Periodic Solutions in Shifts Delta(+/-) for a Nabla Dynamic System of Nicholson's Blowflies on Time Scales

Lili Wang, Pingli Xie, and Meng Hu

***Abstract*—In this paper, based on some properties of nabla exponential function $\hat{e}_p(t,t_0)$ and shift operators $\delta_\pm$ on time scales, by using Krasnoselskii's fixed point theorem in a cone and some mathematical methods, sufficient conditions are established for the existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ for a nabla dynamic system of Nicholson's blowflies on time scales of the following form:**

$$x^{\nabla}(t)=-a(t)x(t)+\sum_{i=1}^{m}b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))},$$

where $t\in\mathbb{T},\mathbb{T}\subset\mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P\in[t_0,\infty)_\mathbb{T}$ and $t_0\in\mathbb{T}$ is nonnegative and fixed. Finally, two numerical examples are presented to illustrate the feasibility and effectiveness of the results.

***Index Terms*—positive periodic solution; Nicholson's blowflies model; nabla dynamic equation; shift operator; time scale.**

I. Introduction

THE theory of time scales was introduced by S. Hilger [1] in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. A time scale is a nonempty arbitrary closed subset of reals. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors [2,3] powerfully. Nowadays the theory on time scales has been widely applied to ecological dynamic systems.

In 1980, Gurney et al. [4] proposed a mathematical model

$$x'(t)=-\delta x(t)+px(t-\tau)e^{-ax(t-\tau)}$$

to describe the dynamics of Nicholson's blowflies, where $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $1/a$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. Nicholson's blowflies model and its analogous equations on time scales have attracted much attention in the past few years; see, for example, [5,6].

The existence problem of periodic solutions is of importance to biologists since most models deal with certain types of populations. In the paper of Kaufmann and Raffoul [7], the authors were the first to define the notion of periodic time scales, by satisfying the additivity "there exists a $\omega>0$ such that $t\pm\omega\in\mathbb{T},\forall t\in\mathbb{T}$." Under this additivity all periodic time scales are unbounded above and below. However, there are many time scales that are of interest to biologists and scientists such as $\overline{q^{\mathbb{Z}}}$ and $\cup_{k=1}^{\infty}[3^{\pm k},2.3^{\pm k}]\cup\{0\}$ which do not satisfy the additivity. To overcome such difficulties, Adıvar introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t\pm\omega$ for a fixed $\omega>0$. He defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [8] and then generalized in [9].

In recent years, periodic solutions in shifts $\delta_\pm$ for some nonlinear dynamic equations on time scales with delta derivative have been studied by many authors; see, for example, [10-13]. However, to the best of our knowledge, there are few papers published on the existence of periodic solutions in shifts $\delta_\pm$ for a dynamic equation on time scales with nabla derivative.

Motivated by the above, in the present paper, we first study some properties of the nabla exponential function $\hat{e}_p(t,t_0)$ and shift operators $\delta_\pm$ on time scales, and then we consider the following nabla dynamic system of Nicholson's blowflies on time scales:

$$x^{\nabla}(t)=-a(t)x(t)+\sum_{i=1}^{m}b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))},\quad (1)$$

where $t\in\mathbb{T},\mathbb{T}\subset\mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P\in[t_0,\infty)_\mathbb{T}$ and $t_0\in\mathbb{T}$ is nonnegative and fixed; $a,b_i\in C_{ld}(\mathbb{T},(0,\infty))$ $(i=1,2,\dots,m)$ are Δ-periodic in shifts $\delta_\pm$ with period ω and $-a\in\mathcal{R}^+$; $c_i\in C_{ld}(\mathbb{T},(0,\infty))$ are periodic in shifts $\delta_\pm$ with period ω for $i=1,2,\cdots,m$; $\tau_i(i=1,2,\dots,m)$ are fixed if $\mathbb{T}=\mathbb{R}$ and $\tau_i\in[P,\infty)_\mathbb{T}$ if $\mathbb{T}$ is periodic in shifts $\delta_\pm$ with period P.

For convenience, we introduce the notation

$$f^*=\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_\mathbb{T}}f(t),\ f_*=\inf_{t\in[t_0,\delta_+^{\omega}(t_0)]_\mathbb{T}}f(t),$$

where f is a positive and bounded periodic function.

Take the initial condition

$$x(s)=\phi(s),\phi\in C_{ld}([\delta_-(\tau^*,0),0]_\mathbb{T},(0,\infty)),\phi\not\equiv0,\quad (2)$$

where $\tau^*=\max_{1\le i\le m}\tau_i$.

It is easy to prove that the initial value problem (1) and (2) has a unique non-negative solution $x(t)$ on $[0,\infty)_\mathbb{T}$.

