The Computation of Fourier Transform via Gauss-Regularized Fourier Series

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Abstract—**In this paper, we construct a new regularized Fourier series, that is the Gauss-regularized Fourier series. Moreover, we find this regularized Fourier series can be used to compute the Fourier transforms of bandlimited signals. When the regularization parameter tends to zero, we prove that this regularized Fourier series is uniform and L^2-convergence. Numerical results demonstrate the superiority of the new method over some previous methods.**

Index Terms—**Gauss-regularization, ill-posedness, bandlimited signal, Fourier transform.**

I. Introduction

THE computation of Fourier transforms (see Definition 1.1) of signals is a highly ill-posed problem [1–2]. It is not reliable to compute the Fourier transforms of signals by their definitions in practice. Therefore, for computing Fourier transforms of signals, many regularization methods were raised [1–5]. In particular, in [2], Chen constructed a polynomial-regularized Fourier series

$$\hat{f}^p_\alpha(\omega) = h\sum_{k\in\mathbb{Z}} \frac{f(kh)}{1+2\pi\alpha+2\pi\alpha(kh)^2} e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega),$$

where $\alpha > 0$, $h = \frac{\pi}{\Omega}$, $f(kh) = f_\Omega(kh) + \eta(kh)$, $\{\eta(kh)\}$ is the noise $|\eta(kh)| \le \delta$ and $f_\Omega \in L^2(\mathbb{R})$ is the exact bandlimited signal (see Definition 1.2). Moreover, he gave that this polynomial-regularized Fourier series is more effective in controlling the noise than Fourier series when we use it to compute the Fourier transforms of bandlimited signals. When the regularization parameter tends to zero, the uniform and L^2-convergence of this regularized Fourier series are proved by Chen.

In this paper, we construct a new regularized Fourier series, that is the Gauss-regularized Fourier series

$$\hat{f}_\alpha(\omega) = h\sum_{k\in\mathbb{Z}} f(kh)e^{-4\pi^2\alpha(kh)^2} e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega).$$

Moreover, we find this regularized Fourier series can be used to compute the Fourier transforms of bandlimited signals. When the regularization parameter tends to zero, we prove the uniform and L^2-convergence of this regularized Fourier series. Numerical results demonstrate the Gauss-regularized Fourier series has the better performance in controlling the noise than the polynomial-regularized Fourier series when we used it to compute the Fourier transforms of bandlimited signals.

Next, we review some definitions, notations and basic results.

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Definition 1.1: For $f \in L^2(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-it\omega}dt. \tag{1}$$

Definition 1.2: Let $f \in L^2(\mathbb{R})$, if there exists a positive Ω such that $\hat{f}(\omega) = 0$ for $|\omega| > \Omega$, then f is said to be Ω-bandlimited.

By Shannon sampling theorem [6–13], the Ω-bandlimited signal $f(x)$ can be exactly reconstructed from its samples $f(kh)$ and

$$f(t) = \sum_{k\in\mathbb{Z}} f(kh)\frac{\sin(\Omega(t-kh))}{(\Omega(t-kh))}, \tag{2}$$

where $h = \frac{\pi}{\Omega}$ and the series (2) converges both uniformly on $\mathbb{R}$ and in $L^2(\mathbb{R})$. By (2) and the time invariance of the bandlimited signals, it follows that

$$h\sum_{k\in\mathbb{Z}} |f(kh)|^2 = \|f\|^2_{L^2}, \tag{3}$$

(For the detail throughout see [13]). Taking the Fourier transform on both sides of (2), we have

$$\hat{f}(\omega) = h\sum_{k\in\mathbb{Z}} f(kh)e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega).$$

For any $f, g \in L^2(\mathbb{R})$, define their convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

This paper is organized as follows: In section 2, we give the convergence property of Gauss-regularized Fourier series, that is the uniform and L^2-convergence. Numerical results are presented in section 3. Section 4 concludes the paper with some outlook. Finally, in section 5, we provide proofs of Theorem 2.1 and Theorem 2.2.

II. The uniform and L^2-convergence of $\hat{f}_\alpha$

In this section, we will give the convergence property of Gauss-regularized Fourier series, that is the uniform and L^2-convergence. The following theorem gives the uniform convergence of Gauss-regularized Fourier series.

