Periodic Solutions for $P-$Laplacian Differential Equation with Singular Forces of Attractive Type

De-xin Lan and Wen-bin Chen*

***Abstract*—This paper is concerned with the periodic solutions for $p-$Laplacian differential equation with singular forces of attractive type. By employing Mawhin's coincidence degree theorem and some analytical techniques, some new results on the existence of periodic solutions are derived. The numerical results demonstrate the remarkable accuracy and efficiency of our method compared with other schemes.**

***Index Terms*—Periodic solution, $p-$Laplacian equation, Continuation theorem, Singular forces.**

I. Introduction

THE aim of this paper is to consider the solvability of periodic boundary value problem for $p-$Laplacian differential equation with singular forces of attractive type as follows

$$(\varphi_p(x'(t)))' + f(x'(t)) + g(t, x(t)) = e(t), \qquad (1)$$

$$x(0) = x(T),\, x'(0) = x'(T), \qquad (2)$$

where $\varphi_p(s) = |s|^{p-2}s$ with $p > 1$, $f : R \times R \to R$ is continuous, $g : R \times (0, +\infty) \to R$ is an L^2-Cauathéodory function, $g(t, x)$ is $T-$periodic with the first variable and can be singular at $x = 0$, i.e., $g(t, x)$ can be unbounded as $x \to 0^+$. The problem (1)-(2) is of attractive type if $g(t, x) \to +\infty$ for $x \to 0^+$.

In the past few years, there were plenty of results on the existence of periodic solutions of Duffing, Liénard or Rayleigh type equation with a singularity, (see [1-8, 11-15] and the references therein). For example, recently, Zhang [3] studied the following Liénard equation with singular forces of repulsive type:

$$x''(t) + f(x(t))x'(t) + g(t, x(t)) = 0, \qquad (3)$$

where $f : R \times R \to R$ is continuous, $g : R \times (0, +\infty) \to R$ is an L^2-Cauathéodory function, $g(t, x)$ is $T-$periodic with the first variable and can be singular at $x = 0$, i.e., $g(t, x)$ can be unbounded as $x \to 0^+$. Equation (3) is repulsive type if $g(t, x) \to -\infty$ for $x \to 0^+$. Meanwhile, let

$$\bar{g}(x) = \frac{1}{T}\int_0^T g(t, x)\, dt,\ x > 0.$$

Assume that

$$\varphi(t) = \limsup_{x\to+\infty} \frac{g(t, x)}{x}$$

exists uniformly a.e. for $t \in [0, T]$. Where $\varphi \in C(R, R)$ and $\varphi(t + T) = \varphi(t)$, $\forall t \in R$.

Assume that the following conditions are satisfied.

(H_1) (Balance condition) There exist constants $0 < D_1 < D_2$ such that if x is a positive continuous $T-$periodic function satisfying

$$\frac{1}{T}\int_0^T g(t, x(t))\, dt = 0,$$

then

$$D_1 \le x(\tau) \le D_2,$$

for some $\tau \in [0, T]$.

(H_2) (Degree condition) $\bar{g}(x) < 0$ for all $x \in (0, D_1)$, and $\bar{g}(x) > 0$ for all $x > D_2$.

(H_3) (Decomposition condition) $g(t, x) = g_0(x) + g_1(t, x)$, where $g_0 \in C((0, +\infty), R)$ and $g_1 : [0, T] \times [0, +\infty) \to R$ is an L^2-Carathéodory function, i.e., g_1 is measurable with respect to the first variable, continuous with respect to the second one, and for any $b > 0$ there is $h_b \in L^2((0, T); [0, +\infty))$ such that $|g_1(t, x)| \le h_b(t)$ for a.e. $t \in [0, T]$ and all $x \in [0, b]$.

(H_4)(Strong force condition at $x = 0$) $\int_0^1 g_0(x)\, dx = -\infty$.

