Numerical Solutions of the Modified Burger's Equation using FTCS Implicit Scheme

Surattana Sungnul, Bubpha Jitsom and Mahosut Punpocha

***Abstract*—In this paper, we investigate the behavior of a modified Burger's equation in the form**

$$u_t + (c + bu)u_x = (\mu_0 + \mu_1 u)u_{xx},$$

where c, b, μ_0 and μ_1 are arbitrary parameters. Numerical solutions of this problem is obtained by the finite difference method in FTCS implicit scheme. The results obtained by advantages of mathematical software are compared between the numerical solutions and the exact solutions for some given initial and boundary conditions.

***Index Terms*—Burger's equation, FTCS implicit scheme, finite difference method.**

I. Introduction

BURGER'S equation is a nonlinear partial differential equation, describing an evolutionary process in which a convective phenomenon is in balance with a diffusive phenomenon. The complete nonlinear Burger's equation is given by [13]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu_0 \frac{\partial^2 u}{\partial x^2}, (x,t) \in D, \tag{1}$$

where u is fluid velocity, μ_0 is viscosity coefficient and D is a continuous domain.

Equation (1) is a parabolic PDE, which can serve as a model equation for the boundary-layer equations. For the steady boundary-layer and "parabolized" Navier-Stokes equation, the independent variables t and x can be replaced by x and y, respectively to give

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = \mu_0 \frac{\partial^2 u}{\partial y^2}, (x,y) \in D, \tag{2}$$

where x, y are the marching direction. For simplicity, the linear Burger's equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \mu_0 \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

is often used in place of equation (1). Note that if $c = 0$, equation (3) represents the heat equation. The exact steady-state solution of equation (3) with the boundary conditions,

$$u(0,t) = u_0 \equiv \text{constant}, \tag{4}$$

$$u(l,t) = 0, \tag{5}$$

is given by

$$u = u_0 \left\{ \frac{1 - \exp[R_l(\frac{x}{l} - 1)]}{1 - \exp(-R_l)} \right\}, \tag{6}$$

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where

$$R_l = \frac{cl}{\mu_0}.$$

The exact unsteady solution of equation (3) and the initial condition [13] can be expressed as

$$u(x,0) = \sin(kx),$$

where k is a constant. The periodic boundary condition is given by

$$u(x,t) = \exp(-k^2\mu_0 t)\sin k(x - ct). \tag{7}$$

Equations (1) and (3) can be combined into a generalized equation as

$$u_t + (c + bu)u_x = \mu_0 u_{xx}, \tag{8}$$

where c and b are free parameters. Note that if $b = 0$, the linear equation is obtained. Moreover if $c = 0$ and $b = 1$, the nonlinear Burger's equation is obtained. For the case that $c = \frac{1}{2}$ and $b = -1$, the generalized Burger's equation has the stationary solution

$$u = -\frac{c}{b}\left[1 + \tanh\frac{c(x - x_0)}{2\mu_0}\right]. \tag{9}$$

Hence, if the initial distribution of u is given by equation (9), the exact solution does not vary with time but it remains fixed at the initial distribution. Additional exact solution of Burger's equation can be found by Platzman (1972), which describes 35 different exact solutions.

Equation (8) can be put into conservative form

$$u_t + \overline{F}_x = 0, \tag{10}$$

where $\overline{F}$ is defined by

$$\overline{F} = cu + \frac{bu^2}{2} - \mu_0 u_x. \tag{11}$$

Alternatively, equation (8) can be rewritten as

$$u_t + F_x = \mu_0 u_{xx}, \tag{12}$$

where F is defined by

$$F = cu + \frac{bu^2}{2}. \tag{13}$$

For the linearized case ($b = 0$), F is reduced into

$$F = cu.$$

In 2010, Blandin et al [1] considered the problem of stabilization of the inviscid Burgers equation using boundary conditions. Wu [2] suggested a fractional Lie group method to solve fractional partial differential equation. A time-fractional Burgers equation is used as an example to illustrate the effectiveness of the Lie group method and a few classes of the exact solution were obtained. Pandey and Verma [3] generated the numerical solutions of the Burger's equation by applying the Crank-Nicolson method directly to the Burger's equation.

In 2012, Jiwari [4] used uniform Haar wavelet and the quasilinearization process to propose for the numerical simulation of time dependent nonlinear Burgers equation. The following year, the study of a fractional Burgers equation arising in nonlinear acoustics was presented by Lombard, et al [5].

In 2014, Wongsaijai et al [6] proposed a compact finite difference method to solve the Rosenau-RLW equation. A numerical tool is applied to the model by using a three-level average implicit finite difference technique.

In 2015, other methods occurred, such as a fourth-order singly diagonally implicit runge-Kutta method for solving one-dimensional Burgers' equation was presented by Deng and Pan [7]. A hybrid numerical scheme based on the Euler implicit method and quasilinearization. Uniform Haar wavelets was developed for the numerical solution of the Burgers equation by Jiwari [8]. Zhanlav et al [9] proposed an explicit finite difference scheme to solve the unsteady Burgers equation. Esen and Tasboza [10] presented a few numerical examples which supported numerical results for the time fractional Burgers equation, where various boundary and initial conditions obtained by collocation method using cubic B-spline. Bhrawy [11] reported a new space-time spectral algorithm to obtain an approximate solution for the space-time fractional Burger's equation. The algorithm was based on a spectral shifted Legendre collocation method in combination with the shifted Legendre operational matrix of fraction derivatives.

