Strong Tensor Non-commutative Residuated Lattices

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***Abstract*—In this paper, we study the properties of tensor operators on non-commutative residuated lattices. We give some equivalent conditions of (strict) strong tensor non-commutative lattices and investigate the relation between state operators on $\mathcal{L}$ and state operators on $\mathcal{L}^T$. Moreover, we give the representation theory for (strict) strong tensor non-commutative residuated lattices and obtain the one to one correspondence between tense filters in $\mathcal{L}$ and tense congruences on $\mathcal{L}$.**

***Index Terms*—tensor operator, non-commutative residuated lattice, frame, filter, congruence.**

I. Introduction

Residuated lattices are an important algebraic structure in mathematics. The works on residuated lattices were initiated by Krull, Dilworth and Ward etc. ([8], [13], [16], [17]). Also, these structures are closely related to logics. BL-algebra are algebras of basic fuzzy logics. MV-algebras are algebras of Łukasiewicz infinite valued logics and Heyting algebras are algebras of intuitionistic logics. Residuated lattices are a common generalization of these algebras.

Classical tense logic is the propositional logic with two tense operators G which reveals the future and H which expresses the past. Burges [2] studied tensor operators on Boolean algebra. Later, many authors have investigated tensor operators on other algebras. Diaconescu and Georgescu [7] studied the tensor operators for MV-algebra and Łukasiewicz-Moisil algebras. Chajda, Kolăřík and Paseka ([5], [6]) studied tense operators for effect algebras for investigating quantum structures dynamically. Recently, Bakhshi [1] studied the algebraic properties of tense operators for non-commutative residuated lattices. The Dedekind-MacNeill completion of non-commutative lattices with involutive is investigated in [1].

In this paper, we will further study the tensor operators on non-commutative residuated lattices. We give some characterizations of tensor non-commutative residuated lattices which extend some results on effect algebras in [6]. The condition $\neg\neg x = x$ is important in studying tense operators for effect algebras. However, this condition is not valid in non-commutative residuated lattices. We have to overcome this difficulty for studying tense operators on non-commutative residulated lattices. in this paper, we get the one to one correspondence between tense filters (not normal tense filters) of $\mathcal{L}$ and tense congruences on $\mathcal{L}$. The paper is constructed as follows: In Section 2, we give some basic properties on tensor non-commutative residuated lattices. In Section 3, we give some equivalent conditions for strong tensor non-commutative residuated lattices and investigate the relation between state operators on $\mathcal{L}$ and state operators on $\mathcal{L}^T$. Finally, we give the representation theory of (strict) strong tensor non-commutative residuated lattices in Section 4. In Section 5, we study the relation between tense filters in $\mathcal{L}$ and tense congruences on $\mathcal{L}$.

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II. Preliminaries

In this section, we give some basic notions and properties on non-commutative lattices which is useful in the paper.

A structure $(\mathcal{L}, \bigcap, \bigcup, *, \rightarrow, \rightsquigarrow, 0, 1)$ is called a non-commutative residuated lattice, if the following conditions are satisfied:

L1) $(\mathcal{L}, \bigcap, \bigcup, 0, 1)$ is a bounded lattice;

L2) $(\mathcal{L}, *, 1)$ is a monoid (not necessarily commutative);

L3) $x * y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$, for all $x, y, z \in \mathcal{L}$.

For $x \in \mathcal{L}$, we denote $x \rightarrow 0$ by $\neg x$ and denote $x \rightsquigarrow 0$ by $\sim x$.

A non-commutative residuated lattice $\mathcal{L}$ is called to be involutive, if $\neg \sim x = \sim \neg x = x$, for all $x \in \mathcal{L}$.

II.1 Proposition. *([1]) Let $\mathcal{L}$ be a non-commutative residuated lattice. For all $x, y, z \in \mathcal{L}$, then*

1) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

2) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$; $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$.

3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$. Particularly, $x \leq y$ implies $\neg y \leq \neg x$ and $\sim y \leq \sim x$.

4) $x * (x \rightsquigarrow y) \leq x \wedge y$; $(x \rightarrow y) * x \leq x \wedge y$;

5) $x * \sim x = 0 = \neg x * x$.

6) $x * y = 0$ iff $x \leq \neg y$ iff $y \leq \sim x$; Particularly, $x \leq \sim \neg x$ and $x \leq \neg \sim x$.

II.2 Definition. ([1]) Let $(\mathcal{L}, \bigcap, \bigcup, *, \rightarrow, \rightsquigarrow, 0, 1)$ be a non-commutative residuated lattice and G, H be maps of $\mathcal{L}$ into itself. We call $(\mathcal{L}, G, H)$ a tensor non-commutative residuated lattice, if the following conditions are satisfied:

TRL1) $G(1) = 1, H(1) = 1$.

TRL2) $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$, $G(x \rightsquigarrow y) \leq G(x) \rightsquigarrow G(y)$, $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$, $H(x \rightsquigarrow y) \leq H(x) \rightsquigarrow H(y)$.