Theorem 2.1: Assume that $f_\Omega \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is Ω-bandlimited. For each fixed $0 < c < \Omega$, if the regularization parameter $\alpha = \alpha(\delta)$ satisfies $\alpha(\delta) \to 0$ and $\frac{\delta}{\sqrt{\alpha(\delta)}} \to 0$ when $\delta \to 0$, then $\hat{f}_\alpha$ converges to $\hat{f}_\Omega$ uniformly on $[-\Omega + c, \Omega - c]$ when $\delta \to 0$.

Proof: See section 5. ■

Next, we give L^2-convergence of Gauss-regularized Fourier series.

Theorem 2.2: Assume that $f_\Omega \in L^2(\mathbb{R})$ is Ω-bandlimited. If the regularization parameter $\alpha = \alpha(\delta)$ satisfies $\alpha(\delta) \to 0$

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and $\frac{\delta^2}{\sqrt{\alpha(\delta)}} \to 0$ when $\delta \to 0$, then $\hat{f}_\alpha$ converges to $\hat{f}_\Omega$ in $L^2([-\Omega,\Omega])$ when $\delta \to 0$.

Proof: See section 5. ■

III. Numerical Results

In this section, we give some numerical results to show that the Gauss-regularized Fourier series is more effective in controlling the noise than the polynomial-regularized Fourier series when we used it to compute the Fourier transforms of bandlimited signals.

For a large N, we use the next formulas in practical computation

$$\hat{f}_\alpha(\omega) = h\sum_{-N}^{N} f(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega), \quad (4)$$

$$\hat{f}_\alpha^p(\omega) = h\sum_{-N}^{N} \frac{f(kh)e^{-ikh\omega}}{1+2\pi\alpha+2\pi\alpha(kh)^2}\chi_{[-\Omega,\Omega]}(\omega) \quad (5)$$

and

$$\hat{f}(\omega) = h\sum_{-N}^{N} f(kh)e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega). \quad (6)$$

Here $f(kh) = f_\Omega(kh) + \eta(kh)$ and $\{\eta(kh)\}$ is the noise $|\eta(kh)| \le \delta$.

Example 3.1: Comparison between different regularization methods.

Let

$$f_\Omega = \frac{1-\cos t}{\pi t^2}.$$

It is easy to see that $\hat{f}_\Omega(\omega) = (1-|\omega|)\chi_{[-1,1]}(\omega)$. We choose the noise that is uniformly distributed in $[-0.0025, 0.0025]$ and $N = 200$. Figure 1 shows the result using the Gauss-regularized Fourier series with $\alpha = 0.0001$, the result using the polynomial-regularized Fourier series with $\alpha = 0.0001$ and the result using the Fourier series. Figure 2 presents the result using the Gauss-regularized Fourier series with $\alpha = 0.00001$, the result using the polynomial-regularized Fourier series with $\alpha = 0.00001$ and the result using the Fourier series. By Figure 1 and Figure 2, the Gauss-regularized Fourier series has good performance at least in some cases when we used it to compute the Fourier transforms of bandlimited signals.

IV. Conclusion

Computing the Fourier transforms of bandlimited signals by Fourier series is an ill-posed problem. As a result, many regularization methods were raised. In this paper, we present a new regularized Fourier series, that is the Gauss-regularized Fourier series. Moreover, we prove the uniform and L^2-convergence of this regularized Fourier series when the regularization parameter tends to zero. Numerical results show that the Gauss-regularized Fourier series is more effective in controlling the noise than the known-regularized Fourier series (the polynomial-regularized Fourier series) when we used it to compute the Fourier transforms of bandlimited signals. Studying other regularization methods is the goal of future work.