In 2016, Seydaoglu et al [12] used a high order splitting method to calculate the numerical solution of the Burger's equation in one dimensional space with periodic boundary conditions.

In this work, we study the modified Burger's equation (8) with modified coefficient of viscosity as

$$u_t + (c + bu)u_x = (\mu_0 + \mu_1 u)u_{xx}, \tag{14}$$

where the parameters c, b, μ_0 and μ_1 are given. Numerical solutions of the modified Burger's equation are obtained by the finite difference method in FTCS implicit scheme.

II. Convergence Theory

Here we consider boundary value problem (BVP) consisting of a partial differential equation with initial and boundary conditions

$$Lu = f, \quad \text{in domain} \quad D, \tag{15}$$

and boundary condition is $u(x,t) = \upsilon(x,t)$ for $(x,t) \in \partial D$, where L is differential operator acting from a space of continuous functions Ω to a continuous function space H $(L : \Omega \to H)$.

We construct the grid $D_h \in D \cup \partial D$ and determine the linear space of discrete functions Ω_h given on grid D_h. Let us consider a finite difference scheme (FDS) which corresponds to the BVP (15)

$$L_h u^h = f^h, \quad \text{on} \quad D_h, \tag{16}$$

and boundary condition is

$$u^h(x_i, t_n) = \upsilon^h(x_i, t_n) \text{ for } (x_i, t_n) \in \partial D_h,$$

$$D_h = \{(x_i, t_n) | i = 0, \pm 1, \pm 2, ..., M, n = 0, 1, ..., [T/\tau] - 1\},$$

τ is time step, where L_h is difference operator acting from a discrete function space Ω_h to a discrete function space H_h $(L_h : \Omega_h \to H_h)$. Assuming u is a function of x and t, we have $u_i^n = u(x_i, t_n)$ and denote f^h, such that

$$f^h = \begin{cases} \varphi_i^n \\ \psi_i \end{cases}, i = 0, \pm 1, \pm 2, ..., M, n = 0, 1, ..., [T/\tau] - 1,$$

where φ_i^n is function on the right hand side of FDS (16) and ψ_i is value of initial condition.

To study stability of numerical methods we need to introduce the norms into the set of discrete functions

$$\|u^h\|_{\Omega_h} = \max_{i,n} |u_i^n|,$$

$$\|f^h\|_{H_h} = \max_i |\psi_i| + \max_{i,n} |\varphi_i^n|.$$

Lax's equivalence theorem [13]

Given a properly posed boundary value problem and a finite difference approximation to it that satisfies the consistent condition, stability is the necessary and sufficient condition for convergence.

1) Consistency: A finite difference representation of a PDE is said to be consistent [13] if we should be able to show that the difference between the PDE and its difference representation vanishes as the mesh is refined, i.e., the truncation error $(T.E.)$ goes to zero as the mesh size go to zero. This should always be the case if the order of the $T.E.$ vanishes under grid refinement.

It is said that FDS (16) approximate with order k of BVP (15) if

$$\|T.E.\|_{H_h} \leq Ch^k \quad \text{as} \quad h \to 0,$$

where the constant C does not depend on h.

2) Stability: The finite difference scheme defined by (16) with linear operator L_h is called stable, if there exist $h_0 > 0$ such that for arbitrary $h < h_0$ and for any discrete function $f^h \in H_h$, the solution of FDS, u^h, which satisfies

$$L_h u^h = f^h,$$

exists uniquely and satisfies the inequality

$$\|u^h\|_{\Omega_h} \leq C \parallel f^h \parallel_{H_h}, \tag{17}$$

where the constant C does not depend on h. The scheme defined by (16) is called stable for $(x,t) \in D \cup \partial D$, if there exist a constant C independent of h and τ such that

$$\max_{i,n} |u_i^n| \leq C[\max_i |\psi_i| + \max_{i,n} |\varphi_i^n|], \tag{18}$$

$$i = 0, \pm 1, \pm 2, .., M, \; n = 0, 1, ..., [T/\tau] - 1.$$

The inequality (18) has to hold for any functions ψ_i and φ_i^n. For a particular case when $\varphi_i^n \equiv 0$, we have only necessary condition for stability.

By Fourier or Von Neumann Analysis [13], we will seek solution to FDS (16) in the form

$$u_i^n = \lambda^n(\alpha) e^{I\alpha i}, \; i = 0, \pm 1, ..., \; n = 0, \pm 1, ..., \; I = \sqrt{-1} \tag{19}$$

where $e^{I\alpha i}$ are eigenvectors corresponding to an eigenvalue λ and α is a wave number.
Necessary condition for stability of FDS (16) will be true for all $\alpha \in$ R which the following inequality (20) holds

$$\mid \lambda(\alpha) \mid \leq 1. \tag{20}$$

III. Numerical Results

This section presents the examples of linear and nonlinear modified Burger's equation (14).