TRL3) $x \leq GP_{\neg}(x) \wedge GP_{\sim}(x), x \leq HF_{\neg}(x) \wedge HF_{\sim}(x)$, where

$$P_{\neg}(x) = \neg H(\sim x), \quad P_{\sim}(x) = \sim H(\neg x),$$

$$F_{\neg}(x) = \neg G(\sim x), \quad F_{\sim}(x) = \sim G(\neg x).$$

II.3 Definition. Let $(\mathcal{L}, G, H)$ be a tensor non-commutative residuated lattice. We call $(\mathcal{L}, G, H)$ to be a strong tensor non-commutative residuated lattice, if $G(0) = H(0) = 0$.

In fact, G and H are strong tense operators on $\mathcal{L}$ in Example 1 in [1] (or see [15]).

II.4 Proposition. *([1]) Let $(\mathcal{L}, \bigcap, \bigcup, *, \to, \rightsquigarrow, 0, 1)$ be a non-commutative residuated lattice. For all $x, y \in \mathcal{L}$, the following conditions are satisfied.*

1) $x \le y$ implies $G(x) \le G(y)$, $H(x) \le H(y)$, $F_\neg(x) \le F_\neg(y)$, $F_\sim(x) \le F_\sim(y)$, $G_\neg(x) \le G_\neg(y)$, $G_\sim(x) \le G_\sim(y)$.

2) $G(x) * G(y) \le G(x * y)$, $H(x) * H(y) \le H(x * y)$.

II.5 Proposition. *Let $(\mathcal{L}, \bigcap, \bigcup, *, \to, \rightsquigarrow, 0, 1)$ be a non-commutative residuated lattice and G, H be maps of $\mathcal{L}$ into itself. Then $(\mathcal{L}, G, H)$ is a strong tensor non-commutative residuated lattice if and only if*

STRL1) $G(0) = 0, H(0) = 0, G(1) = 1, H(1) = 1$.

STRL2) $x \le y$ implies $G(x) \le G(y)$ and $H(x) \le H(y)$; $G(x) * G(y) \le G(x * y)$, $H(x) * H(y) \le H(x * y)$.

STRL3) $x \le GP_\neg(x) \wedge GP_\sim(x), x \le GF_\neg(x) \wedge GF_\sim(x)$, where

$$P_\neg(x) = \neg H(\sim x), \quad P_\sim(x) = \sim H(\neg x),$$

$$F_\neg(x) = \neg G(\sim x), \quad F_\sim(x) = \sim G(\neg x).$$

Proof: $\Longrightarrow$: By Definition II.3 and Proposition II.4, we get the desired result.

$\Longleftarrow$: We only need to prove TRL2). For all $x, y \in \mathcal{L}$, we have $(x \to y) * x \le y$ by Proposition 2.1 4). Hence,

$$G(x \to y) * G(x) \le G((x \to y) * x) \le G(y).$$

This implies $G(x \to y) \le G(x) \to G(y)$.

Similarly, we can get $G(x \rightsquigarrow y) \le G(x) \rightsquigarrow G(y)$; $H(x \to y) \le H(x) \to H(y)$; $H(x \to y) \le H(x) \to H(y)$. ■

II.6 Lemma. *Let $(\mathcal{L}, \bigcap, \bigcup, *, \to, \rightsquigarrow, 0, 1)$ be a strong tensor non-commutative residuated lattice. For all $x, y \in \mathcal{L}$, the following conditions are satisfied.*

1) $G(\neg x) \le \neg G(x)$, $G(\sim x) \le \sim G(x)$, $H(\neg x) \le \neg H(x)$, $H(\sim x) \le \sim H(x)$.

2) $P_\neg(\neg x) \ge P_\sim(x)$, $P_\sim(\sim x) \ge \sim P_\neg(x)$, $F_\neg(\neg x) \ge \neg F_\sim(x)$, $F_\sim(\sim x) \ge \sim F_\neg(x)$.

Proof: 1) By $(x \to 0) * x \le 0$, we have

$$G(x \to 0) * G(x) \le G((x \to 0) * x) \le G(0).$$

Hence, $G(\neg x) \le \neg G(x)$.

Similarly, $G(\sim x) \le \sim G(x)$, $H(\neg x) \le \neg H(x)$, $H(\sim x) \le \sim H(x)$.

3) Using Proposition II.3, we get $P_\neg(\neg x) = \neg H(\sim \neg x) \ge \neg \sim H(\neg x) = \neg P_\sim(x)$.

Similarly, we have $P_\sim(\sim x) \ge \sim P_\neg(x)$, $F_\neg(\neg x) \ge \neg F_\sim(x)$, $F_\sim(\sim x) \ge \sim F_\neg(x)$. □ ■

Notation Let $(\mathcal{L}, \bigcap, \bigcup, *, \Rightarrow, 0, 1)$ be a non-commutative residuated lattice. The following condition

$$\forall\, x, y \in \mathcal{L}, \quad \neg y \le \neg x \Leftrightarrow x \le y \Leftrightarrow \sim y \le \sim x.$$

is denoted by (C).