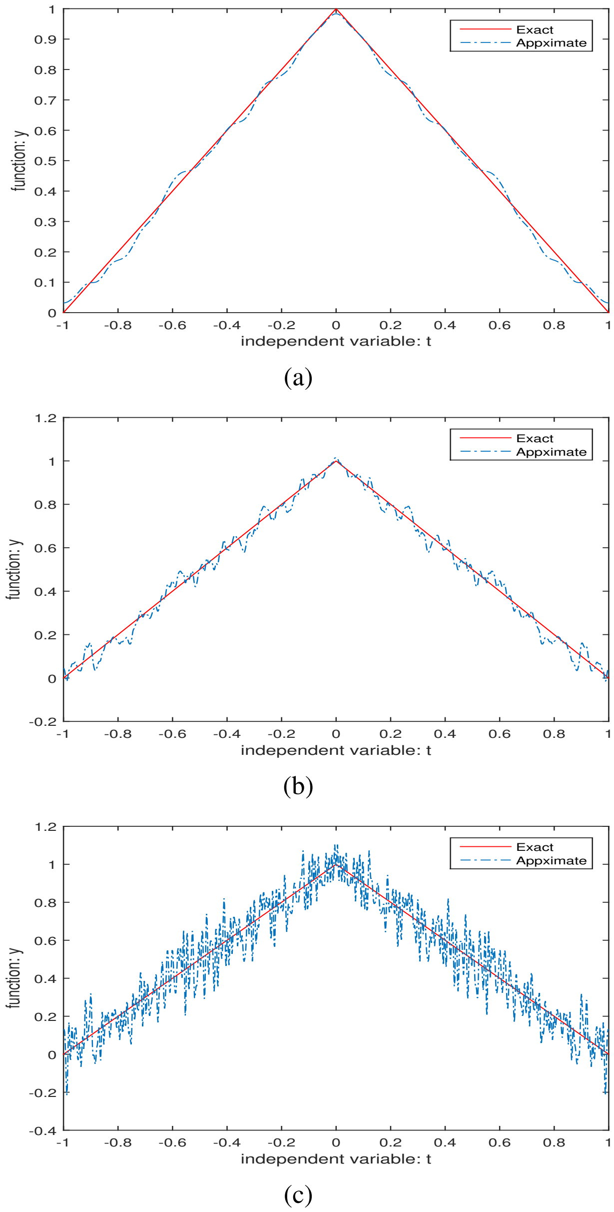


Fig. 1. Comparison of three algorithms with $\alpha = 0.0001$. (a) is the result using Gauss-regularized Fourier series (formula (4)). (b) is the result using polynomial-regularized Fourier series (formula (5)). (c) is the result using Fourier series (formula (6)).

V. Proofs of Theorem 2.1 and Theorem 2.2

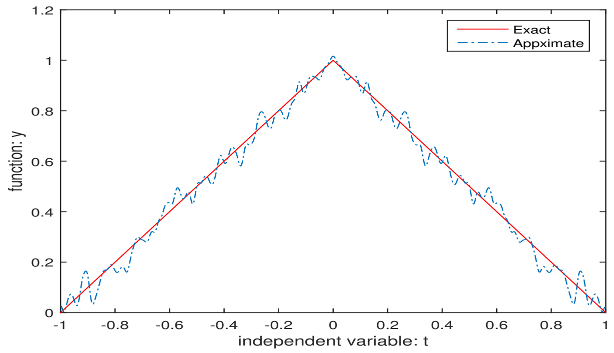
A. *Proof of Theorem 2.1*

To prove Theorem 2.1, we need the following lemma.

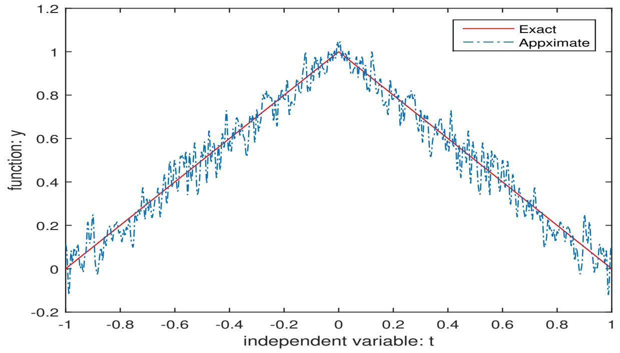
Lemma 5.1: Suppose that $g \in L^2(\mathbb{R})$ is Ω-bandlimited. Then for each fixed $0 < c < \Omega$ and for any $-\Omega + c < \omega < \Omega - c$, we have

$$\left| h\sum_{k\in\mathbb{Z}} g(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega) - \int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt\right|^2 \le \frac{\|g\|_{L^2}^2}{3h\pi\alpha}e^{-c^2/8\pi^2\alpha} + \frac{4h\|g\|_{L^2}^2}{\pi}\left(\int_{c/4\pi\sqrt{\alpha}}^{+\infty} e^{-t^2}dt\right)^2.$$