The main purpose of this paper is to establish sufficient conditions for the existence and nonexistence of positive

Manuscript received June 5, 2017; revised September 6, 2017. This work was supported in part by the Key Project of Scientific Research in Colleges and Universities in Henan Province (Nos.16A110008; 18A110005).

L. Wang and M. Hu are with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000 China e-mail: ay_wanglili@126.com.

P. Xie is with the School of Science, Henan University of Technology, Zhengzhou, Henan, 450001 China.

periodic solutions in shifts $\delta_{\pm}$ of system (1) using Krasnoselskii's fixed point theorem in a cone and some mathematical methods.

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$.

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. The backwards graininess function $\nu : \mathbb{T}_k \to [0, +\infty)$ is defined by $\nu(t) = t - \rho(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left-dense point in $\mathbb{T}$ and its right-side limits exist at right-dense points in $\mathbb{T}$.

The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if $1-\nu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$. The set of all ν-regressive and ld-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}_\nu^+ = \{p \in \mathcal{R}_\nu : 1 - \nu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If p is a ν-regressive function, then the nabla exponential function $\hat{e}_r$ is defined by

$$\hat{e}_p(t, s) = \exp\left\{ \int_s^t \hat{\xi}_{\nu(\tau)}(p(\tau))\nabla\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\hat{\xi}_h(z) = \begin{cases} -\frac{\mathrm{Log}(1-hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Lemma 1. [14] *If $p \in \mathcal{R}_\nu$, and $a, b, c \in \mathbb{T}$, then*

(i) $\hat{e}_0(t, s) \equiv 1$ *and* $\hat{e}_p(t, t) \equiv 1$*;*
(ii) $\hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t, s)$*;*
(iii) $\hat{e}_p(t, s)\hat{e}_p(s, r) = \hat{e}_p(t, r)$*;*
(iv) $(\hat{e}_p(t, s))^\nabla = p(t)\hat{e}_p(t, s)$*;*
(v) $(\hat{e}_p(c, \cdot))^\nabla = -p(\hat{e}_p(c, \cdot))^\rho$ *and* $\int_a^b p(t)\hat{e}_p(c, \rho(t))\nabla t = \hat{e}_p(c, a) - \hat{e}_p(c, b)$*;*
(vi) $\int_a^b f(\rho(t))g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(t)\nabla t.$

For more details about the calculus on time scales, see [14].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, +\infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, +\infty)_\mathbb{T}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, +\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, +\infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 1. [15] *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, +\infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, +\infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 2. [15] *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, +\infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, +\infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 3. *(∇-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is ∇-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, +\infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are ∇-differentiable with ld-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\nabla\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, +\infty)_{\mathbb{T}^*}$ is called the period of f.*

Similar to the proofs of Lemma 2, Corollary 1 and Theorem 2 in [15], we can get the following two lemmas.

Lemma 2. *$\delta_+^\omega(\rho(t)) = \rho(\delta_+^\omega(t))$ and $\delta_-^\omega(\rho(t)) = \rho(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 3. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a ∇-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, +\infty)_{\mathbb{T}^*}$. Assume that $f \in C_{ld}(\mathbb{T})$, then*

$$\int_{t_0}^t f(s)\nabla s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\nabla s.$$

Lemma 4. [16] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Assume that the shifts $\delta_\pm^\omega$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, +\infty)_{\mathbb{T}^*}$. Then the ν-graininess function $\nu : \mathbb{T} \to [0, +\infty)$ satisfies*

$$\nu(\delta_\pm^\omega(t)) = \delta_\pm^{\nabla\omega}(t)\nu(t).$$

Lemma 5. [16] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Assume that the shifts $\delta_\pm^\omega$ are ∇-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, +\infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}_\nu$ is ∇-periodic in shifts $\delta_\pm$ with the period ω. Then*

(i) $\hat{e}_p(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = \hat{e}_p(t, t_0)$ *for* $t, t_0 \in \mathbb{T}^*$*;*
(ii) $\hat{e}_p(\delta_\pm^\omega(t), \rho(\delta_\pm^\omega(s))) = \hat{e}_p(t, \rho(s)) = \frac{\hat{e}_p(t,s)}{1-\nu(t)p(t)}$ *for* $t, s \in \mathbb{T}^*$.

Lemma 6. [14] *Assume that r is ν-regressive and $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\nabla = r(t)y + f(t),\ y(t_0) = y_0$$

is given by

$$y(t) = \hat{e}_r(t, t_0)y_0 + \int_{t_0}^t \hat{e}_r(t, \rho(\tau))f(\tau)\nabla\tau.$$

Set

$$X = \left\{x : x \in C_{ld}(\mathbb{T}, \mathbb{R}), x(\delta_+^\omega(t)) = x(t)\right\}$$

with the norm $\|x\| = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|$, then X is a Banach space.

