big\} \\ &\leq \frac{a_2}{2}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2 + T_1 + T_2 + \cdots + T_{12}. \end{aligned} \tag{19}$$

By the Lipschitz continuity of $b, \bar{\boldsymbol{c}}$ and f, some terms on the right-hand side of (19) can be estimated as follows:

$$T_1 \leq K\|\xi_u^n\|^2 + Kh^2, \tag{20}$$

$$T_2 \leq K\Delta t^n \|\frac{\partial^2 u}{\partial t^2}\|^2_{L^2(t^{n-1}, t^n; H^1)} + K\|\xi_u^n\|^2, \tag{21}$$

$$T_4 \leq K\Delta t^n \|\frac{\partial^2 u}{\partial t^2}\|^2_{L^2(t^{n-1}, t^n; H^1)} + \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2, \tag{22}$$

$$T_5 \leq K\|\boldsymbol{\eta}_{\boldsymbol{z}}^n\|^2 + \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2, \tag{23}$$

$$T_6 \leq K\|\xi_u^n\|^2 + Kh^2, \tag{24}$$

$$T_7 \leq K\|u^n \tilde{\boldsymbol{Z}}^n\|\|\xi_u^n\| + K\|\xi_u^n\|^2 \leq K\|\xi_u^n\|^2, \tag{25}$$

$$T_8 \leq Kb_1\|\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n\|^2 + \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 \leq \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 + Kh^2, \quad (26)$$

$$T_{10} \leq \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 + Kh^2 + K\|\xi_u^n\|^2, \quad (27)$$

$$T_{11} \leq K\|\xi_u^n\tilde{\boldsymbol{z}}^n\| + \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 \leq K\|\xi_u^n\|^2 + \frac{\varepsilon}{10}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2, \quad (28)$$

where ε is a sufficiently small positive constant.

We apply the similar technique in [17] to consider T_3. Let $\pi\boldsymbol{g}^n \in V_h^n$ denote the π-projection of $\boldsymbol{g}^n$, i.e.

$$(\nabla\cdot(\boldsymbol{g}^n - \pi\boldsymbol{g}^n), w^n) = 0, \ \forall\ w^n \in\ W_h^n. \quad (29)$$

Taking $\boldsymbol{v^n} = \pi\boldsymbol{g}^n - \boldsymbol{G}^n$ in the second equation of (11), then we have

$$(\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \pi\boldsymbol{g}^n - \boldsymbol{G}^n) = (\nabla\cdot(\boldsymbol{g}^n - \boldsymbol{G}^n), \xi_u^n)$$

so that

$$\begin{aligned}&(\nabla\cdot(\boldsymbol{g}^n - \boldsymbol{G}^n), \xi_u^n) = (\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \pi\boldsymbol{g}^n - \boldsymbol{G}^n)\\ &\leq \frac{1}{2}(b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + Kb_0^{-1}\|\pi\boldsymbol{g}^n - \boldsymbol{G}^n\|^2.\end{aligned} \quad (30)$$

Le l be the common interior edge between elements e_1 and e_2, and h_l denote the length of this edge. Let $\boldsymbol{\gamma}_l$ denote the unit vector normal to l and x_l denote the midpoint of the edge. By the property of π-projection [18], we see

$$\int_l \pi\boldsymbol{g}^n\cdot\boldsymbol{\gamma}_l ds = \int_l u^n(\boldsymbol{c}^n\cdot\boldsymbol{\gamma}_l)ds. \quad (31)$$