A. Linear Burger's equation

In the modified Burger's equation (14) if $b = 0$ and $\mu_1 = 0$ is called the linear Burger's equation and written in the form

$$u_t + cu_x = \mu_0 u_{xx} \tag{21}$$

In paper [14], B. Jitsom and et. al presented that the numerical solution by FTCS implicit scheme converge to an exact solution. They found that FTCS implicit scheme has properties of consistency with $T.E. = \left(O(\Delta t, (\Delta x)^2)\right)$ and unconditional stability with

$$\lambda = \frac{1}{1 + 4Q\sin^2\left(\frac{\alpha}{2}\right) + IP\sin\alpha},$$

then

$$|\lambda| = \frac{1}{\sqrt{\left[1 + 4Q\sin^2\left(\frac{\alpha}{2}\right)\right]^2 + P^2\sin^2\alpha}} \leq 1,$$

where $P = \frac{c\Delta t}{\Delta x}$ and $Q = \frac{\mu_0 \Delta t}{(\Delta x)^2}$.
An example of linear Burger's equation is shown in problem A1.
Problem A1 : Let us consider linear Burger's equation (22)

$$u_t + \frac{1}{10}u_x = \frac{1}{2}u_{xx},\ 0 \leq x \leq 1,\ 0 \leq t \leq 1. \tag{22}$$

With initial condition :

$$u(x,0) = \frac{1}{2}\left[1 - \tanh\left\{\frac{1}{2}(x - 15)\right\}\right],$$

and boundary conditions :

$$u(0,t) = \frac{1}{2}\left[1 - \tanh\left\{\frac{1}{2}(15 - \frac{1}{2}t)\right\}\right],$$

$$u(1,t) = \frac{1}{2}\left[1 - \tanh(-7 - t)\right].$$

Grid system :

$$D_h = \{(x_i, t_n)|x_i = (i-1)\Delta x,\ t_n = (n-1)\Delta t),$$

$$i = 1,2,...,M,\ n = 1,2,...,T\}.$$

By FTCS implicit scheme, we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{10}\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2(\Delta x)} - \frac{1}{2}\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0, \tag{23}$$

$$u_i^0 = \frac{1}{2}\left[1 - \tanh\left\{\frac{1}{2}(x_i - 15)\right\}\right],$$

$$u_1^n = \frac{1}{2}\left[1 - \tanh\left\{\frac{1}{2}(15 - \frac{1}{2}t_n)\right\}\right],$$

$$u_M^n = \frac{1}{2}\left[1 - \tanh(-7 - t_n)\right].$$

The exact solution of (22) with the initial and boundary conditions is

$$u(x,t) = \frac{1}{2}\left[1 - \tanh\left\{\frac{1}{2}(x - 15 - \frac{1}{2}t)\right\}\right]. \tag{24}$$

Table I presents the absolute error between exact and numerical solutions of problem A1 with $\Delta t = \Delta x = 0.05$ and the graphs of exact solution and numerical solutions of linear Burger's equation for FTCS implicit scheme are shown in Fig. 1 and Fig. 2 respectively. Moreover, The graph of absolute error between exact and numerical solutions of problem A1 is shown in Fig. 3. We can see that maximum of absolute error occurred at the middle space x in each time step t.

TABLE I
Absolute error between exact and numerical solutions for Problem A1 with $\Delta t = \Delta x = 0.05$

t	x			
	0.25	**0.50**	**0.505**	**0.75**
0	0.00	0.00	0.00	0.00
0.10	2.16×10^{-4}	2.76×10^{-4}	2.77×10^{-4}	2.55×10^{-4}
0.20	3.46×10^{-4}	4.63×10^{-4}	4.63×10^{-4}	4.13×10^{-4}
0.30	4.28×10^{-4}	5.81×10^{-4}	5.81×10^{-4}	5.11×10^{-4}
0.40	4.78×10^{-4}	6.54×10^{-4}	6.53×10^{-4}	5.71×10^{-4}
0.50	5.08×10^{-4}	6.98×10^{-4}	6.96×10^{-4}	6.07×10^{-4}
0.60	5.24×10^{-4}	7.23×10^{-4}	7.21×10^{-4}	6.27×10^{-4}
0.70	5.33×10^{-4}	7.36×10^{-4}	7.34×10^{-4}	6.37×10^{-4}
0.80	5.36×10^{-4}	7.41×10^{-4}	7.39×10^{-4}	6.41×10^{-4}
0.90	5.36×10^{-4}	7.41×10^{-4}	7.39×10^{-4}	6.41×10^{-4}
1.00	5.34×10^{-4}	7.38×10^{-4}	7.36×10^{-4}	6.38×10^{-4}

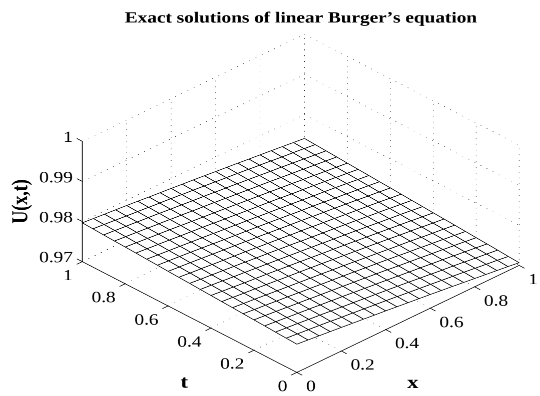

Fig. 1. The plot of exact solution for Problem A1 with $c = 1/10$, $b = 0$ and $\mu_0 = 1/2$.