Lemma 7. *The function $x(t) \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_t^{\delta_+^\omega(t)} G(t,s)\sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s, \quad (3)$$

where

$$G(t,s) = \frac{\hat{e}_{-a}(t,\rho(s))}{\hat{e}_{-a}(t_0,\delta_+^\omega(t_0)) - 1}.$$

Proof: If $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1). By using Lemmas 1 and 6, for any $s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}$, we have

$$x(s) = \hat{e}_{-a}(s,t)x(t) + \int_t^s \hat{e}_{-a}(s,\rho(\theta)) \times \sum_{i=1}^m b_i(\theta)x(\delta_-(\tau_i,\theta))e^{-c_i(\theta)x(\delta_-(\tau_i,\theta))}\nabla\theta.$$

Let $s = \delta_+^\omega(t)$ in the above equality, we have

$$x(\delta_+^\omega(t)) = \hat{e}_{-a}(\delta_+^\omega(t),t)x(t) + \int_t^{\delta_+^\omega(t)} \hat{e}_{-a}(\delta_+^\omega(t),\rho(\theta)) \times \sum_{i=1}^m b_i(\theta)x(\delta_-(\tau_i,\theta))e^{-c_i(\theta)x(\delta_-(\tau_i,\theta))}\nabla\theta.$$

Noticing that $\hat{e}_{-a}(t,\delta_+^\omega(t)) = \hat{e}_{-a}(t_0,\delta_+^\omega(t_0))$, $x(\delta_+^\omega(t)) = x(t)$, by Lemma 1, then $x(t)$ satisfies (3).

Let $x(t)$ be an ω-periodic solution in shifts $\delta_\pm$ of (3). By (3) and Lemmas 1, 2 and 5, we have

$$\begin{aligned} x^\nabla(t) &= -a(t)x(t) \\ &\quad + G(\rho(t),\delta_+^\omega(t))\sum_{i=1}^m b_i(\delta_+^\omega(t))\delta_+^{\nabla\omega}(t) \\ &\quad \times x(\delta_-(\tau_i,\delta_+^\omega(t)))e^{-c_i(\delta_+^\omega(t))x(\delta_-(\tau_i,\delta_+^\omega(t)))} \\ &\quad - G(\rho(t),t)\sum_{i=1}^m b_i(t)x(\delta_-(\tau_i,t)) \\ &\quad \times e^{-c_i(t)x(\delta_-(\tau_i,t))} \\ &= -a(t)x(t) \\ &\quad + \sum_{i=1}^m b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))}. \end{aligned}$$

So, $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1). This completes the proof. ■

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$0 < \frac{1}{\xi-1} \le G(t,s) \le \frac{\xi}{\xi-1}, \ \forall s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}, \quad (4)$$

where $\xi = \hat{e}_{-a}(t_0,\delta_+^\omega(t_0))$. By Lemma 5, we have

$$G(\delta_+^\omega(t),\delta_+^\omega(s)) = G(t,s), \ \forall t \in \mathbb{T}^*, s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}. \quad (5)$$

Define K, a cone in X, by

$$K = \left\{x \in X : x(t) \ge \frac{1}{\xi}\|x\|, \forall t \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}\right\} \quad (6)$$

and an operator $H : K \to X$ by

$$(Hx)(t) = \int_t^{\delta_+^\omega(t)} G(t,s)\sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s. \quad (7)$$

In the following, we shall give some lemmas concerning K and H defined by (6) and (7), respectively.

Lemma 8. *$H : K \to K$ is well defined.*

Proof: For any $x \in K$, $t \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$. In view of (7), by Lemma 3 and (5), we have

$$\begin{aligned} &(Hx)(\delta_+^\omega(t)) \\ &= \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t),s)\sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s \\ &= \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t),\delta_+^\omega(s))\sum_{i=1}^m b_i(\delta_+^\omega(s))\delta_+^{\nabla\omega}(s) \\ &\quad \times x(\delta_-(\tau_i,\delta_+^\omega(s)))e^{-c_i(\delta_+^\omega(s))x(\delta_-(\tau_i,\delta_+^\omega(s)))}\nabla s \\ &= \int_t^{\delta_+^\omega(t)} G(t,s)\sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s \\ &= (Hx)(t), \end{aligned}$$

that is, $Hx \in X$.