(H_5)(Small force condition at $x = \infty$)

$$||\varphi^+||_1 < \frac{\sqrt{3}}{T},\ (\varphi^+(t) = \max(\varphi(t), 0)).$$

In [3], the following theorem was proved.

Theorem 1. Assume that the conditions $(H_1) - (H_5)$ are satisfied. Then Eq. (3) has at least one positive $T-$periodic solution.

On the basis of work of Zhang, Wang [4] further studied periodic solutions for the Liénard equation with a singularity and a deviating argument, which is different from the literature [2],

$$x''(t) + f(x(t))x'(t) + g(t, x(t - \sigma)) = 0, \qquad (4)$$

where $0 \le \sigma < T$ is a constant, $f : R \times R \to R$ is continuous, $g : R \times (0, +\infty) \to R$ is an L^2-Cauathéodory function, $g(t, x)$ is $T-$periodic with the first variable and can be singular at $x = 0$, i.e., $g(t, x)$ can be unbounded as $x \to 0^+$. Eq. (4) is repulsive type if $g(t, x) \to -\infty$ for $x \to 0^+$. By using Mawhin's continuation theorem, the authors proved the following theorem.

Theorem 2. Assume that the conditions $(H_1) - (H_4)$ are satisfied. If further assumed:

(H_5') (Small force condition at $x = \infty$)

$$||\varphi||_\infty < \left(\frac{\pi}{T}\right)^2.$$

Then Eq.(4) has at least one positive $T-$periodic solution.

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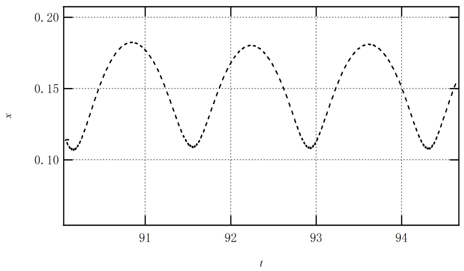

Fig. 1: Numerical solutions for Eq. (6)

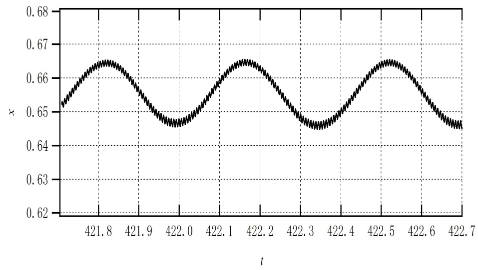

Fig. 2: Numerical solutions for Eq. (7)

Inspired by the above mentioned works, in this paper, we study the existence of positive periodic solution for $p-$Laplacian differential equation with singular forces of attractive type. To the best of our knowledge, there are fewer papers dealing with the periodic solutions for $p-$Laplacian differential equation with singular forces of attractive type in the literature. By applying Mawhin's continuation theorem, we prove the following theorem.

Theorem 3. Assume that the following conditions are satisfied:
(h_1) $\lim\limits_{x\to 0}\inf\limits_{t\in[0,\,T]} g(t,\,x)=+\infty$,
(h_2) There exist nonnegative constants $m_1,\,m_2,\,f(0)=0$ and $\alpha\leq p-1$ such that

$$|f(u)|\leq m_1|u|^{\alpha}+m_2,\,\forall u\in R.$$

(h_3) There exist constants $0<\bar{D}_1<\bar{D}_2$ such that $g(t,\,u)-e(t)>0,\,(t,\,u)\in[0,\,T]\times(0,\,\bar{D}_1]$, and $g(t,\,u)-e(t)<0,\,(t,u)\in[0,\,T]\times[\bar{D}_2,\,\infty)$.
Then the problem (1)-(2) has at least one positive $T-$periodic solution.