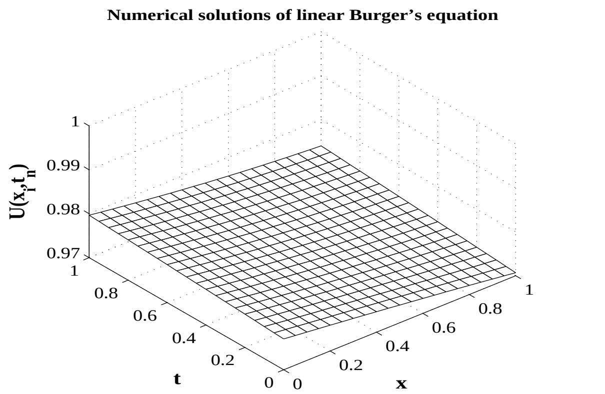

Fig. 2. The plot of numerical solutions for Problem A1 with $c = 1/10$, $b = 0$, $\mu_0 = 1/2$, $\Delta t = 0.05$ and $\Delta x = 0.05$.

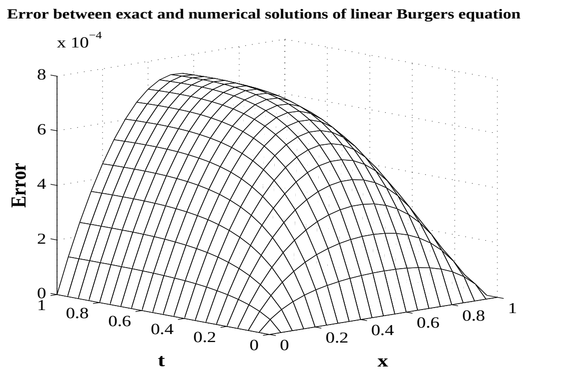

Fig. 3. The plot of absolute error between exact and numerical solutions for Problem A1.

TABLE II
MAXIMUM OF ABSOLUTE ERRORS FOR PROBLEM A1 WITH $\Delta x = \Delta t = 0.05, 0.01, 0.005, 0.001$

Δx	Δt	Maximum of absolute errors
0.05	0.05	7.162×10^{-3}
0.01	0.01	7.141×10^{-3}
0.005	0.005	6.979×10^{-3}
0.001	0.001	7.178×10^{-4}

Table II presents the maximum of absolute errors, we can see that maximum of absolute error goes to zero as the grid sizes Δx and Δt go to zero.

B. Nonlinear Burger's equations

Nonlinear Burger's equations are investigated by comparing the results between numerical solutions in FTCS implicit scheme and exact solutions. We then study the behaviour of solutions of modified Burger's equation (14) as follows,

$$u_t + (c + bu)u_x = (\mu_0 + \mu_1 u)u_{xx}. \quad (25)$$

In this work, nonlinear Burger's equation of 3 cases are studied as follows,

Case B1 : Nonlinear modified Burger's equation in conservative forms with $\mu_1 = 0$,
Case B2 : Nonlinear modified Burger's equation with $0 \le \mu_1 \le 1$,
Case B3 : Nonlinear modified Burger's equation for stationary solution with $\mu_1 = 0$.

Case B1 : Nonlinear modified Burger's equation in conservative form with $\mu_1 = 0$.
We consider nonlinear generalized Burger's equation (8) in the conservative form,

$$u_t + F_x = \mu_0 u_{xx} \quad \text{where} \quad F = cu + \frac{bu^2}{2}. \quad (26)$$

To obtain numerical solutions, we use FTCS implicit scheme as follows,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} - \mu_0 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0. \quad (27)$$

Where $\Delta t = \tau$ and $\Delta x = h$, we have

$$u_i^{n+1} - u_i^n + \tfrac{\tau}{2h}\left(F_{i+1}^n - F_{i-1}^n\right) - \tfrac{\mu_0 \tau}{h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) = 0$$

$$-\tfrac{\mu_0\tau}{h^2}u_{i-1}^{n+1} - \left(-1 - \tfrac{2\mu_0\tau}{h^2}\right)u_i^{n+1} - \tfrac{2\mu\tau}{h^2}u_{i+1}^{n+1} = u_i^n - \tfrac{\tau}{2h}\left(F_{i+1}^n - F_{i-1}^n\right),$$

or

$$a_i u_{i-1}^{n+1} - b_i u_i^{n+1} - c_i u_{i+1}^{n+1} = u_i^n + d_i\left(F_{i+1}^n - F_{i-1}^n\right), \quad (28)$$

$i = 2, ..., M-1,\ n = 1, ..., [T/\tau] - 1,$
where

$$a_i = -\tfrac{\mu_0\tau}{h^2},\quad b_i = -1 - \tfrac{2\mu_0\tau}{h^2},\quad c_i = \tfrac{\mu_0\tau}{h^2},\quad d_i = -\tfrac{\tau}{2h},\ i = 2, ..., M-1.$$

The equation (28) can be written as the system of tridiagonal matrix, which we can solve these system by the sweep method.