If $\boldsymbol{g}^n$ is smooth enough, by the midpoint rule of integration

$$\frac{1}{h_l}\int_l \pi\boldsymbol{g}^n\cdot\boldsymbol{\gamma}_l ds - (\boldsymbol{c}(u^n)\cdot\boldsymbol{\gamma}_l)u^n(x_l) = O(h_l^2),$$

we derive

$$\begin{aligned}&\frac{1}{h_l}\int_l(\pi\boldsymbol{g}^n - \boldsymbol{G}^n)\cdot\boldsymbol{\gamma}_l ds\\ &= u^n(x_l)(\boldsymbol{c}(u^n) - \boldsymbol{c}(u^{n-1}))\cdot\boldsymbol{\gamma}_l\\ &\quad + u^n(x_l)(\boldsymbol{c}(u^{n-1}) - \boldsymbol{c}(U^{n-1}))\cdot\boldsymbol{\gamma}_l\\ &\quad + (u^n(x_l) - U_e^n)\boldsymbol{c}(U^{n-1})\cdot\boldsymbol{\gamma}_l + O(h_l^2).\end{aligned} \quad (32)$$

Furthermore, if u^n is smooth enough, we have

$$|u^n(x_l) - U_e^n| \leq |\xi_u^n| + O(h^n). \quad (33)$$

Noticing that for $\forall\ \boldsymbol{v}\in V_h$, the function $\boldsymbol{v}$ is specified in the interior of Ω and $\boldsymbol{v}\cdot\boldsymbol{\gamma}$ is a constant on each edge of element e. From (31) to (33), we have

$$\|\pi\boldsymbol{g}^n - \boldsymbol{G}^n\|^2 \leq K\|\xi_u^n\|^2 + K\big\{(\Delta t)^2 + h^2\big\}, \quad (34)$$

then

$$T_3 \leq \frac{1}{2}(b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + Kb_0^{-1}\big\{\|\xi_u^n\|^2 + (\Delta t)^2 + h^2\big\}. \quad (35)$$

Substituting (20)-(28) and (35) into (19), we obtain

$$\begin{aligned}&\frac{1}{2\Delta t^n}\Big\{[\|\xi_u^n\|^2 - \|\xi_u^{n-1}\|^2 + \|\xi_u^n - \xi_u^{n-1}\|^2\Big\}\\ &+\frac{1}{2}(b(U^n)\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) + \frac{a_0}{2\Delta t^n}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2\\ &+\frac{1}{2\Delta t^n}[(a^n\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n) - (a^{n-1}\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1})]\\ &\leq \frac{\varepsilon}{2}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 + \frac{a_2}{2}\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2 + K\big\{\|\xi_u^n\|^2 + (\Delta t)^2 + h^2\big\}\\ &+K(\Delta t^n)\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2_{L^2(t^{n-1},t^n;H^1)} + T_9 + T_{12}.\end{aligned} \quad (36)$$

Let N be the time-step at which $\|\xi_u^n\|$ is maximal, that is,

$$\|\xi_u^N\|^2 = \max_{1\leq n\leq N^*}\|\xi_u^n\|^2.$$

Multiplying (36) by $2\Delta t^n$ and summing on n from 1 to N, we obtain

$$\begin{aligned}&\|\xi_u^N\|^2 + \sum_{n=1}^N\|\xi_u^n - \xi_u^{n-1}\|^2\Delta t^n + b_0\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2\Delta t^n\\ &+a_0\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2 + a_0\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^N\|^2\\ &\leq K\big\{(\Delta t)^2 + h^2\big\} + \sum_{n=1}^N\big\{K\|\xi_u^n\|^2 + \varepsilon\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2\big\}\Delta t^n\\ &+a_2\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2\Delta t^n + \|\xi_u^0\|^2 + a_1\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^0\|^2\\ &+2\sum_{n=1}^N\big\{(\frac{\eta_u^n - \eta_u^{n-1}}{\Delta t^n}, \xi_u^n) + (a^n\frac{\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n)\big\}\Delta t^n.\end{aligned} \quad (37)$$