Remark 1. When $p=2$, Eq. (1) reduces to the following second-order differential equation

$$x''(t)+f(t,x'(t))+g(t,x(t))=0. \tag{5}$$

We can construct concrete functions g and f such that all conditions of Theorem 3 are satisfied. For example, consider the following equation

$$x''(t)+(x'(t))^3-\frac{1}{2}(1+0.5\sin(1000t))\frac{1}{x^3(t)}=\sin(100t). \tag{6}$$

Corresponding to Theorem 3 and (5), we have $f(x)=x^3$, $g(t,x(t))=-\frac{1}{2}+(1+0.5\sin(1000t))\frac{1}{x^3(t)}$, $e(t)=\sin(1000t)$, $m_1=1,\alpha=3,m_2=\frac{1}{2}$, $\bar{D}_1=0.1$, $\bar{D}_2=2$. By using Theorem 3, Eq. (6) has at least one positive $T-$periodic solution. It is not difficult to find that the singular item $g_0(x)$ do not include the independent variables t in Eq. (3) and Eq. (4). But, the singular item $g(t,x)$ in Eq. (6) has the independent variables t. This can also be illustrated by numerical simulation. By using MATLAB (R2013a) toolkit: $ode45$. Eq. (6) is simulated on tspan=[90.06, 94.65] with history=[2, 2]. Fig.1 shows that the equation admits one positive $T-$periodic solution.

Remark 2. When $p\neq 2$, We can consider the following equation

$$\begin{aligned}&(\varphi_3(x'(t)))'+(x'(t))^3-\frac{1}{2}-x(t)\\&+(9+0.5\sin(1000t))\frac{1}{x^{17}(t)}=\sin(1000t).\end{aligned} \tag{7}$$

By Theorem 3 and (1), we have $f(x)=x^3$, $g(t,x(t))=-\frac{1}{2}-x(t)+(9+0.5\sin(1000t))\frac{1}{x^{17}(t)}$, $e(t)=\sin(1000t)$, $m_1=1,\alpha=3,m_2=\frac{1}{2}$, $\bar{D}_1=\frac{1}{2}$, $\bar{D}_2=2$. According to Theorem 3, Eq. (7) has at least one positive $T-$periodic solution. Similarly, Eq. (7) can also be illustrated by numerical simulation, which is simulated on tspan=[421.7, 422.7] with history=[1.5, -1.5] see Fig. 2. Therefore, the results achieved in this paper are significant.

The rest of the paper is organized as follows. In section II, some necessary definitions and Lemmas are introduced. In section III, the existence of positive periodic solutions conditions are presented.

II. PRELIMINARIES

Lemma 1. [9] Let L be a Fredholm operator of index zero and let N be $L-$compact on $\bar{\Omega}$. Suppose that the following conditions are satisfied:
(a1) $Lx\neq\lambda Nx,\,\forall(x,\,\lambda)\in\partial\Omega\times(0,1)$;
(a2) $QNv\notin ImL,\,\forall x\in KerL\cap\partial\Omega$;
(a3) $deg\{JQN,\,\Omega\cap KerL,\,0\}\neq 0$, where $Q:Z\to Z$ is a projection with $ImL=KerQ$, $J:ImQ\to KerL$ is a isomorphism with $J(\theta)=\theta$, where θ is the zero element of Z.
Then $Lx=Nx$ has at least one solution in $D(L)\cap\bar{\Omega}$.

Lemma 2. [10] (Generalized Bellman's Inequality). Consider the following inequality:

$$|y(t)|\leq C+M\int_0^t|y(s)|^{\beta}\,ds, \tag{8}$$

where C,M,β are nonnegative constants and $t>0$. If $\beta\leq 1$, then for $t\in(0,T_0]$, we have $|y(s)|\leq D$, where

$$D=\begin{cases}Ce^{MT_0}, & if\ \beta=1;\\(C^{1-\beta}+MT_0(1-\beta))^{\frac{1}{1-\beta}}, & if\ \beta<1.\end{cases}$$

In order to use Mawhin's continuation theorem, we should consider the following system:

$$\begin{cases}x'(t)=\varphi_q(y(t)),\\y'(t)=-f(\varphi_q(y(t)))-g(t,x(t))+e(t),\end{cases} \tag{9}$$

where $\varphi_q(s)=|s|^{q-2}s$, $\frac{1}{p}+\frac{1}{q}=1$, $y(t)=\varphi_p(x'(t))$. Obviously, if $(x(t),y(t))^{\top}$ is a solution of (9), then $x(t)$ is a solution of (1)-(2).