Problem B1 : Let us consider nonlinear modified Burger's equation (29) for $c = 0,\ b = 1,$ and $\mu_0 = \frac{1}{4}$,

$$u_t + uu_x = \frac{1}{4}u_{xx},\ 0 \le x \le 1,\ 0 \le t \le 1. \quad (29)$$

With initial condition : $u(x, 0) = \frac{1}{1+e^{2x}}$,
and boundary conditions :

$$u(0,t) = \frac{1}{1+e^{-t}}, u(1,t) = \frac{1}{1+e^{2-t}}.$$

Grid system :

$$D_h = \{(x_i, t_n) | x_i = (i-1)\Delta x,\ t_n = (n-1)\Delta t),$$
$$i = 1, 2, ..., M,\ n = 1, 2, ..., T\}.$$

By FTCS implicit scheme, we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} - \frac{1}{4}\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0 \quad (30)$$

$$u_i^0 = \frac{1}{1+e^{2x_i}},$$

$$u_1^n = \frac{1}{1+e^{-t_n}},\quad u_M^n = \frac{1}{1+e^{2-t_n}}.$$

The exact solution of (29) with the initial and boundary conditions is

$$u(x,t) = \frac{1}{1 + e^{\frac{2x-t}{4\mu_0}}}.$$

Table III presents the absolute error between exact and numerical solutions for problem B1 with $\Delta t = \Delta x = 0.05$ and the graphs of an exact and numerical solutions are shown in Fig. 4 and Fig. 5 respectively. Moreover, the graph of absolute errors is shown in Fig. 6. We can see that maximum of absolute error occurred at the middle space x in each time step t.

TABLE III
ABSOLUTE ERROR BETWEEN EXACT AND NUMERICAL SOLUTIONS FOR PROBLEM B1 WITH $\Delta t = \Delta x = 0.05$

t	x			
	0.25	**0.50**	**0.505**	**0.75**
0	0.00	0.00	0.00	0.00
0.10	4.91×10^{-4}	4.08×10^{-4}	3.67×10^{-4}	1.92×10^{-4}
0.20	8.32×10^{-4}	8.21×10^{-4}	7.59×10^{-4}	4.39×10^{-4}
0.30	1.10×10^{-3}	1.20×10^{-3}	1.10×10^{-3}	7.12×10^{-4}
0.40	1.30×10^{-3}	1.60×10^{-3}	1.50×10^{-3}	9.91×10^{-4}
0.50	1.40×10^{-3}	1.90×10^{-3}	1.80×10^{-3}	1.30×10^{-3}
0.60	1.50×10^{-3}	2.10×10^{-3}	2.20×10^{-3}	1.50×10^{-3}
0.70	1.60×10^{-4}	2.30×10^{-3}	2.30×10^{-3}	1.80×10^{-3}
0.80	1.60×10^{-3}	2.50×10^{-3}	2.50×10^{-3}	2.20×10^{-3}
0.90	1.60×10^{-3}	2.60×10^{-3}	2.70×10^{-3}	2.10×10^{-3}
1.00	1.50×10^{-3}	2.70×10^{-3}	2.70×10^{-3}	2.30×10^{-3}

Table IV shows the maximum of absolute errors, we can see that the maximum of absolute error goes to zero as the mesh sizes Δx and Δt go to zero.

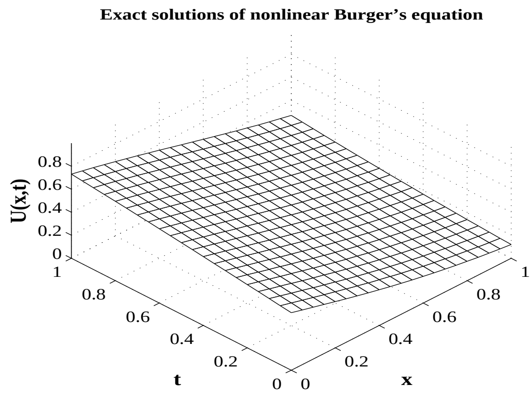

Fig. 4. The plot of exact solution for Problem B1 with $c = 0$, $b = 1$ and $\mu_0 = 1/4$.