Furthermore, for any $x \in K$, $t \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned} (Hx)(t) &\ge \frac{1}{\xi-1}\int_t^{\delta_+^\omega(t)} \sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s \\ &= \frac{1}{\xi}\cdot\frac{\xi}{\xi-1}\int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s \\ &\ge \frac{1}{\xi}\|Hx\|, \end{aligned}$$

that is, $Hx \in K$. This completes the proof. ■

Lemma 9. *$H : K \to K$ is completely continuous.*

Proof: Clearly, H is continuous on $[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$. For any $x \in K$, $t \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned} \|Hx\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} (Hx)(t) \\ &\le \frac{\xi}{\xi-1}\int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s)x(\delta_-(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s \\ &< \frac{\xi}{\xi-1}\cdot\frac{B}{c_*} := M_1, \end{aligned} \quad (8)$$

where

$$c_* = \min_{1\le i\le m} c_{i*}, \ B := \int_{t_0}^{\delta_+^\omega(t_0)} \sum_{i=1}^m b_i(s)\nabla s.$$

Furthermore, for $t \in \mathbb{T}$, we have

$$(Hx)^\nabla(t) = -a(t)(Hx)(t) + \sum_{i=1}^m b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))},$$

and

$$\begin{aligned}\|(Hx)^{\nabla}(t)\| &= \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} |-a(t)(Hx)(t)\\ &\quad+\sum_{i=1}^{m} b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))}|\\ &\le a^*M_1+\frac{1}{c_*}\sum_{i=1}^{m} b_i^*.\end{aligned}$$

To sum up, $\{Hx : x \in K\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, the functional H is completely continuous. This completes the proof. ■

III. Existence result

In this section, we shall state and prove our main result about the existence of at least one positive periodic solution in shifts $\delta_{\pm}$ of system (1).

Lemma 10. *(Guo-Krasnoselskii [17]) Let X be a Banach space and $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $H : K\cap(\overline{\Omega}_2\setminus\Omega_1) \to K$ is a completely continuous operator such that, either*

(1) $\|Hx\| \le \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Hx\| \ge \|x\|, x \in K \cap \partial\Omega_2$*; or*
(2) $\|Hx\| \ge \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Hx\| \le \|x\|, x \in K \cap \partial\Omega_2$.

Then H has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 11. *Let*

$$\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}. \tag{9}$$

Then there exist positive constants M_1 and M_2 such that for $x \in K$,

$$M_2 \le \|Hx\| \le M_1. \tag{10}$$

Proof: From (8), for any $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$\|Hx\| \le M_1. \tag{11}$$

From (9), there exists a $q > 1$ such that

$$\sum_{i=1}^{m} b_i(t) > qa(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}. \tag{12}$$

For any $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$\begin{aligned}(Hx)(t) &= \int_t^{\delta_+^{\omega}(t)} G(t,s)\sum_{i=1}^{m} b_i(s)x(\delta_-(\tau_i,s))\\ &\quad\times e^{-c_i(s)x(\delta_-(\tau_i,s))}\nabla s\\ &> q\int_{t_0}^{\delta_+^{\omega}(t_0)} \frac{a(s)\hat{e}_{-a}(t_0,\rho(s))}{\hat{e}_{-a}(t_0,\delta_+^{\omega}(t_0))-1}\\ &\quad\cdot \min_{1\le i\le m}\inf_{s\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\{x(\delta_-(\tau_i,s))\\ &\quad\times e^{-c_i(s)x(\delta_-(\tau_i,s))}\}\nabla s\\ &= q\int_{t_0}^{\delta_+^{\omega}(t_0)} \frac{1}{\hat{e}_{-a}(t_0,\delta_+^{\omega}(t_0))-1}\\ &\quad\cdot \min_{1\le i\le m}\inf_{s\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\{x(\delta_-(\tau_i,s))\\ &\quad\times e^{-c_i(s)x(\delta_-(\tau_i,s))}\}\nabla[\hat{e}_{-a}(t_0,s)]\\ &\ge q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\},\end{aligned} \tag{13}$$

where $c^* = \max_{1\le i\le m} c_i^*$.

Comparing (3) with (7), we also have for $x \in K, t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$x(t) > q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\},$$

which implies that

$$x_* > q\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\}. \tag{14}$$

In the same way as (8), $x(t) \le M_1$, which implies that

$$x^* \le M_1. \tag{15}$$

If $\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\} = x^*e^{-c^*x^*}$, then

$$(Hx)(t) > qM_1e^{-c^*M_1} := M_{21} > 0. \tag{16}$$

If $\min\{x_*e^{-c^*x_*}, x^*e^{-c^*x^*}\} = x_*e^{-c^*x_*}$, from (14), $x_* > qx_*e^{-c^*x_*}$, which implies that

$$x_* > \frac{\ln q}{c^*}.$$

From (13), we obtain

$$(Hx)(t) > q\frac{\ln q}{c^*}e^{-c^*\cdot\frac{\ln q}{c^*}} = \frac{\ln q}{c^*} := M_{22} > 0. \tag{17}$$

Let $M_2 = \min\{M_{21}, M_{22}\}$, then for $x \in K$,

$$\|Hx\| \ge M_2. \tag{18}$$

This completes the proof. ■

Theorem 1. *Assume that*

$$\sum_{i=1}^{m} b_i(t) > a(t), t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}.$$

Then system (1) has at least one positive ω-periodic solution in shifts $\delta_{\pm}$.