If $\mathcal{L}$ is involutive, then $\mathcal{L}$ satisfies condition (C).

II.7 Proposition. *Let $(\mathcal{L}, \bigcap, \bigcup, *, \to, \rightsquigarrow, 0, 1)$ be a non-commutative residuated lattice with condition (C). For all $x, y \in \mathcal{L}$, we have*

$P_\neg G(x) \le x$, $P_\sim G(x) \le x$, $F_\neg H(x) \le x$, $F_\sim H(x) \le x$.

Proof: By D4) of Definition II.2, we have

$$\begin{aligned}\neg x \le HF_\neg(\neg x) = H(\neg G(\sim \neg x)) \le H(\neg G(x)) \\ \le \neg \sim H(\neg G(x)) = \neg P_\sim(G(x)).\end{aligned}$$

This proves $P_\sim G(x) \le x$.

Similarly, we have $P_\neg G(x) \le x$, $F_\neg H(x) \le x$, $F_\sim H(x) \le x$. ■

III. Characterizations of Strong Tensor Non-Commutative Residuated Lattices

In this section, we give some equivalent characterizations of (strict) strong tensor non-commutative residuated lattices. Some techniques in [6] are used.

III.1 Theorem. *Let $\mathcal{L}$ be a non-commutative residuated lattice with condition (C) and $G, H : \mathcal{L} \longrightarrow \mathcal{L}$ be mappings. Then the followings are equivalent.*

1) $(\mathcal{L}, G, H)$ is a strong tensor non-commutative residuated lattice.

2) i) $G(x) * G(y) \le G(x * y), H(x) * H(y) \le H(x * y)$.

ii) G has two left adjoints $P_\neg$ and $P_\sim$ such that $P_\neg(x) = \neg H(\sim x)$, $P_\neg(1) = 1$ $P_\sim(x) = \sim H(\neg x)$, $P_\sim(1) = 1$ and H has two left adjoints $F_\neg$ and $F_\sim$ such that $F_\neg(x) = \neg G(\sim x)$, $F_\neg(1) = 1$, $F_\sim(x) = \sim G(\neg x)$, $F_\sim(1) = 1$, for all $x \in \mathcal{L}$.

Proof: 1) $\Longrightarrow$ 2) Suppose that $(\mathcal{L}, G, H)$ is a strong tensor non-commutative residuated lattice. Then i) is valid by Proposition II.5.

ii) For all $x, y \in \mathcal{L}$, if $P_\neg(x) \le y$, then we get $x \le GP_\neg(x) \le G(y)$ by STRL2) and STRL3). If $x \le G(y)$, then $P_\neg(x) \le P_\neg G(y) \le y$ by Proposition II.4. This proves that $P_\neg$ is a left adjoint of G. It is easy to check that $P_\neg(1) = \neg H(\sim 1) = \neg H(0) = \neg 0 = 1$. Analogously, $P_\sim$ is also a left adjoint of G and $P_\sim(1) = 1$. Similarly, H has two left adjoints $F_\neg$ and $F_\sim$ such that $F_\neg(x) = \neg G(\sim x)$, $F_\neg(1) = 1$, $F_\sim(x) = \sim G(\neg x)$, $F_\sim(1) = 1$, for all $x \in \mathcal{L}$.

2) $\Longrightarrow$ 1) Note that G and H are the right adjoints, we have that G and H preserve infima and order. Hence, $G(1) = H(1) = 1$. This proves that STRL2) holds.

For all $y \in \mathcal{L}$, since $G(\sim y) \le G(\sim y)$ and G is right adjoint, we get $P_\sim(G(\sim y)) \le \sim y$. That is, $\sim H(\neg G(\sim y)) \le \sim y$. By condition (C), we have $y \le H(\neg G(\sim y)) = HF_\neg(y)$. Similarly, $y \le HF_\sim(y)$, $y \le GP_\neg(y)$, $y \le GP_\sim(y)$. Hence, STRL3) holds.

$\neg H(0) = \neg H(\sim 1) = P_\neg(1) = 1$ implies $H(0) = 0$. Similarly, $G(0) = 0$. This proves STRL1). ■

III.2 Lemma. *Let $(\mathcal{L}, G, H)$ be a non-commutative residuated lattice. For all $a_i, b_i \in \mathcal{L}, i \in I$, we have*

$$\bigwedge\{a_i \mid i \in I\} * \bigwedge\{b_i \mid i \in I\} \le \bigwedge\{a_i * b_i \mid i \in I\}.$$

Proof: For all $i \in I$, $\bigwedge\{a_i \mid i \in I\} \le a_i$ and $\bigwedge\{b_i \mid i \in I\} \le b_i$ hold. This concludes the desired result. ■

A pair (T, R) is called a frame if T is a nonempty set and R is a binary relation on T.