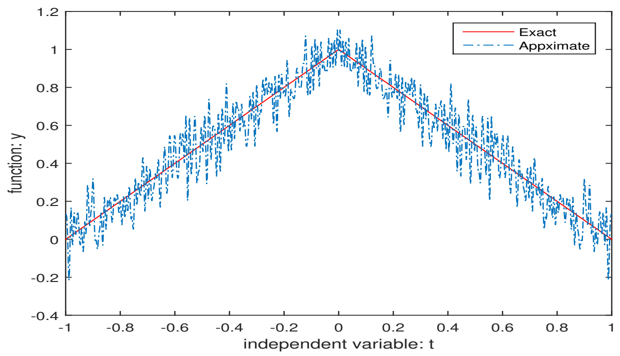
Proof: For $-\Omega+c < \omega < \Omega-c$, by Shannon's sampling



(d)



(e)



(f)

Fig. 2. Comparison of three algorithms with $\alpha = 0.00001$. (d) is the result using Gauss-regularized Fourier series (formula (4)). (e) is the result using polynomial-regularized Fourier series (formula (5)). (f) is the result using Fourier series (formula (6)).

formula (2), we obtain that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt \\
&= \sum_{k\in\mathbb{Z}} g(kh)\int_{-\infty}^{+\infty}\frac{\sin(\Omega(t-kh))}{\Omega(t-kh)}e^{-4\pi^2\alpha t^2}e^{-it\omega}dt \\
&= h\sum_{k\in\mathbb{Z}} g(kh)\frac{1}{2\pi}e^{-i\omega kh}\chi_{[-\Omega,\Omega]}(\omega) * \frac{1}{2\sqrt{\pi\alpha}}e^{-\omega^2/16\pi^2\alpha} \\
&= h\sum_{k\in\mathbb{Z}} g(kh)e^{-ikh\omega}\frac{1}{4\pi\sqrt{\pi\alpha}}\int_{\omega-\Omega}^{\omega+\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&h\sum_{k\in\mathbb{Z}} g(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega) \\
&\quad -\int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt \\
&= h\sum_{k\in\mathbb{Z}} g(kh)e^{-ikh\omega}\Big(e^{-4\pi^2\alpha(kh)^2} \\
&\quad -\frac{1}{4\pi\sqrt{\pi\alpha}}\int_{\omega-\Omega}^{\omega+\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\Big).
\end{aligned}$$

Note that $e^{-4\pi^2\alpha(kh)^2} = \frac{1}{4\pi\sqrt{\pi\alpha}}\int_{-\infty}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{itkh}dt$ (see [15]), we have

$$\begin{aligned}
&h\sum_{k\in\mathbb{Z}} g(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega) \\
&\quad -\int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt \\
&= h\sum_{k\in\mathbb{Z}} g(kh)\frac{e^{-ikh\omega}}{4\pi\sqrt{\pi\alpha}}\Big(\int_{-\infty}^{\omega-\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt \\
&\quad +\int_{\omega+\Omega}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\Big).
\end{aligned}$$

Therefore, for $\omega\in[-\Omega+c,\Omega-c]$, by Cauchy-Schwarz's inequality and the inequality $\|a+b\|^2 \le 2\left(\|a\|^2+\|b\|^2\right)$