Throughout this paper, let $X = Y = \{u = (x(t), y(t))^\top \in C(R, R^2),\ u(t) = u(t+T)\}$, where the norm $|u|_0 = \max\{|x|_0, |y|_0\}$, and $|x|_0 = \max_{t\in[0,T]} |x(t)|$, $|y|_0 = \max_{t\in[0,T]} |y(t)|$. It is obvious that X and Y are Banach spaces.

Now we define the operator

$$L : D(L) \subset X \to Y, Lu = u' = (x'(t), y'(t))^\top,$$

where $D(L) = \{u|u = (x(t), y(t))^\top \in C^1(R, R^2), u(t) = u(t+T)\}$.

Let $Z = \{u|u = (x(t), y(t))^\top \in C^1(R, R\times\Gamma), u(t) = u(t+T)\}$, where $\Gamma = \{x \in R, x(t) = x(t+T)\}$. Define a nonlinear operator $N : \bar{\Omega} \to Y$ as follows:

$$Nv = (\varphi_q(y(t)), -f(\varphi_q(y(t))) - g(t, x(t)) + e(t))^\top,$$

where $\Omega \subset (X \cap Z) \subset X$ and Ω is an open and bounded set. Then problem (1)-(2) can be written as $Lv = Nv$ in $\bar{\Omega}$.

We know

$$KerL = \{u|u \in X, u' = (x'(t), y'(t))^\top = (0, 0)^\top\},$$

then $x'(t) = 0$, $y'(t) = 0$. Obviously $x \in R$, $y \in R$, thus $KerL = R^2$, and it is also easy to prove that $ImL = \{z \in Y, \int_0^T z(s)ds = 0\}$. So, L is a Fredholm operator of index zero.

Let

$$P : X \to KerL, Pv = \frac{1}{T}\int_0^T v(s)\, ds,$$

$$Q : Y \to ImQ, Qz = \frac{1}{T}\int_0^T z(s)\, ds.$$

Let $K_p = L|^{-1}_{Kerp\cap D(L)}$, then it is easy to see that

$$(K_p z)(t) = \int_0^T G(t, s)z(s)\, ds,$$

where

$$G(t, s) = \begin{cases} \frac{s-T}{T}, & 0 \le t \le s; \\ \frac{s}{T}, & s \le t \le T. \end{cases}$$

For all Ω such that $\bar{\Omega} \subset (X \cap Z) \subset X$, we have $K_p(I - Q)N(\bar{\Omega})$ is a relative compact set of X, $QN(\bar{\Omega})$ is a bounded set of Y. Thus the operator N is L-compact in $\bar{\Omega}$.

III. Proof of Theorem 3

Firstly, let $\Omega_1 = \{x \in \Omega, Lx = \lambda Nx, \forall\lambda \in (0, 1)\}$. If $\forall\lambda \in \Omega_1$, we have

$$\begin{cases} x'(t) = \lambda\varphi_q(y(t)), \\ y'(t) = -\lambda f(\varphi_q(y(t))) - \lambda g(t, x(t)) + \lambda e(t). \end{cases} \tag{10}$$

By substituting $y(t) = \varphi_p(\frac{1}{\lambda}x'(t))$ into the second equation of (10), we have

$$(\varphi_p(\frac{1}{\lambda}x'(t)))' + \lambda f(\frac{1}{\lambda}x'(t)) + \lambda g(t, x(t)) = \lambda e(t), \lambda \in (0, 1) \tag{11}$$