Assuming that the grids change at most M times in the time interval $[0, T]$, and $M \leq M^*$, where M^* is independent of h and Δt, we can get the following two analysis [3]:

$$2\sum_{n=1}^N(\frac{\eta_u^n - \eta_u^{n-1}}{\Delta t^n}, \xi_u^n)\Delta t^n \leq K(M^*h)^2 + \frac{1}{4}\|\xi_u^N\|^2, \quad (38)$$

$$2\sum_{n=1}^N(a^n\frac{\tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\eta}}_{\boldsymbol{z}}^{n-1}}{\Delta t^n}, \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n)\Delta t^n \leq Kh^2 + \varepsilon\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2\Delta t^n. \quad (39)$$

Substituting (38) and (39) into (37), we find

$$\begin{aligned}&\frac{3}{4}\|\xi_u^N\|^2 + \sum_{n=1}^N\|\xi_u^n - \xi_u^{n-1}\|^2\Delta t^n + b_0\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2\Delta t^n\\ &+a_0\sum_{n=1}^N\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n - \tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2 + a_0\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^N\|^2\\ &\leq K\big\{(\Delta t)^2 + h^2 + (M^*h)^2\big\} + \|\xi_u^0\|^2 + a_1\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^0\|^2\\ &+\sum_{n=1}^N\big\{K\|\xi_u^n\|^2 + 2\varepsilon\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 + a_2\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^{n-1}\|^2\big\}\Delta t^n.\end{aligned} \quad (40)$$

Choosing $0 < \varepsilon < b_0/2$ and using the discrete Gronwall's lemma, we have

$$\max_n\|\xi_u^n\|^2 \leq K\big\{(\Delta t)^2 + h^2 + (M^*h)^2\big\}, \quad (41)$$

and

$$\max_n\|\tilde{\boldsymbol{\xi}}_{\boldsymbol{z}}^n\|^2 \leq K\big\{(\Delta t)^2 + h^2 + (M^*h)^2\big\}. \quad (42)$$

By the triangle inequality and (9), we obtain the following result:

Theorem 3.1: Assuming that the coefficients satisfy the condition (2), and the grids change at most M times in the time interval $[0, T]$, $M \leq M^*$ and $u \in L^2(H^1), u_t \in L^2(H^1), u_{tt} \in L^2(H^1)$, then we derive

$$\max_n\|u^n - U^n\| \leq K\big\{h + \Delta t + M^*h\big\},$$

$$\max_n\|\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{Z}}^n\| \leq K\big\{h + \Delta t + M^*h\big\},$$

where K and M^* are positive constants independent of h and Δt.

IV. Numerical Example

In this section, we present numerical experiments to illustrate the efficiency of our upwind-mixed finite element method with moving grids. We consider the model (1) on $(x,t) \in [0,1]\times[0,1]$, where $a(x,t) = t^2(x+0.25)+0.25, b(x,t,u) = 0.005ut^2(x+0.05), c(x,t) = 0.05t(x+1)+1$ and $f(x,t,u)$ is chosen properly so that the exact solution is $u = \mathrm{e}^{-t}\sin\pi x$.

To compare the computations and show the convergence rate easily, we set h and Δt change according to the following four cases:

Case i: If $t \in [0,0.4]$, set $h = \Delta t = 0.1$; If $t \in (0.4,0.6]$, set $h = \Delta t = 0.05$; If $t \in (0.6,1.0]$, set $h = \Delta t = 0.1$ and calculate 40 steps in every time interval.

Case ii: If $t \in [0,0.4]$, set $h = \Delta t = 0.05$; If $t \in (0.4,0.6]$, set $h = \Delta t = 0.025$; If $t \in (0.6,1.0]$, set $h = \Delta t = 0.05$ and calculate 80 steps in every time interval.

Case iii: If $t \in [0,0.4]$, set $h = \Delta t = 0.025$; If $t \in (0.4,0.6]$, set $h = \Delta t = 0.0125$; If $t \in (0.6,1.0]$, set $h = \Delta t = 0.025$ and calculate 160 steps in every time interval.