$$\begin{aligned}
&\Big|h\sum_{k\in\mathbb{Z}} g(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega) \\
&\quad -\int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt\Big|^2 \\
&\le \frac{h^2}{8\pi^3\alpha}\left(\sum_{k\in\mathbb{Z}}|g(kh)|\left|\int_{-\infty}^{\omega-\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|\right)^2 \\
&\quad +\frac{h^2}{8\pi^3\alpha}\left(\sum_{k\in\mathbb{Z}}|g(kh)|\left|\int_{\omega+\Omega}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|\right)^2 \\
&\le \frac{h^2}{8\pi^3\alpha}\sum_{k\in\mathbb{Z}}|g(kh)|^2\Bigg(\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\omega-\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|^2 \\
&\quad +\sum_{k\in\mathbb{Z}}\left|\int_{\omega+\Omega}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|^2\Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k\in\mathbb{Z},k\neq 0}\left|\int_{-\infty}^{\omega-\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|^2 \\
&= \sum_{k\in\mathbb{Z},k\neq 0}\left|\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{-itkh}dt\right|^2 \\
&= \sum_{k\in\mathbb{Z},k\neq 0}\Bigg|\frac{e^{-itkh}}{-ikh}e^{-t^2/16\pi^2\alpha}\Big|_{\Omega-\omega}^{+\infty} \\
&\quad -\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}\frac{te^{-itkh}}{ikh8\pi^2\alpha}dt\Bigg|^2 \\
&= \sum_{k\in\mathbb{Z},k\neq 0}\Bigg|\frac{e^{-i(\Omega-\omega)kh}}{ikh}e^{-(\Omega-\omega)^2/16\pi^2\alpha} \\
&\quad -\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}\frac{te^{-itkh}}{ikh8\pi^2\alpha}dt\Bigg|^2 \\
&\le \sum_{k\in\mathbb{Z},k\neq 0}\frac{2e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2} \\
&\quad +2\left|\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}\frac{te^{-itkh}}{ikh8\pi^2\alpha}dt\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k\in\mathbb{Z},k\neq 0}\left[\frac{2e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2}\right.\\
&\quad\left.+\frac{2}{(kh8\pi^2\alpha)^2}\left(\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}tdt\right)^2\right]\\
&= \sum_{k\in\mathbb{Z},k\neq 0}\left[\frac{2e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2}\right.\\
&\quad\left.+\frac{2}{(kh)^2}\left(e^{-t^2/16\pi^2\alpha}\Big|_{+\infty}^{\Omega-\omega}\right)^2\right]\\
&= \sum_{k\in\mathbb{Z},k\neq 0}\left[\frac{2e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2}+\frac{2e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2}\right]\\
&= \sum_{k\in\mathbb{Z},k\neq 0}\frac{4e^{-(\Omega-\omega)^2/8\pi^2\alpha}}{(kh)^2}\\
&\leq \sum_{k\in\mathbb{Z},k\neq 0}\frac{4e^{-c^2/8\pi^2\alpha}}{(kh)^2},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\omega-\Omega} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|^2\\
&\quad\leq \sum_{k\in\mathbb{Z},k\neq 0}\frac{4e^{-c^2/8\pi^2\alpha}}{(kh)^2}+\left(\int_{\Omega-\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2\\
&\quad\leq \sum_{k\in\mathbb{Z},k\neq 0}\frac{4e^{-c^2/8\pi^2\alpha}}{(kh)^2}+\left(\int_{c}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left|\int_{\Omega+\omega}^{+\infty} e^{-t^2/16\pi^2\alpha}e^{itkh}dt\right|^2\\
&\quad\leq \sum_{k\in\mathbb{Z},k\neq 0}\frac{4e^{-c^2/8\pi^2\alpha}}{(kh)^2}+\left(\int_{c}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2.
\end{aligned}
$$

Therefore, for $\omega\in[-\Omega+c,\Omega-c]$,

$$
\begin{aligned}
&\left|h\sum_{k\in\mathbb{Z}} g(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\chi_{[-\Omega,\Omega]}(\omega)\right.\\
&\quad\left.-\int_{-\infty}^{+\infty} g(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt\right|^2\\
&\leq \frac{h^2}{8\pi^3\alpha}\sum_{k\in\mathbb{Z}}|g(kh)|^2\left[\sum_{k\in\mathbb{Z},k\neq 0}\frac{8e^{-c^2/8\pi^2\alpha}}{(kh)^2}\right.\\
&\quad\left.+2\left(\int_{c}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2\right]\\
&= \frac{h}{8\pi^3\alpha}\|g\|_{L^2}^2\left[\sum_{k\in\mathbb{Z},k\neq 0}\frac{8e^{-c^2/8\pi^2\alpha}}{(kh)^2}\right.\\
&\quad\left.+2\left(\int_{c}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2\right]\\
&= \frac{\|g\|_{L^2}^2}{3h\pi\alpha}e^{-c^2/8\pi^2\alpha}+\frac{h\|g\|_{L^2}^2}{4\pi^3\alpha}\left(\int_{c}^{+\infty} e^{-t^2/16\pi^2\alpha}dt\right)^2\\
&= \frac{\|g\|_{L^2}^2}{3h\pi\alpha}e^{-c^2/8\pi^2\alpha}+\frac{4h\|g\|_{L^2}^2}{\pi}\left(\int_{c/4\pi\sqrt{\alpha}}^{+\infty} e^{-t^2}dt\right)^2,
\end{aligned}
$$