Let $x(t)$ be an arbitrary $T-$periodic solution of (11). Assume that

$$x(t_1) = \max_{t\in[0,T]} x(t), x(t_2) = \min_{t\in[0,T]} x(t), t_1, t_2 \in [0, T]. \tag{12}$$

Then we get

$$x'(t_1) = 0, x''(t_1) \le 0. \tag{13}$$

$$x'(t_2) = 0, x''(t_2) \ge 0. \tag{14}$$

From (11), (14), and using $(\varphi_p(x'(t)))' = (p-1)|x'(t)|^{p-2}x''(t)$, we obtain

$$g(t_2, x(t_2)) = -\frac{1}{\lambda}(\varphi_p(\frac{1}{\lambda}x'(t_2)))' + e(t) \le e(t).$$

Then from the assumption (h_3), we must have that there exists $\bar{D}_1 > 0$ such that

$$x(t_2) > \bar{D}_1. \tag{15}$$

Similarly, substituting (11) and (13), we can see that there exist positive $\bar{D}_2$ such that

$$x(t_1) < \bar{D}_2. \tag{16}$$

(15) and (16) implies that $x(t)$ is bounded and

$$\bar{D}_1 < x(t_2) \le x(t) \le x(t_1) < \bar{D}_2. \tag{17}$$

Next we show that $y(t)$ is bounded. From the second equation of (10) and $x'(t_1) = 0$, hence $y(t_1) = 0$. Thus for any $t \in [0, T]$ such that $0 \le t_1 \le t$, by the assumption (h_2), we can write

$$\begin{aligned} & |y(t)| \\ = & \left|y(t_1) + \int_{t_1}^t y'(s)ds\right| = \int_{t_1}^t |y'(s)|ds \\ \le & \int_0^t |y'(s)|ds \\ \le & \int_0^t |-\lambda f(\varphi_q(y(s))) - \lambda g(s, x(s)) + \lambda e(s)|ds \\ \le & \int_0^t |f(\varphi_q(y(s)))| + |g(s, x(s)) - e(s)|ds \\ \le & \int_0^t |f(\varphi_q(y(s)))|ds + \int_0^T |g(s, x(s)) - e(s)|ds \\ \le & T \max_{t\in[0,T], x\in(\bar{D}_1,\bar{D}_2)} |g(t, x) - e(t)| \\ & + Tm_2 + m_1\int_0^t |y(s)|^{(q-1)\alpha}ds. \end{aligned} \tag{18}$$

Let $D_1 = T \max_{t\in[0,T], x\in(\bar{D}_1,\bar{D}_2)} |g(t, x) - e(t)|$ and $\beta = (q-1)\alpha = \frac{\alpha}{p-1}$. Then from (18) we get

$$|y(t)| \le D_1 + m_1\int_0^t |y(s)|^\beta ds.$$

Note that $0 \le \beta \le 1$ since $0 \le \alpha \le p - 1$ from the assumption (h_3). Now from the generalized Bellman's inequality (8), we obtain $|y(t)| \le D_2$ for all $t \in [t_1, T]$, where

$$D_2 = \begin{cases} D_1 e^{m_1 T}, & if\ \alpha = p-1; \\ (D_1^{\frac{p-1-\alpha}{p-1}} + \frac{m_1 T(p-1-\alpha)}{p-1})^{\frac{p-1}{p-1-\alpha}}, & if\ \alpha < p-1. \end{cases}$$