Let $\mathcal{L}$ be a non-commutative residuated lattice and T be a nonempty set. We denote the set of all mappings from T to $\mathcal{L}$ by $\mathcal{L}^T$.

For $f, g \in \mathcal{L}^T$, define operations on $\mathcal{L}^T$ by

$$f * g(x) = f(x) * g(x), \quad (f \vee g)(x) = f(x) \vee g(x),$$

$$(f \wedge g)(x) = f(x) \wedge g(x), \quad (f \to g)(x) = f(x) \to g(x).$$

we can see that $\mathcal{L}^T$ is a non-commutative residuated lattice (see [1]).

Let 0 and 1 be the elements in $\mathcal{L}^T$ such that $0(x) = 0$ and $1(x) = 1$, for all $x \in T$.

Similarly to Theorem 3 in [1], we also have the following theorem.

III.3 Theorem. *Let $(\mathcal{L}, G, H)$ be a strong tensor non-commutative residuated lattice and (T, R) be a frame. For all $p \in \mathcal{L}^T$, we can define $\widehat{G}, \widehat{H} : \mathcal{L}^T \to \mathcal{L}^T$ as*

$$\widehat{G}(p)(u) = \bigwedge\{p(v) \mid uRv\},$$

$$\widehat{H}(p)(u) = \bigwedge\{p(v) \mid vRu\}.$$

Then $(\mathcal{L}^T, \widehat{G}, \widehat{H})$ is a strong tensor non-commutative residuated lattice.

Proof: By Theorem 3 in [1], we only need to check that $\widehat{G}(0) = 0$, $\widehat{H}(0) = 0$, $\widehat{G}(1) = 1$ and $\widehat{H}(1) = 1$. ■

III.4 Definition. $(\mathcal{L}, G, H)$ is called a strict strong tensor non-commutative residuated lattice, if for all $x, y \in \mathcal{L}$,

$$G(x \to y) = G(x) \to G(y),$$

$$G(x \rightsquigarrow y) = G(x) \rightsquigarrow G(y),$$

$$H(x \to y) = H(x) \to H(y),$$

$$H(x \rightsquigarrow y) = H(x) \rightsquigarrow H(y).$$

III.5 Lemma. *Let $(\mathcal{L}, G, H)$ be a strict strong dynamic non-commutative residuated lattice. For all $x \in \mathcal{L}$, we have $G(\neg x) = \neg G(x)$, $G(\sim x) =\sim G(x)$, $H(\neg x) = \neg H(x)$, $H(\sim x) =\sim H(x)$.*

Proof: By Definition III.4, we have $G(x \to 0) = G(x) \to G(0) = G(x) \to 0$. So $G(\neg x) = \neg G(x)$. Similarly, $G(\sim x) =\sim G(x)$, $H(\neg x) = \neg H(x)$, $H(\sim x) =\sim H(x)$. ■

III.6 Theorem. *If $\mathcal{L}$ is a non-commutative residuated lattice with condition (C), $G, H : \mathcal{L} \to \mathcal{L}$ are mappings. Then the following conditions are equivalent.*

1) $(\mathcal{L}, G, H)$ is a strict strong tensor non-commutative residuated lattice.

2) G and H satisfy the following properties:

i) $G(0) = H(0) = 0$.

ii) G is both a left adjoint and a right adjoint to H.

iii) $G(x \to y) = G(x) \to G(y)$, $G(x \rightsquigarrow y) = G(x) \rightsquigarrow G(y)$.

iv) $H(x \to y) = H(x) \to H(y)$, $H(x \rightsquigarrow y) = H(x) \rightsquigarrow H(y)$.

Proof: 1) $\Longrightarrow$ 2) : i) By Definition II.3, we obviously have $G(0) = H(0) = 0$.

ii) For all $x, y \in \mathcal{L}$, if $x \le G(y)$, By Proposition II.7 we get $P_\neg(x) \le y$, i.e. $\neg H(\sim x) \le y \le \neg \sim y$. This implies $\sim y \le H(\sim x) =\sim H(x)$. Hence $H(x) \le y$. Conversely, if $H(x) \le y$, we have

$$\sim y \le\sim H(x) = H(\sim x) \le\sim \neg H(\sim x) =\sim P_\neg(x).$$

Then $P_\neg(x) \le y$. By Theorem III.1 again, $x \le G(y)$ holds. Hence, G is a right adjoint to H. Similarly, we can prove that H is also a right adjoint to G.

iii) and iv) are obvious.

2) $\Longrightarrow$ 1) : Since G is a right adjoint, G preserves infima and $G(1) = 1$. For all $x, y \in \mathcal{L}$, we have $G(x) \le G(y \to x * y) = G(y) \to G(x * y)$ by $x \le y \to x * y$. This implies $G(x) * G(y) \le G(x * y)$. Similarly, H preserves order and $H(x) * H(y) \le H(x * y)$. Then STRL2) holds.