Proof: Let

$$\Omega_1 = \{x \in X : \|x\| \le M_2\},$$

and

$$\Omega_2 = \{x \in X : \|x\| \le M_1\}.$$

Clearly, Ω_1 and Ω_2 are open bounded subsets in X, and $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. From Lemma 8, $H : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous.

If $x \in K \cap \partial\Omega_2$, which implies that $\|x\| = M_1$, from Lemma 11, $\|Hx\| \le M_1$. Hence $\|Hx\| \le \|x\|$ for $x \in K \cap \partial\Omega_2$.

If $x \in K \cap \partial\Omega_1$, which implies that $\|x\| = M_2$, from Lemma 11, $\|Hx\| \le M_2$. Hence $\|Hx\| \ge \|x\|$ for $x \in K \cap \partial\Omega_1$.

From the cone fixed point theorem (Lemma 10), the operator H has at least one fixed point lying in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$, i.e., system (1) has at least one positive ω-periodic solution in shifts $\delta_{\pm}$. This completes the proof. ■

IV. Nonexistence result

In this section, we shall state and prove our main result about the nonexistence of positive periodic solution in shifts $\delta_{\pm}$ of system (1).

Lemma 12. *Assume that*

$$\sum_{i=1}^{m} b_i(t) \leq \frac{1}{2}a(t), t \in [t_0, \delta_{+}^{\omega}(t_0)]_{\mathbb{T}}. \tag{19}$$

Then every positive solution of system (1) tends to zero as $t \to \infty$.

Proof: Let $x(t)$ be any positive solution of system (1). By using Lemma 5, integrating system (1) from t_0 to $t(> t_0)$, we have

$$\begin{aligned} x(t) &= \hat{e}_{-a}(t,t_0)x(t_0) \\ &\quad + \int_{t_0}^{t} \hat{e}_{-a}(t,\rho(s)) \sum_{i=1}^{m} b_i(s)x(\delta_{-}(\tau_i,s)) \\ &\quad \times e^{-c_i(s)x(\delta_{-}(\tau_i,s))}\nabla s. \end{aligned} \tag{20}$$

From (19),

$$\begin{aligned} x(t) &\leq \hat{e}_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}\int_{t_0}^{t} a(s)\hat{e}_{-a}(t,\rho(s))\nabla s \\ &= \hat{e}_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}\int_{t_0}^{t} \nabla[\hat{e}_{-a}(t,s)] \\ &= \hat{e}_{-a}(t,t_0)x(t_0) + \frac{1}{2c_*}[1-\hat{e}_{-a}(t,t_0)]. \end{aligned}$$

Let $\beta = \limsup_{t\to\infty} x(t)$, then $0 \leq \beta < \infty$.

Next, we shall prove $\beta = 0$. We have some possible cases to consider.

Case 1. $x^{\nabla}(t) > 0$ eventually. Choose $t_0 > 0$ such that $x^{\nabla}(t) > 0$ for $t \geq t_0$. Let $\eta > 0$ be a sufficient large number with $\delta_{-}(\tau_i, t) > t_0, i = 1,2,\ldots,m$ for $t > t_0 + \eta$. Then $0 < x(\delta_{-}(\tau_i,t)) < x(t)$ for $t \geq t_0 + \eta$ and $i = 1,2,\ldots,m$. From (1), for $t \geq t_0 + \eta$,

$$\begin{aligned} 0 &< -a(t)x(t) + \sum_{i=1}^{m} b_i(t)x(\delta_{-}(\tau_i,t))e^{-c_i(t)x(\delta_{-}(\tau_i,t))} \\ &< \left[\sum_{i=1}^{m} b_i(t) - a(t)\right]x(t) < 0. \end{aligned}$$

This contradiction shows that Case 1 is impossible.