If $0 \le t \le t_1$, we have $0 \le t_1 \le t + T \le 2T$ and from the

$T-$periodicity of $y(t)$ and the assumption (h_3), we have

$$
\begin{aligned}
&|y(t)|\\
= & |y(t+T)| = \left|y(t_1) + \int_{t+T}^{t_1} y'(s)ds\right|\\
= & \left|\int_{t+T}^{t_1} y'(s)ds\right|\\
\leq & \int_{t_1}^{t+T} |y'(s)|ds\\
\leq & \int_{0}^{t+T} |y'(s)|ds\\
\leq & \int_{0}^{t+T} |\lambda f(\varphi_q(y(s))) + \lambda g(s,x(s)) - e(s)|ds\\
\leq & \int_{0}^{t+T} |f(\varphi_q(y(s)))| + |g(s,x(s)) - e(s)|ds\\
\leq & \int_{0}^{t+T} |f(\varphi_q(y(s)))|ds + \int_{0}^{2T} |g(s,x(s)) - e(s)|ds\\
\leq & 2T \max_{t\in[0,T], x\in(\bar{D}_1,\bar{D}_2)} |g(t,x) - e(t)| + 2Tm_2\\
& + m_1 \int_{0}^{t+T} |y(s)|^{(q-1)\alpha} ds. \quad (19)
\end{aligned}
$$

From the above inequality, it follows that for $0 \leq t \leq t_1$, we have $|y(t)| = |y(t+T)| \leq 2D_1 + m_1 \int_0^{t+T} |y(s)|^\beta ds$. This implies that for $0 \leq t \leq t_1$, we have $|y(t)| \leq D_3$, where

$$
D_3 = \begin{cases} 2D_1 e^{m_1 T}, & if\ \alpha = p-1;\\ (2D_1^{\frac{p-1-\alpha}{p-1}} + \frac{2m_1 T(p-1-\alpha)}{p-1})^{\frac{p-1}{p-1-\alpha}}, & if\ \alpha < p-1. \end{cases}
$$

Since $D_2 \leq D_3$, the above inequalities imply that

$$|y|_0 := \max_{t\in[0,T]} |y(t)| \leq D_3, \quad (20)$$

which again implies

$$|x'|_0 \leq D_3^{q-1} = D_3^{\frac{1}{p-1}}. \quad (21)$$

Let $\Omega_2 = \{u : u \in kerL, QNu = 0\}$. If $u \in R^2$ is a constant vector with

$$
\begin{cases} |y(t)|^{q-2} y(t) = 0,\\ \frac{1}{T}\int_0^T [-f(\varphi_q(y(t))) - g(t,x(t)) + e(t)]dt = 0. \end{cases}
$$

So $y(t) = 0$ and by assumption (h_2), we can see that there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$M_1 < x(t) < M_2.$$

Let us define $0 < A_1 = \min(M_1, \bar{D}_1)$ and $A_2 = \max(M_2, \bar{D}_2)$, then

$$A_1 < x(t) < A_2,$$

which implies $\Omega_2 \subset \Omega_1$. Now, if we set $\Omega = \{u : u = (x,y)^\top \in X, A_1 < x < A_2, |y|_0 < D_3 + 1\}$, then $\Omega \supset \Omega_1 \bigcup \Omega_2$. So from (17) and (20), we see that conditions (a_1) and (a_2) of Lemma 1 are satisfied. The remainder is to verify condition (a_3) of Lemma 1. In order to do it, let

$$z = Kx = K\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} x - \frac{A_1+A_2}{2}\\ y\end{pmatrix},$$

then, we have

$$x = z + \begin{pmatrix} \frac{A_1+A_2}{2}\\ 0\end{pmatrix}.$$

Define J : ImQ $\rightarrow$ KerL is a linear isomorphism with

$$J(x,y) = \begin{pmatrix} y\\ x\end{pmatrix},$$

and define

$$H(\mu,u) = \mu Ku + (1-\mu)JQNu, \quad \forall(u,\mu) \in \Omega \times [0,1].$$

Then,

$$
\begin{aligned}
H(\mu,u) = & \begin{pmatrix} \mu x - \frac{\mu(A_1+A_2)}{2}\\ \mu y\end{pmatrix} + \frac{1-\mu}{T}\\
& \cdot \begin{pmatrix} \int_0^T [-f(\varphi_q(y(t))) - g(t,x(t)) + e(t)]dt\\ \int_0^T \varphi_q(y(t))dt\end{pmatrix}. \quad (22)
\end{aligned}
$$

Now we claim that $H(\mu,u)$ is a homotopic mapping. By way of contradiction, assume that there exist $\mu_0 \in [0,1]$ and $u_0 = \begin{pmatrix} x_0\\ y_0\end{pmatrix} \in \partial\Omega$ such that $H(\mu_0,u_0) = \mathbf{0}$.