Case iv: If $t \in [0,0.4]$, set $h = \Delta t = 0.0125$; If $t \in (0.4,0.6]$, set $h = \Delta t = 0.00625$; If $t \in (0.6,1.0]$, set $h = \Delta t = 0.0125$ and calculate 320 steps in every time interval.

The numerical solutions $U^n, \tilde{\boldsymbol{Z}}^n$ are computed and the L^2-norm error estimates of $U^n - u^n, \tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{z}}^n$ are obtained, see Tables I and II below, respectively.

Table I. L^2-norm error estimates of $U^n - u^n$

time	Case i	Case ii	Case iii	Case iv
t=0.4	0.0454	0.0232	0.0117	0.0057
t=0.6	0.0250	0.0134	0.0064	0.0031
t=1.0	0.0289	0.0158	0.0079	0.0039

Table II. L^2-norm error estimates of $\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{z}}^n$

time	Case i	Case ii	Case iii	Case iv
t=0.4	0.0531	0.0303	0.0159	0.0068
t=0.6	0.0637	0.0348	0.0161	0.0074
t=1.0	0.0627	0.0384	0.0190	0.0093

The convergence rates of $U^n - u^n$, $\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{z}}^n$ are given in Table III, which are also shown by Figure 1. From Fig. (a) and (b), we can see that the convergence rate of $U^n - u^n$ is one order and that of $\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{z}}^n$ is little smaller than one order at the beginning. But the convergence rate $\tilde{\boldsymbol{Z}}^n - \tilde{\boldsymbol{z}}^n$ will be close to one order when h decreases, which is consistent with the analysis in this paper.

Table III. Convergence rate of L^2-norm

time	Rates of $\|u^n - U^n\|$			Rates of $\|\tilde{\boldsymbol{z}}^n - \tilde{\boldsymbol{Z}}^n\|$		
	i/ii	ii/iii	iii/iv	i/ii	ii/iii	iii/iv
t=0.4	0.9683	0.9857	1.0383	0.8082	0.9276	1.2178
t=0.6	0.9024	1.0620	1.0388	0.8718	1.1121	1.1300
t=1.0	0.8512	1.0072	1.0153	0.7074	1.0136	1.0352

For Case iv, we compare the exact solution u, $\tilde{\boldsymbol{z}}$ with the approximate solution $U, \tilde{\boldsymbol{Z}}$ at time $t = 0.25,\ 0.5,\ 0.75$ respectively, see Figure 2. From Fig. (c) and Fig. (d), we can see that the approximate solutions are very close to the exact solutions.

V. Conclusions

We have considered the upwind-mixed finite element method with moving grids for quasi-nonlinear Sobolev equations. This method is constructed by two methods. The convection term is approximated by the upwind method and the diffusion term is discretized by an expanded mixed finite element method. This method can simultaneously approximate the scalar unknown function and the adjoint vector function effectively. We have proved optimal error estimates in L^2-norm for both the scalar unknown function and the adjoint vector function, and presented numerical experiments to verify the validity of this method.

In this paper, the Sobolev equations we have considered are of quasi-nonlinear type. We can extend our method to the whole nonlinear Sobolev equations. The results for this case will be presented in a forthcoming paper.