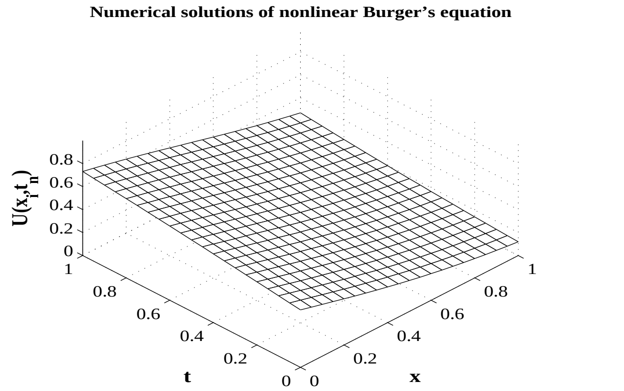

Fig. 5. The plot of numerical solutions for Problem B1 with $c = 0$, $b = 1$, $\mu_0 = 1/4$, $\Delta t = 0.05$ and $\Delta x = 0.05$.

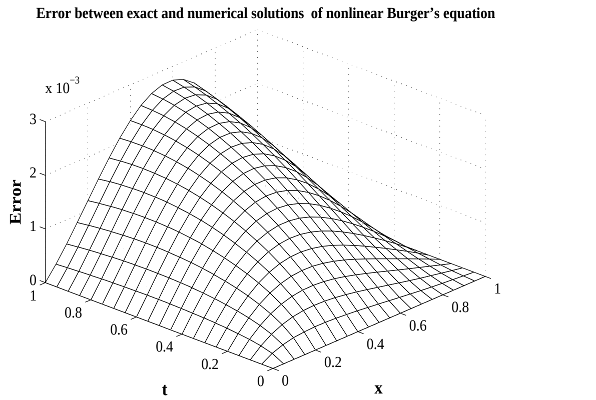

Fig. 6. The plot of absolute error between exact and numerical solutions for Problem B1.

TABLE IV
MAXIMUM OF ABSOLUTE ERRORS FOR PROBLEM B1 WITH $\Delta x = \Delta t = 0.05, 0.01, 0.005, 0.001$

Δx	Δt	Maximum of absolute errors
0.05	0.05	1.855×10^{-3}
0.01	0.01	3.794×10^{-4}
0.005	0.005	1.902×10^{-4}
0.001	0.001	3.811×10^{-5}

Case B2 : Nonlinear modified Burger's equation with $0 \leq \mu_1 \leq 1$.
We consider a modified Burger's equation (14) in the form,

$$u_t + R(u)u_x = S(u)u_{xx}, \tag{31}$$

where $R(u) = c + bu$ and $S(u) = \mu_0 + \mu_1 u$.
FTCS implicit scheme is used in numerical solution, we get

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + R\left(u_i^n\right)\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - S\left(u_i^n\right)\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0. \tag{32}$$

Where $\Delta t = \tau$ and $\Delta x = h$, we have

$$u_i^{n+1} - u_i^n + R(u_i^n)\tfrac{\tau}{2h}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) - S(u_i^n)\tfrac{\tau}{h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) = 0$$

$$\left[-R(u_i^n)\frac{\tau}{2h} - S(u_i^n)\frac{\tau}{h^2}\right]u_{i-1}^{n+1} - \left[-1 - 2S(u_i^n)\frac{\tau}{h^2}\right]u_i^{n+1} + \left[R(u_i^n)\tfrac{\tau}{2h} - S(u_i^n)\tfrac{\tau}{h^2}\right]u_{i+1}^{n+1} = u_i^n,$$

or

$$a_i u_{i-1}^{n+1} - b_i u_i^{n+1} + c_i u_{i+1}^{n+1} = u_i^n, \quad (33)$$
$$i = 2, ..., M-1, \; n = 1, ..., [T/\tau] - 1,$$

where

$$\begin{aligned} a_i &= -R(u_i^n)\tfrac{\tau}{2h} - S(u_i^n)\tfrac{\tau}{h^2}, \\ b_i &= -1 - 2S(u_i^n)\tfrac{\tau}{h^2}, \\ c_i &= R(u_i^n)\tfrac{\tau}{2h} - S(u_i^n)\tfrac{\tau}{h^2}, \; i = 2, ..., M-1. \end{aligned}$$

We can see that equation (33) is the system of tridiagonal matrix. Numerical solutions are obtained by the sweep method.

Problem B2 : Let us consider nonlinear modified Burger's equation (34) for $c = 0, \; b = 1, \; \mu_0 = \frac{1}{4}$ and $0 \le \mu_1 \le 1$ presents in the form,

$$u_t + uu_x = \left(\frac{1}{4} + \mu_1 u\right)u_{xx}, \; 0 \le x \le 1, \; 0 \le t \le 10 \quad (34)$$

With initial condition : $u(x,0) = \frac{1}{1+e^{2x}}$,
and boundary conditions :

$$u(0,t) = \frac{1}{1+e^{-t}}, u(1,t) = \frac{1}{1+e^{2-t}}.$$

Grid system :

$$D_h = \{(x_i, t_n) | x_i = (i-1)\Delta x, \; t_n = (n-1)\Delta t),$$
$$i = 1, 2, ..., M, \; n = 1, 2, ..., T\}.$$