the first equality holds by (3). ■

Proof of Theorem 2.1: Since for any $\omega\in[-\Omega+c,\Omega-c]$

$$
\begin{aligned}
&\hat{f}_\alpha(\omega)-\hat{f}_\Omega(\omega)\\
&= h\sum_{k\in\mathbb{Z}}(f_\Omega(kh)+\eta(kh))e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}-\hat{f}_\Omega(\omega)\\
&= h\sum_{k\in\mathbb{Z}} f_\Omega(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\\
&\quad+h\sum_{k\in\mathbb{Z}}\eta(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\\
&\quad-\int_{-\infty}^{+\infty} f_\Omega(t)e^{-it\omega}dt,
\end{aligned}
$$

therewith, by Lemma 5.1

$$
\begin{aligned}
&|\hat{f}_\alpha(\omega)-\hat{f}_\Omega(\omega)|\\
&\leq \left|\int_{-\infty}^{+\infty} f_\Omega(t)e^{-4\pi^2\alpha t^2}e^{-it\omega}dt-\int_{-\infty}^{+\infty} f_\Omega(t)e^{-it\omega}dt\right|\\
&\quad+\left|h\sum_{k\in\mathbb{Z}}\eta(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\right|\\
&\quad+\frac{\|f_\Omega\|_{L^2}}{\sqrt{3h\pi\alpha}}e^{-c^2/16\pi^2\alpha}\\
&\quad+\frac{2\sqrt{h}\|f_\Omega\|_{L^2}}{\sqrt{\pi}}\int_{c/4\pi\sqrt{\alpha}}^{+\infty} e^{-t^2}dt\\
&= \mathrm{I}+\mathrm{II}+\frac{\|f_\Omega\|_{L^2}}{\sqrt{3h\pi\alpha}}e^{-c^2/16\pi^2\alpha}\\
&\quad+\frac{2\sqrt{h}\|f_\Omega\|_{L^2}}{\sqrt{\pi}}\int_{c/4\pi\sqrt{\alpha}}^{+\infty} e^{-t^2}dt,
\end{aligned}
$$

where

$$
\mathrm{I}=\left|\int_{-\infty}^{+\infty} f_\Omega(t)\left(1-e^{-4\pi^2\alpha t^2}\right)e^{-it\omega}dt\right|
$$

and

$$
\mathrm{II}=\left|h\sum_{k\in\mathbb{Z}}\eta(kh)e^{-4\pi^2\alpha(kh)^2}e^{-ikh\omega}\right|.
$$

We treat I first: Since $\left|f_\Omega(t)\left(1-e^{-4\pi^2\alpha t^2}\right)\right|\leq|f_\Omega(t)|$ and $f_\Omega(t)\in L^1(\mathbb{R})$, by Lebesgue's dominated convergence theorem (see [16])

$$
\mathrm{I}\leq\int_{-\infty}^{+\infty}\left|f_\Omega(t)\left(1-e^{-4\pi^2\alpha t^2}\right)\right|dt\to 0,\quad\text{as}\quad\alpha\to 0.
$$

We now treat II: Since

$$
\begin{aligned}
\mathrm{II} &\leq h\sum_{k\in\mathbb{Z}}|\eta(kh)|e^{-4\pi^2\alpha(kh)^2}\\
&\leq h\delta\sum_{k\in\mathbb{Z}}e^{-4\pi^2\alpha(kh)^2}\\
&= h\delta+2h\delta\sum_{k=1}^{+\infty}e^{-4\pi^2\alpha(kh)^2}\\
&\leq h\delta+2h\delta\sum_{k=1}^{+\infty}\int_{k-1}^{k}e^{-4\pi^2\alpha(th)^2}dt\\
&= h\delta+2h\delta\int_0^{+\infty}e^{-4\pi^2\alpha(th)^2}dt\\
&= h\delta+\frac{2h\delta}{\sqrt{\alpha}}\int_0^{+\infty}e^{-4\pi^2(th)^2}dt,
\end{aligned}
$$

then II converges to 0 uniformly on $\mathbb{R}$ if the regularization parameter $\alpha = \alpha(\delta)$ is chosen such that $\frac{\delta}{\sqrt{\alpha(\delta)}} \to 0$ when $\delta \to 0$.