Now, we prove STRL3). For all $x \in \mathcal{L}$, we have

$$H(x) \le H(\neg \sim x) \le \neg H(\sim x) = P_\neg(x).$$

By $H(x) \le H(x)$, then $x \le GH(x) \le GP_\neg(x)$. Similarly, $x \le GP_\sim(x)$, $x \le HF_\neg(x)$, $x \le HF_\sim(x)$. ■

In the following, we discuss the relation between state operators on commutative residuated lattice $\mathcal{L}$ and state operators on $\mathcal{L}^T$.

III.7 Definition. ([10]) Let $(\mathcal{L}, \bigcap, \bigcup, *, \to, 0, 1)$ be a residuated lattice and $\tau : \mathcal{L} \longrightarrow \mathcal{L}$ a map. If the following conditions are satisfied

1) $\tau(0) = 0$,
2) $x \to y = 1 \Longrightarrow \tau(x) \to \tau(y) = 1$,
3) $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$,
4) $\tau(x * y) = \tau(x) * \tau(x \to (x * y))$,
5) $\tau(\tau(x) * \tau(y)) = \tau(x) * \tau(y)$,
6) $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$,
7) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$,
8)$\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$,

for all $x, y \in \mathcal{L}$, then τ is called to be a state operator on $\mathcal{L}$.

The following proposition is Proposition 3.5 in [10], which is useful.

III.8 Proposition. *([10]) Let $\mathcal{L}$ be a residuated lattice and $\tau : \mathcal{L} \longrightarrow \mathcal{L}$ a state operator on $\mathcal{L}$. We have*

1) $\tau(1) = 1$,
2) $x \le y$ implies $\tau(x) \le \tau(y)$.

For $x, y \in \mathcal{L}$, we have $x \le y \Longleftrightarrow x \to y = 1$ ([1]).

III.9 Theorem. *If $\tau : \mathcal{L} \longrightarrow \mathcal{L}$ is a state operator on $\mathcal{L}$, the mapping $\bar{\tau} : \mathcal{L}^T \longrightarrow \mathcal{L}^T$ defined by $\bar{\tau}(f) = \tau f$ is also a state operator on $\mathcal{L}^T$.*

Proof: 1) Obviously, $\bar{\tau}(0) = 0$.
2) For $f, g \in \mathcal{L}^T$, if $f \to g = 1$, we have $f \le g$. For every $x \in \mathcal{L}$, we get $\tau f(x) \le \tau g(x)$ by Proposition III.8. It concludes that $\bar{\tau}(f) \le \bar{\tau}(g)$. That is, $\bar{\tau}(f) \to \bar{\tau}(g) = 1$.

3)

$$\begin{aligned} c\bar{\tau}(f \to g)(x) &= \tau(f \to g)(x) \\ &= \tau(f(x) \to g(x)) \\ &= \tau f(x) \to \tau(f(x) \wedge g(x)) \\ &= \tau f(x) \to \tau(f \wedge g)(x) \\ &= (\bar{\tau} \to \bar{\tau}(f \wedge g))(x). \end{aligned}$$

Therefore, $\bar{\tau}(f \to g) = \bar{\tau}(f) \to \bar{\tau}(f \wedge g)$. Similarly, we have

$$\bar{\tau}(f) * g) = \bar{\tau}(f) * \bar{\tau}(f \to f * g),$$

$$\bar{\tau}(\bar{\tau}(f) * \bar{\tau}(g)) = \bar{\tau}(f) * \bar{\tau}(g),$$

$$\bar{\tau}(\bar{\tau}(f) \to \bar{\tau}(g)) = \bar{\tau}(f) \to \bar{\tau}(g),$$

$$\bar{\tau}(\bar{\tau}(f) \wedge \bar{\tau}(g)) = \bar{\tau}(f) \wedge \bar{\tau}(g),$$

$$\bar{\tau}(\bar{\tau}(f) \vee \bar{\tau}(g)) = \bar{\tau}(f) \vee \bar{\tau}(g).$$

Hence, $\bar{\tau}$ is a state operator on $\mathcal{L}^T$. ■

IV. Representations of strong tensor non-commutative residuated lattices

In this section, we shall give representation theorems for strong tensor non-commutative residuated lattices and strict strong tensor non-commutative residuated lattices. Some proofs are similar to those in [6].

Let P and $P^{'}$ be two bounded posets. A map $f : P \longrightarrow P^{'}$ is called to be morphism, if f preserves order, top element and bottom element. A map $f : P \longrightarrow P^{'}$ is called to be order reflecting, if f is a morphism and

$$f(x) \leq f(y) \Longleftrightarrow x \leq y, \quad \forall x, y \in P.$$

IV.1 Definition. Let $\mathcal{L}$ and $\mathcal{L}^{'}$ be two non-commutative residuated lattices. A map $f : \mathcal{L} \longrightarrow \mathcal{L}^{'}$ is called a semi-morphism from $\mathcal{L}$ into $\mathcal{L}^{'}$, if f satisfies the followings:

1) f preserves order.
2) $f(x) * f(y) \leq f(x * y), \forall x, y \in \mathcal{L}$.
3) $f(0) = 0, f(1) = 1$.