By FTCS implicit scheme, we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + R(u_i^n)\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - S(u_i^n)\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0 \quad (35)$$

$$u_i^0 = \frac{1}{1+e^{2x_i}},$$

$$u_1^n = \frac{1}{1+e^{-t_n}}, \quad u_M^n = \frac{1}{1+e^{2-t_n}}.$$

Numerical solutions of a nonlinear modified Burger's equation (34) at fixed $\mu_1 = 0.5$ with vary time $0 \le t \le 10$ are presented in Table V. We found that the numerical solutions will be increased and converge to 1.00 whereas time increases. In the case of fixed time, $t = 5$ and vary $0 \le \mu_1 \le 1$ are shown in Fig. 7, we can see that the numerical solutions will be slightly decreased when μ_1 increased. The graphs of numerical solutions of nonlinear modified Burger's equation for $\mu_1 = 0, \; 0.1, \; 0.5, \; 1.0$ are shown in Fig. 8 - Fig. 11.

TABLE V
NUMERICAL SOLUTIONS OF NONLINEAR MODIFIED BURGER'S EQUATION (34) AT FIXED $\mu_1 = 0.5$ AND $0 \le t \le 10$

	$x = 0.00$	$x = 0.25$	$x = 0.50$	$x = 1.00$
$t = 0$	0.500000	0.377540	0.268941	0.119202
$t = 2$	0.880797	0.800901	0.708335	0.500000
$t = 5$	0.999331	0.985310	0.975615	0.952574
$t = 10$	0.999954	0.999897	0.999827	0.999664

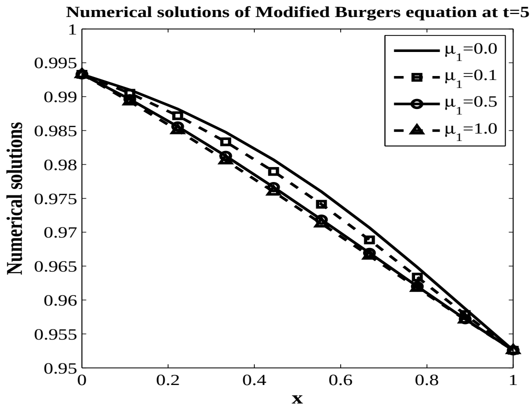

Fig. 7. The plot of numerical solutions of a nonlinear modified Burger's equation (34) with t=5.

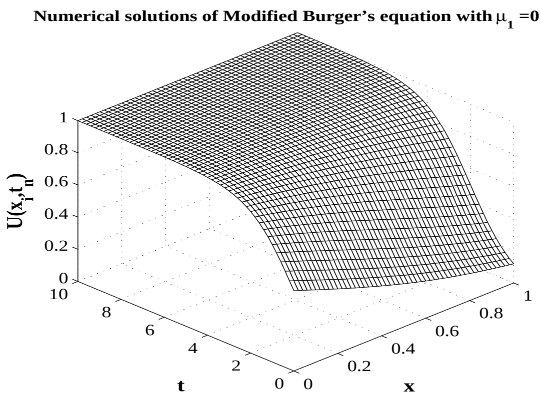

Fig. 8. The plot of numerical solutions of a nonlinear modified Burger's equation (34) at $\mu_1 = 0$.

Case B3 : Nonlinear modified Burger's equation for stationary solution with $\mu_1 = 0$.
We consider a modified Burger's equation (14) in the form,

$$u_t + R(u)u_x = S(u)u_{xx}, \quad (36)$$

where $R(u) = c + bu$ and $S(u) = \mu_0 + \mu_1 u$.
In case $c = \frac{1}{2}, \; b = -1$ and $\mu_0 = \frac{1}{4}$ and $\mu_1 = 0$. We have,

$$u_t + (\frac{1}{2} - u)u_x = \frac{1}{4}u_{xx}, \quad (37)$$

FTCS implicit scheme is used in numerical solutions, we have

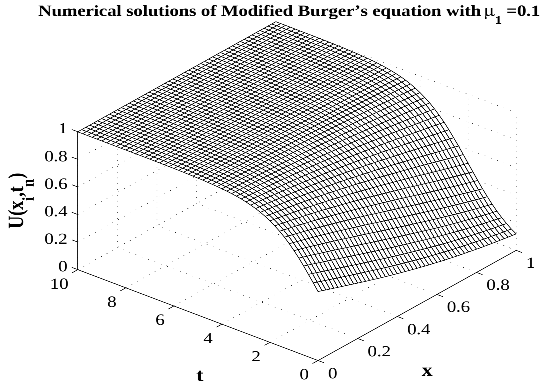

Fig. 9. The plot of numerical solutions of a nonlinear modified Burger's equation (34) at $\mu_1 = 0.1$.