Case 2. $x^{\nabla}(t) < 0$ eventually. Choose $t_0 > 0$ such that $x^{\nabla}(t) < 0$ for $t \geq t_0$. Then $\beta < x(\delta_{-}(\tau_i,t)) < x(\delta_{-}(\tau_i,t_0))$ for $t \geq t_0 + \eta$ and $i = 1,2,\ldots,m$. From (19) and (20), we have

$$\begin{aligned} x(t) &\leq \hat{e}_{-a}(t,t_0)x(t_0) + \frac{1}{2}\max_{1\leq i\leq m} x(\delta_{-}(\tau_i,t_0))e^{-c_*\beta} \\ &\quad \times[1-\hat{e}_{-a}(t,t_0)]. \end{aligned} \tag{21}$$

Let $t \to \infty$ in (21), we obtain

$$\beta \leq \frac{1}{2}\max_{1\leq i\leq m} x(\delta_{-}(\tau_i,t_0))e^{-c_*\beta}. \tag{22}$$

Again let $t_0 \to \infty$ in (22), we have that $\beta \leq \beta(\frac{1}{2}e^{-c_*\beta})$, which implies that $\beta = 0$.

Case 3. $x^{\nabla}(t)$ is oscillatory. By the definition of oscillatory, then

(i) there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$x^{\nabla}(t_n) = 0 \text{ and } \lim_{n\to\infty} x(t_n) = \beta;$$

or

(ii) there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned} &x^{\nabla}(t_n)x^{\nabla}(\rho(t_n)) < 0 \text{ for } n = 1,2,\ldots, \\ &\text{and } \lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} x(\rho(t_n)) = \beta. \end{aligned}$$

In case (i), from (1),

$$\begin{aligned} &a(t_n)x(t_n) \\ &= \sum_{i=1}^{m} b_i(t_n)x(\delta_{-}(\tau_i,t_n))e^{-c_i(t_n)x(\delta_{-}(\tau_i,t_n))} \\ &\leq x(\delta_{-}(\tau_l,t_n))e^{-c_*x(\delta_{-}(\tau_l,t_n))}\sum_{i=1}^{m} b_i(t_n), \end{aligned} \tag{23}$$

where $l = l(n) \in \{1,2,\ldots,m\}$ such that

$$\begin{aligned} &x(\delta_{-}(\tau_l,t_n))e^{-c_*x(\delta_{-}(\tau_l,t_n))} \\ &= \max_{1\leq i\leq m} x(\delta_{-}(\tau_i,t_n))e^{-c_i(t_n)x(\delta_{-}(\tau_i,t_n))}. \end{aligned}$$

From (19) and (23), we have

$$2x(t_n)e^{c_*x(\delta_{-}(\tau_l,t_n))} \leq x(\delta_{-}(\tau_l,t_n)). \tag{24}$$

Set $\alpha = \limsup_{n\to\infty} x(\delta_{-}(\tau_l,t_n))$, then $\alpha \leq \beta$. Finding the superior limit of both sides of (24), we obtain

$$\beta(2e^{c_*\alpha}) \leq \alpha,$$

then

$$\beta(2e^{c_*\alpha}) \leq \alpha \leq \beta,$$

which implies that $\beta = \alpha = 0$.

In case (ii), from (1),

$$\begin{aligned} &a(t_n)a(\rho(t_n))x(t_n)x(\rho(t_n)) \\ &+\sum_{i=1}^{m} b_i(t_n)x(\delta_{-}(\tau_i,t_n))e^{-c_i(t_n)x(\delta_{-}(\tau_i,t_n))} \\ &\times\sum_{i=1}^{m} b_i(\rho(t_n))x(\delta_{-}(\tau_i,\rho(t_n))) \\ &\times e^{-c_i(\rho(t_n))x(\delta_{-}(\tau_i,\rho(t_n)))} \\ <\; &a(t_n)x(t_n)\sum_{i=1}^{m} b_i(\rho(t_n))x(\delta_{-}(\tau_i,\rho(t_n))) \\ &\times e^{-c_i(\rho(t_n))x(\delta_{-}(\tau_i,\rho(t_n)))} \\ &+a(\rho(t_n))x(\rho(t_n))\sum_{i=1}^{m} b_i(t_n)x(\delta_{-}(\tau_i,t_n)) \\ &\times e^{-c_i(t_n)x(\delta_{-}(\tau_i,t_n))} \\ \leq\; &[a(t_n)x(t_n)\sum_{i=1}^{m} b_i(\rho(t_n)) \\ &+a(\rho(t_n))x(\rho(t_n))\sum_{i=1}^{m} b_i(t_n)] \\ &\times x(\delta_{-}(\tau_l,\hat{t}_n))e^{-c_*x(\delta_{-}(\tau_l,\hat{t}_n))}, \end{aligned} \tag{25}$$

where $l = l(n) \in \{1,2,\ldots,m\}$, $\hat{t}_n = \{t_n, \rho(t_n)\}$, such that

$$\begin{aligned} &x(\delta_{-}(\tau_l,\hat{t}_n))e^{-c_*x(\delta_{-}(\tau_l,\hat{t}_n))} \\ &= \max_{1\leq i\leq m}\{x(\delta_{-}(\tau_i,t_n))e^{-c_i(t_n)x(\delta_{-}(\tau_i,t_n))}, \\ &\quad x(\delta_{-}(\tau_i,\rho(t_n)))e^{-c_i(\rho(t_n))x(\delta_{-}(\tau_i,\rho(t_n)))}\}. \end{aligned}$$