A semi-morphism $f : \mathcal{L} \longrightarrow \mathcal{L}^{'}$ is called to be strict, if for all $x, y \in \mathcal{L}$,

$$f(x \rightarrow y) = f(x) \rightarrow f(y), f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y).$$

If f is a strict semi-morphism, for all $x \in \mathcal{L}$, we have

$$f(\neg x) = \neg f(x), \quad f(\sim x) = \sim f(x).$$

Let S be a set of semi-morphisms from $\mathcal{L}$ into $\mathcal{L}^{'}$. A subset $T \subseteq S$ is full, if for $x, y \in \mathcal{L}$,

$$x \leq y \Longleftrightarrow t(x) \leq t(y), \forall t \in T.$$

IV.2 Theorem. *Let $(\mathcal{L}, G, H)$ be a dynamic non-commutative residuated lattice with a full set S of semi-morphisms into a non-commutative residuated lattice $\mathcal{C}$. Then*

1) There exists a semi-morphisms set T satisfies the following conditions:

i) $S \subseteq T$;

ii) the map $\iota_{\mathcal{L}}^{T} : (\mathcal{L}, G, H) \longrightarrow (\mathcal{L}^{T}, \widehat{G}, \widehat{H})$ which sends x to $\iota_{\mathcal{L}}^{T}(x)$ is order reflecting, where $\iota_{\mathcal{L}}^{T}(x)(t) = t(x)$, for all $x \in \mathcal{L}, t \in T$.

2) There exists a frame (T, R) satisfies:

for all $s, t \in T$, $(s, t) \in R$ iff $\forall x \in \mathcal{L}, s(G(x)) \leq t(x)$. Moreover,

$$s(G(x)) = \bigwedge\{t(x) \mid sRt\}.$$

Proof: Let T be the smallest set consisting of semi-morphisms into $\mathcal{C}$ such that $S \subseteq T$ and $s \circ G, s \circ H \in T$. Let $R = \{(s, t) \in T \times T \mid s(G(x)) \leq t(x)\}$.

1) For all $x, y \in \mathcal{L}, s \in S$, we have

$$x \leq y \Longrightarrow s \circ G(x) \leq s \circ G(y), \quad s \circ H(x) \leq s \circ H(y).$$

$$(s \circ G)(x) * (s \circ G)(y) \leq s \circ (G(x) * G(y)) \leq (s \circ G)(x * y).$$

$$s \circ G(0) = s(G(0)) = s(0) = 0,$$

$$s \circ G(1) = s(G(1)) = s(1) = 1.$$

This concludes $s \circ G \in T$.

Similarly, $s \circ H \in T$.

2) For all $x, y \in \mathcal{L}, t \in T$, since S is a full set and $S \subseteq T$, we have

$$\begin{aligned} x \leq y \quad \Longleftrightarrow \quad & t(x) \leq t(y) \Longleftrightarrow \iota_{L}^{T}(x)(t) \leq \iota_{L}^{T}(y)(t) \\ & \Longleftrightarrow \iota_{L}^{T}(x) \leq \iota_{L}^{T}(y). \end{aligned}$$

Hence, ι_{L}^{T} is order reflecting.

Since $(\iota_{L}^{T}(x) * \iota_{L}^{T}(y))(t) = t(x) * t(y) \leq t(x * y) = \iota_{L}^{T}(x * y)(t)$, we have $\iota_{L}^{T}(x) * \iota_{L}^{T}(y) \leq \iota_{L}^{T}(x * y)$. Hence, ι_{L}^{T} is an order reflecting semi-morphisms into $\mathcal{C}$.

3) For all $s \in T$, there is $t \in T$ such that $t_s = s \circ G$. Therefore,

$$s(G(x)) = t_s(x) \geq \bigwedge\{t(x) \mid sRt\} \geq s(G(x)).$$

This implies that $\iota_{L}^{T}(G(x)) = \bigwedge\{t(x) \mid sRt\}$. That is, $\iota_{L}^{T}(G(x)) = \widehat{G}(\iota_{L}^{T}(x))$. Also, we have

$$s(G(x)) = \bigwedge\{t(x) \mid sRt\}.$$

■

Similarly, we can give the representation theorem for strict strong non-commutative residuated lattices.

IV.3 Theorem. *Let $(\mathcal{L}, G, H)$ be a strict strong non-commutative residuated lattice with a full set S of strict semi-morphisms into a non-commutative residuated lattice $\mathcal{C}$ and ($\mathcal{L}$ satisfy condition (C). Then*

1) There exists a strict semi-morphisms set T satisfies the following conditions:

i) $S \subseteq T$;

ii) the map $\iota_{\mathcal{L}}^{T} : (\mathcal{L}, G, H) \longrightarrow (\mathcal{L}^{T}, \widehat{G}, \widehat{H})$ which sends x to $\iota_{\mathcal{L}}^{T}(x)$ is order reflecting, where $\iota_{\mathcal{L}}^{T}(x)(s) = s(x)$, for all $x \in \mathcal{L}, s \in S$.