Substituting μ_0 and u_0 into (22), we have

$$
\begin{aligned}
& H(\mu_0,u_0)\\
= & \begin{pmatrix} \mu_0 x_0 - \frac{\mu_0(A_1+A_2)}{2}\Phi\\ \mu_0 y_0 + (1-\mu_0)\varphi_q(y_0))\end{pmatrix}, \quad (23)
\end{aligned}
$$

where $\Phi = -(1-\mu_0)f(\varphi_q(y_0)) - (1-\mu_0)[g(t,x_0) - e(t)]$. Since $H(\mu_0,x_0) = 0$, then we can see that

$$\mu_0 y_0 + (1-\mu_0)\varphi_q(y_0)) = 0.$$

Combining with $\mu_0 \in [0,1]$, we obtain $y_0 = 0$. Thus $x_0 = A_1$ or A_2.

If $x_0 = A_1$, it follows from (h_2) that $g(t,x_0) - e(t) > 0$, then substituting $y_0 = 0$ into (23), we have

$$
\begin{aligned}
& \mu_0 x_0 - \frac{\mu_0(A_1+A_2)}{2}\\
& -(1-\mu_0)f(\varphi_q(y_0)) - (1-\mu_0)[g(t,x_0) - e(t)]\\
= & \mu_0 x_0 - \frac{\mu_0(A_1+A_2)}{2} - (1-\mu_0)[g(t,x_0) - e(t)]\\
< & \mu_0(x_0 - \frac{(A_1+A_2)}{2}).\\
< & 0. \quad (24)
\end{aligned}
$$

If $x_0 = A_2$, it follows from (h_2) that $g(t,x_0) - e(t) < 0$, then substituting $y_0 = 0$ into (23), we have

$$
\begin{aligned}
& \mu_0 x_0 - \frac{\mu_0(A_1+A_1)}{2}\\
& -(1-\mu_0)f(\varphi_q(y_0)) - (1-\mu_0)[g(t,x_0) - e(t)]\\
= & \mu_0 x_0 - \frac{\mu_0(A_1+A_2)}{2} - (1-\mu_0)[g(t,x_0) - e(t)]\\
> & \mu_0(x_0 - \frac{(A_1+A_2)}{2}).\\
> & 0. \quad (25)
\end{aligned}
$$

Combining with (24) and (25), we can see that $H(\mu_0,u_0) \neq 0$, which contradicts the assumption. Therefore $H(\mu,u)$ is a homotopic mapping and $u^\top H(\mu,u) \neq 0$,

$\forall (u,\mu) \in (\partial\Omega \cap \mathrm{KerL}) \times [0,1]$. Then

$$\begin{aligned}
& \deg(\mathrm{JQN}, \Omega \cap \mathrm{KerL}, 0) \\
= \;& \deg(H(0,u), \Omega \cap \mathrm{KerL}, 0) \\
= \;& \deg(H(1,u), \Omega \cap \mathrm{KerL}, 0) \\
= \;& \deg(Kx, \Omega \cap \mathrm{KerL}, 0) \\
= \;& \sum_{u \in K^{-1}(0)} \mathrm{sgn}|K'(u)| \\
= \;& 1 \neq 0.
\end{aligned}$$

Thus, the condition (a_3) of Lemma 1 is also satisfied. So, by applying Lemma 1, we can conclude that the problem (1)-(2) has at least one positive T-periodic solution.

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