From (19) and (25), we have

$$\begin{aligned} &2x(t_n)x(\rho(t_n))e^{c_* x(\delta_-(\tau_l,\hat{t}_n))} \\ \leq\ & [x(t_n)+x(\rho(t_n))]x(\delta_-(\tau_l,\hat{t}_n)). \quad (26) \end{aligned}$$

Set $\alpha = \limsup_{n\to\infty} x(\delta_-(\tau_l,t_n))$, then $\alpha \leq \beta$. Finding the superior limit of both sides of (26), we obtain

$$\beta e^{c_*\alpha} \leq \alpha,$$

then

$$\beta e^{c_*\alpha} \leq \alpha \leq \beta,$$

which implies that $\beta = \alpha = 0$. This completes the proof. ■

From Lemma 12, we can get the following Theorem.

Theorem 2. *Assume that the condition (19) hold. Then system (1) has no positive ω-periodic solution in shifts $\delta_\pm$.*

V. Numerical examples

Consider the following Nicholson's blowflies model on time scales $\mathbb{T}$

$$\begin{aligned} x^\nabla(t) &= -a(t)x(t) \\ &\quad + \sum_{i=1}^{2} b_i(t)x(\delta_-(\tau_i,t))e^{-c_i(t)x(\delta_-(\tau_i,t))}. \quad (27) \end{aligned}$$

Example 1. Take

$$\begin{aligned} &a(t) = a_0 + \frac{|\sin 2t + \cos 3t|}{2}, \\ &b_1(t) = e^{e-1}(10+0.005|\sin t|), \\ &b_2(t) = e^{e-1}(10+0.005|\cos t|), \\ &c_1(t) = c_2(t) = 0.25 + 0.025|\sin 3t + \cos 2t|. \end{aligned}$$

Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then $\omega = \pi$ and $\delta_+^\omega(t) = t + \pi$. It is easy to verify $a(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ satisfy

$$\begin{aligned} &a(\delta_+^\omega(t))\delta_+^{\nabla\omega}(t) = a(t),\ b_i(\delta_+^\omega(t))\delta_+^{\nabla\omega}(t) = b_i(t), \\ &c_i(\delta_+^\omega(t)) = c_i(t),\ \forall t \in \mathbb{T}^*,\ i = 1, 2, \end{aligned}$$

and $-a \in \mathcal{R}^+$.

Case I. If $a_0 = 18$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) \geq 20e^{e-1} > a(t), t \in \mathbb{R}.$$

According to Theorem 1, when $\mathbb{T} = \mathbb{R}$, system (27) exists at least one positive π-periodic solution in shifts $\delta_\pm$.

Case II. If $a_0 = 240$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) \leq 20.02e^{e-1} < \frac{1}{2}a(t), t \in \mathbb{R}.$$

According to Theorem 2, when $\mathbb{T} = \mathbb{R}$, system (27) has no positive periodic solution in shifts $\delta_\pm$.

Example 2. Take

$$\begin{aligned} &a(t) = \frac{1}{a_0 t}, b_1(t) = \frac{1}{2t},\ b_2(t) = \frac{1}{3t}, \\ &c_1(t) = c_2(t) = 0.25. \end{aligned}$$

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, then $\omega = 4$ and $\delta_+^\omega(t) = 4t$. It is easy to verify $a(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ satisfy

$$\begin{aligned} &a(\delta_+^\omega(t))\delta_+^{\nabla\omega}(t) = a(t),\ b_i(\delta_+^\omega(t))\delta_+^{\nabla\omega}(t) = b_i(t), \\ &c_i(\delta_+^\omega(t)) = c_i(t),\ \forall t \in \mathbb{T}^*,\ i = 1, 2, \end{aligned}$$

and $-a \in \mathcal{R}^+$.

Case I. If $a_0 = 6$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} > a(t), t \in 2^{\mathbb{N}_0}.$$

According to Theorem 1, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (27) exists at least one positive 4-periodic solution in shifts $\delta_\pm$.