2) There is a frame (T, R) satisfies:

for all $s, t \in R$, $(s, t) \in R$ iff $\forall x \in \mathcal{L}, s(G(x)) \leq t(x)$. Moreover,

$$s(G(x)) = \bigwedge\{t(x) \mid sRt\},$$

$$s(H(x)) = \bigwedge\{t(x) \mid tRs\}.$$

Proof: Firstly, we show that, if s and t are morphisms, we have

$$s(G(x)) \leq t(x) \Longleftrightarrow t(H(x)) \leq s(x), \forall x \in \mathcal{L}.$$

For all $x \in \mathcal{L}$, suppose $s(G(x)) \leq t(x)$, then

$$s(\sim x) \leq s(\circ G P_{\neg}(\sim x))) \leq t(P_{\neg}(\sim x)) \leq \sim t P_{\neg}(x).$$

Since $P_{\neg}(x) = \neg H(\sim x) = \neg \sim H(x) \geq H(x)$, we have $t(H(x)) \leq t(P_{\neg}(x))$, hence,

$$\sim t(H(x)) \geq \sim t(P_{\neg}(x)) \geq s(\sim x) = \sim s(x).$$

It follows $t(H(x)) \leq s(x)$. Similarly, we can prove the other direction.

Define

$$\begin{aligned} R &= \{(s, t) \in T \times T \mid s(G(x)) \leq t(x)\} \\ &= \{(s, t) \in T \times T \mid t(H(x)) \leq s(x)\}. \end{aligned}$$

Analogously to the above theorem, we can get the desired result. ■

V. Congruences on commutative residuated lattices

There is a bijection correspondence between normal filters of $\mathcal{L}$ and congruences on $\mathcal{L}$. In [1], the author proves that there is a bijection correspondence between tense normal filters of $\mathcal{L}$ and tense congruences on $\mathcal{L}$. In this section, we will prove that there is a bijection correspondence between filters of $\mathcal{L}$ and congruences on $\mathcal{L}$ when $G(x*y)=G(x)*G(y), H(x*y)=H(x)*H(y)$, for $x,y\in\mathcal{L}$.

Recall that a filter F of $(\mathcal{L},G,H)$ is called to be a tense filter, if $G(x),H(x)\in F$, for all $x\in F$.

A congruence θ on $(\mathcal{L},G,H)$ is called to be a tense congruence, if $x\theta y$, then $G(x)\theta G(y)$ and $H(x)\theta H(y)$, for $x,y\in\mathcal{L}$.

In paper [14], the author gives the one to one correspondence between the ideals in quasi-Mv algebras and ideal congruences on quasi-Mv algebras. Inspired this fact, we will define a relation on $\mathcal{L}$, which can be used to construct congruences on $\mathcal{L}$. Further, we can give the one to one correspondence between tense filters in $\mathcal{L}$ and tense congruences on $\mathcal{L}$ under certain conditions.

Let F be a subset of $\mathcal{L}$. The relation $\mathcal{C}(F)$ on $\mathcal{L}$ is defined as following: for $x,y\in\mathcal{L}$,

$$x\mathcal{C}(F)y \Longleftrightarrow$$
$$G(x\to y)\wedge G(y\to x), H(x\to y)\wedge H(y\to x)\in F.(R)$$

V.1 Proposition. *Let $(\mathcal{L},G,H)$ be a tense commutative residuated lattice such that $G(x*y)=G(x)*G(y), H(x*y)=H(x)*H(y)$ and F be a tense filter of $\mathcal{L}$. The relation $\mathcal{C}(F)$ is a tense congruence on $\mathcal{L}$.*

Proof: 1) Since F is a filter, we have $1\in F$. For all $x\in\mathcal{L}$,

$$G(x)\to G(x)=1\in F$$

holds. This concludes that $\mathcal{C}(F)$ is reflexive.

2) The symmetry is obvious.

3) Suppose $x\mathcal{C}(F)y$ and $y\mathcal{C}(F)z$. We have

$$G(x\to y), G(y\to x), G(y\to z), G(z\to y)\in F.$$

Hence,

$$\begin{aligned} cG(x\to z) &\le G((x\to y)\to(z\to y)) \\ &\le G(x\to y)\to G(z\to y)\in F. \end{aligned}$$

Similarly, $G(z\to x)\in F$. So $\mathcal{C}(F)$ is transitive.