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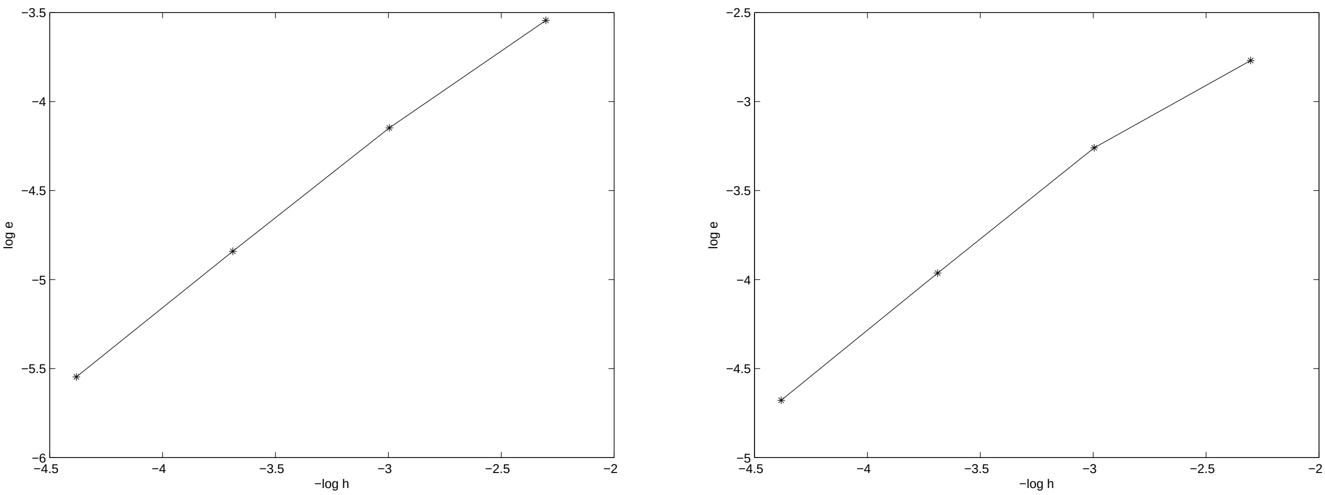
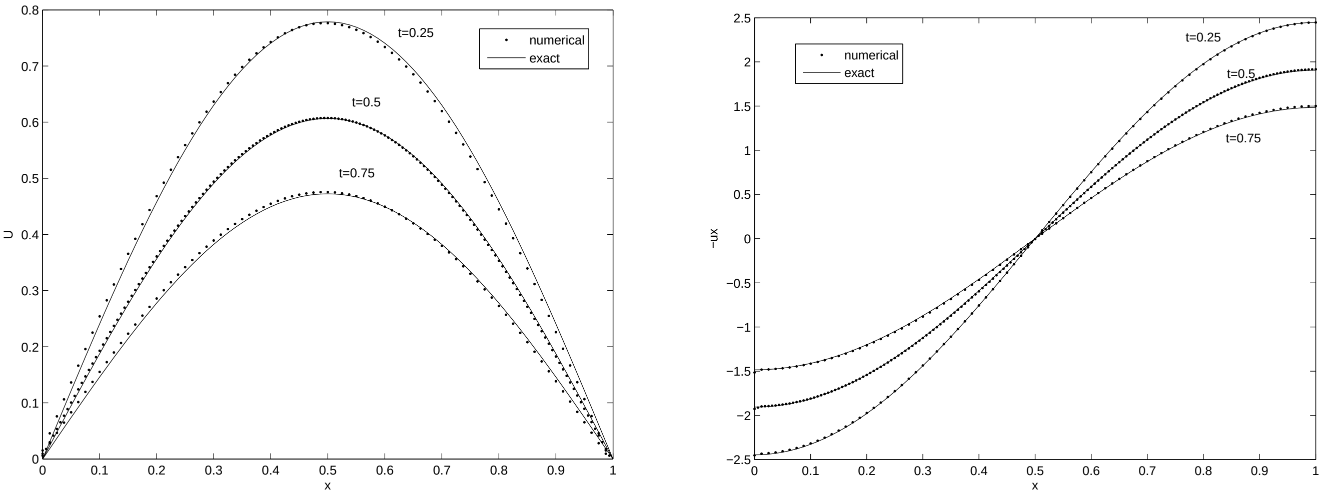


(a) convergence rates of $\|U-u\|$ (b) convergence rates of $\|\tilde{Z}-\tilde{z}\|$

Figure 1: Convergence rate figures



(c) comparing u with U (d) comparing $\tilde{z}$ with $\tilde{Z}$

Figure 2: Compare figures for Case iv at time $t = 0.25,\, 0.5,\, 0.75$

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