Case II. If $a_0 = \frac{1}{2}$, by a direct calculation, we can get

$$\sum_{i=1}^{2} b_i(t) = \frac{5}{6t} < \frac{1}{2}a(t), t \in 2^{\mathbb{N}_0}.$$

According to Theorem 2, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (27) has no positive periodic solution in shifts $\delta_\pm$.

VI. Conclusion

Two problems for a Nicholson's blowflies model with time delays on time scales have been studied, namely, existence and nonexistence of positive periodic solutions in shifts $\delta_\pm$ on time scales. It is important to notice that the methods used in this paper can be extended to other types of biological models; see, for example, [18-20]. Future work will include biological dynamic systems modeling and analysis on time scales.

References

[1] S. Hilger, *Ein Masskettenkalkl mit Anwendug auf Zentrumsmanning-faltigkeiten [Ph.D. thesis]*, Universität Würzburg, 1988.

[2] B. Jain, A. Sheng, "An exploration of the approximation of derivative functions via finite differences", *Rose-Hulman Undergraduate Math J.*, vol. 8, pp. 1-19, 2007.

[3] F. Atici, D. Biles, A. Lebedinsky, "An application of time scales to economics", *Math. Comput. Model.*, vol. 43, pp. 718-726, 2006.

[4] W. Gurney, S. Blythe, R. Nisbet, "Nicholson's blowflies (revisited)", *Nature*, vol. 287, pp. 17-21, 1980.

[5] C. Wang, "Existence and exponential stability of piecewise mean-square almost periodic solutions for impulsive stochastic Nicholson's blowflies model on time scales", *Appl. Math. Comput.*, vol. 248, no. 1, pp. 101-112, 2014.

[6] Z. Yao, "Existence and global asymptotic stability of almost periodic positive solution for Nicholson's blowflies model with linear harvesting term on time scales", *Math. Appl.*, vol. 28, no. 1, pp. 224-232, 2015.

[7] E. Kaufmann, Y. Raffoul, "Periodic solutions for a neutral nonlinear dynamical equation on a time scale", *J. Math. Anal. Appl.*, vol. 319, no. 1, pp. 315-325, 2006.

[8] M. Adıvar, "Function bounds for solutions of Volterra integro dynamic equations on the time scales", *Electron. J. Qual. Theo.*, vol. 7, pp. 1-22, 2010.

[9] M. Adıvar, Y. Raffoul, "Existence of resolvent for Volterra integral equations on time scales", *B. Aust. Math. Soc.*, vol. 82, no. 1, pp. 139-15, 2010.

[10] E. Çetin, "Positive periodic solutions in shifts $\delta_\pm$ for a nonlinear first-order functional dynamic equation on time scales", *Adv. Differ. Equ.*, vol. 2014, 2014:76.

[11] E. Çetin, F. Topal, "Periodic solutions in shifts $\delta_\pm$ for a nonlinear dynamic equation on time scales", *Abstr. Appl. Anal.*, vol. 2012, Article ID 707319.

[12] M. Hu, L. Wang, Z. Wang, "Positive periodic solutions in shifts $\delta_\pm$ for a class of higher-dimensional functional dynamic equations with impulses on time scales", *Abstr. Appl. Anal.*, vol. 2014, Article ID 509052.

[13] M. Adıvar, H. Koyuncuoğlu, Y. Raffoul, "Existence of periodic solutions in shifts $\delta_\pm$ for neutral nonlinear dynamic systems", *Appl. Math. Comput.*, vol. 242, pp. 328-339, 2014.

[14] M. Bohner, A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*, Birkhauser, Boston, 2001.

[15] M. Adıvar, "A new periodicity concept for time scales", *Math. Slovaca*, vol. 63, no. 4, pp. 817-828, 2013.

[16] M. Hu, L. Wang, "Multiple positive periodic solutions in shifts $\delta_\pm$ for an impulsive nabla dynamic equation on time scales", *Int. J. Dyn. Syst. Diff. Equ.*, vol. 7, no. 1, pp. 1-17, 2017.

[17] D. Guo, V. Lakshnikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
[18] C. Kang, H. Miao, X. Chen, "Global Stability Analysis for a Delayed HIV Infection Model with General Incidence Rate and Cell Immunity," Engineering Letters, vol. 24, no. 4, pp392-398, 2016.
[19] L. Xu, Y. Liao, "On an Almost Periodic Gilpin-Ayala Competition Model of Phytoplankton Allelopathy," IAENG International Journal of Applied Mathematics, vol. 47, no. 2, pp182-190, 2017.
[20] S. Lo, C. Lin, "Cooperative-Competitive Analysis and Tourism Forecasting of Southern Offshore Islands in Taiwan by Grey Lotka-Volterra Model," Engineering Letters, vol. 25, no. 2, pp183-190, 2017.