4) Suppose $x\mathcal{C}(F)y$ and $a\mathcal{C}(F)b$. We have

$$x*(x\to y)\le y,\quad a*(a\to b)\le b.$$

Hence,

$$x*a*(x\to y)*(a\to b)\le y*b.$$

It concludes that

$$(x\to y)*(a\to b)\le x*a\to y*b.$$

Then

$$\begin{aligned} G(x\to y)*G(a\to b) &= G((x\to y)*(a\to b)) \\ &\le G((x*a)\to(y*b)). \end{aligned}$$

By $x\to y\le (x\to z)\to(y\to z)$, we get

$$G(x\to y)\le G(x\to z)\to(y\to z)).$$

Then $(x\to z)\mathcal{C}(F)(y\to z)$.

5) Suppose $x\mathcal{C}(F)y$. Then $G(x\to y), G(y\to x)\in F$. By $G(x\to y)\le G(x)\to G(y)$, we have $G(G(x\to y))\le G(G(x)\to G(y))$. Since F is a tensor filter, we concludes $G(G(x\to y))\in F$ and so $G(G(x)\to G(y))\in F$. Similarly, $G(G(y)\to G(x))\in F$. This proves that $G(x)\mathcal{C}(F)G(y)$.

By above, we get that $\mathcal{C}(F)$ is a tense congruence on $\mathcal{L}$. ■

V.2 Proposition. *Let $(\mathcal{L},G,H)$ be a tense commutative residuated lattice and F be a subset of $\mathcal{L}$. If $\mathcal{C}(F)$ is a tensor congruence on $\mathcal{L}$, then $\mathcal{C}(F)(1)$ is a filter of $\mathcal{L}$.*

Proof: For $x,y\in\mathcal{C}(F)(1)$, we have

$$G(1)=G(x\to 1), G(x)=G(1\to x)\in F.$$

Hence,

$$G(1\to(x*y))=G(x*y)=G(x)*G(y)\in F,$$
$$G((x*y)\to 1)=G(1)\in F.$$

This implies that $x*y\in\mathcal{C}(F)(1)$.

If $x\in\mathcal{C}F(1)$ and $x\le y$, we have $(1\to x)\le(1\to y)$ and so $G(1\to x)\le G(1\to y)\in F$. Since $G(x\to 1)=G(y\to 1)=1\in F$, we get $y\in\mathcal{C}F(1)$. ■

V.3 Proposition. *Let θ be a tense congruence on $(\mathcal{L},G,H)$. Then $\theta(1)$ is a tense filter.*

Proof: Let $x,y\in\theta(1)$. We have $G(x),G(y)\in\theta(1)$.

By $x\theta 1, y\theta 1$, we concludes that $x*y\theta 1$, i.e. $x*y\in\theta(1)$.

If $x\le y$ and $x\theta 1$, then $x\to y\theta 1\to y$, i.e. $1\theta y$. Hence, $\theta(1)$ is a tensor filter. ■

V.4 Theorem. *Let $(\mathcal{L},G,H)$ be a tense commutative residuated lattice. There is a bijection between the tense filters of $\mathcal{L}$ and tense congruences on $\mathcal{L}$.*

Let A be a subset of $\mathcal{L}$. Denote by $\mathrm{Fil}(A)$ the filter generated by A. Ciung [3] proved that

$$\begin{aligned} \mathrm{Fil}(A) &= \{x\in\mathcal{L}\mid x\ge a_1*a_2*\cdots*a_n, n\in N, \\ &\qquad a_1,a_2,\cdots,a_n\in A\}. \end{aligned}$$

If F is a filter of $\mathcal{L}$ and $a\in\mathcal{L}$, then

$$\begin{aligned} \mathrm{Fil}(F,a)=\{x\in\mathcal{L}\mid x\ge (f_1*a^{n_1})*(f_2*a^{n_2})*\cdots* \\ (f_m*a^{n_m}), m\in N, n_1,n_2,\cdots,n_m\in N^+, \\ f_1,f_2,\cdots,f_m\in F\}. \end{aligned}$$

Similar to Proposition 5.1 of [12], we have the following proposition.

V.5 Proposition. *Let $\mathcal{L}$ be a tense residuated lattice and $a\in\mathcal{L}$ such that $G(a)=H(a)=a$. Then $Fil(F,a)$ is a tense filter of $\mathcal{L}$.*

Proof: For $x\in\mathrm{Fil}(F,a)$, there exist $y_1,y_2,\cdots,y_t\in F$, $m_1,m_2,\cdots,m_t\in N^+$ such that $x\ge y_1*a^{m_1}*y_2*a^{m_2}*\cdots*y_t*a^{m_t}$. Thus

$$\begin{aligned} cG(x) &\ge G(y_1*a^{m_1}*y_2*a^{m_2}*\cdots*y_t*a^{m_t}) \\ &\ge G(y_1)*G(a)^{m_1}*G(y_2)*G(a^{m_2})*\cdots* \\ &\quad G(y_t)*G(a)^{m_t}. \end{aligned}$$

This proves that $G(x)\in\mathrm{Fil}(F,a)$. ■

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