Finally, since

$$\frac{\|f_\Omega\|_{L^2}}{\sqrt{3h\pi\alpha}} e^{-c^2/16\pi^2\alpha} + \frac{2\sqrt{h}\|f_\Omega\|_{L^2}}{\sqrt{\pi}} \int_{c/4\pi\sqrt{\alpha}}^{+\infty} e^{-t^2} dt \to 0$$

when $\alpha \to 0$, the proof is finished. ■

B. *Proof of Theorem 2.2*

Since

$$\begin{aligned} &\hat{f}_\alpha(\omega) - \hat{f}_\Omega(\omega) \\ &= h \sum_{k\in\mathbb{Z}} \eta(kh) e^{-4\pi^2\alpha(kh)^2} e^{-ikh\omega} \chi_{[-\Omega,\Omega]}(\omega) \\ &\quad -h \sum_{k\in\mathbb{Z}} f_\Omega(kh) \left(1 - e^{-4\pi^2\alpha(kh)^2}\right) e^{-ikh\omega} \chi_{[-\Omega,\Omega]}(\omega), \end{aligned}$$

using the inequality $\|a+b\|^2 \le 2\left(\|a\|^2 + \|b\|^2\right)$, we obtain

$$\begin{aligned} &\|\hat{f}_\alpha(\omega) - \hat{f}_\Omega(\omega)\|_{L^2}^2 \\ &\le 2h^2 \left\| \sum_{k\in\mathbb{Z}} \eta(kh) e^{-4\pi^2\alpha(kh)^2} e^{-ikh\omega} \chi_{[-\Omega,\Omega]}(\omega) \right\|_{L^2}^2 \\ &\quad +2h^2 \left\| \sum_{k\in\mathbb{Z}} f_\Omega(kh) \times \right. \\ &\qquad \left. \left(1 - e^{-4\pi^2\alpha(kh)^2}\right) e^{-ikh\omega} \chi_{[-\Omega,\Omega]}(\omega) \right\|_{L^2}^2 \\ &= 8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} \left| \eta(kh) e^{-4\pi^2\alpha(kh)^2} \right|^2 \\ &\quad +8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} \left| f_\Omega(kh) \left(1 - e^{-4\pi^2\alpha(kh)^2}\right) \right|^2. \end{aligned}$$

It follows similar lines of the treatment of II in the proof of Theorem 2.1 that

$$8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} \left| \eta(kh) e^{-4\pi^2\alpha(kh)^2} \right|^2 \to 0,$$

if the regularization parameter $\alpha = \alpha(\delta)$ satisfies

$$\frac{\delta^2}{\sqrt{\alpha(\delta)}} \to 0$$

when $\delta \to 0$.

By (3), we have

$$\begin{aligned} &8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} \left| f_\Omega(kh) \left(1 - e^{-4\pi^2\alpha(kh)^2}\right) \right|^2 \\ &\le 8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} |f_\Omega(kh)|^2 \\ &= 8\Omega^2 h \|f\|_{L^2}^2 < \infty. \end{aligned}$$

Using Lebesgue's dominated convergence theorem

$$8\Omega^2 h^2 \sum_{k\in\mathbb{Z}} \left| f_\Omega(kh) \left(1 - e^{-4\pi^2\alpha(kh)^2}\right) \right|^2 \to 0$$

when $\alpha \to 0$. Hence we obtain $\|\hat{f}_\alpha(\omega) - \hat{f}_\Omega(\omega)\|_{L^2} \to 0$ if the regularization parameter $\alpha = \alpha(\delta)$ is chosen such that $\alpha(\delta) \to 0$ and $\frac{\delta^2}{\sqrt{\alpha(\delta)}} \to 0$ when $\delta \to 0$. This finishes the proof.

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