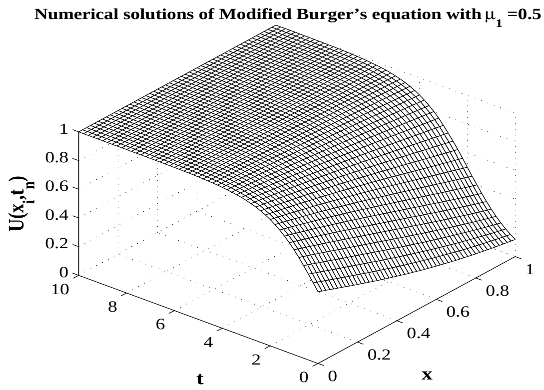

Fig. 10. The plot of numerical solutions of a nonlinear modified Burger's equation (34) at $\mu_1 = 0.5$.

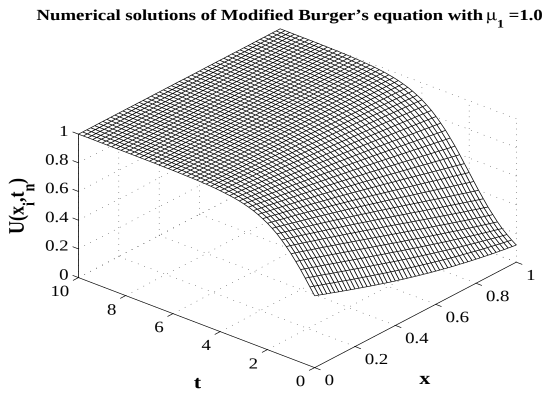

Fig. 11. The plot of numerical solutions of a nonlinear modified Burger's equation (34) at $\mu_1 = 1.0$.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + R\left(u_i^n\right) \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - S\left(u_i^n\right) \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0 \qquad (38)$$

Problem B3 : Let us consider a nonlinear modified Burger's equation (39)

$$u_t + (\frac{1}{2} - u)u_x = \frac{1}{4}u_{xx}, \ 0 \leq x \leq 10, \ 0 \leq t \leq 1. \qquad (39)$$

With initial condition : $u(x, 0) = \frac{1}{2}\left(1 + \tanh(x - 5)\right)$, and boundary conditions : $u(0, t) = 0$, $u(1, t) = 1$.

Grid system :

$$D_h = \{(x_i, t_n) | x_i = (i-1)\Delta x, \ t_n = (n-1)\Delta t)\},$$

$$i = 1, 2, ..., M, \ n = 1, 2, ..., T\}.$$

By FTCS implicit scheme, we get

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + R\left(u_i^n\right) \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - S\left(u_i^n\right) \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0. \qquad (40)$$

$$u_i^0 = \frac{1}{2}\left(1 + \tanh(x_i - 5)\right),$$

$$u_1^n = 0, \quad u_M^n = 1.$$

The stationary solution of (39) with initial and boundary conditions has

$$u = \frac{1}{2}\left(1 + \tanh(x - 5)\right).$$

The graph of a stationary solution and absolute error of problem B3 at $t = 1$ are shown in Fig. 12, Fig. 13, Fig. 14 respectively.

TABLE VI
MAXIMUM OF ABSOLUTE ERRORS FOR PROBLEM B3 WITH $\Delta x = \Delta t = 0.05, 0.01, 0.005, 0.001$ AT $t = 1$

Δx	Δt	Maximum of absolute errors
0.05	0.05	1.054×10^{-3}
0.01	0.01	5.742×10^{-4}
0.005	0.005	7.257×10^{-5}
0.001	0.001	5.832×10^{-8}

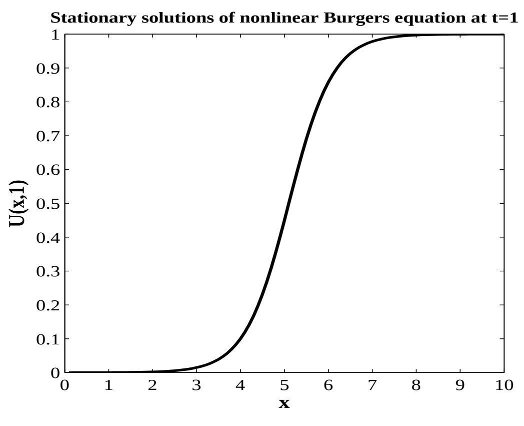

Fig. 12. The plot of stationary solutions for Problem B3 with $c = \frac{1}{2}$, $b = -1$ and $\mu_0 = 1/4$ at $t = 1$.

Table VI shows the maximum of absolute errors, we can see that the maximum of absolute error goes to zero as the grid sizes Δx and Δt go to zero.

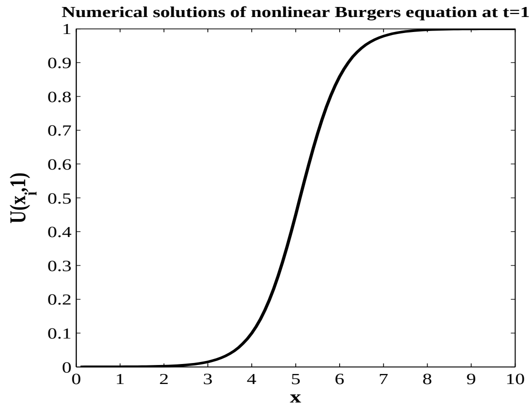

Fig. 13. The plot of numerical solution for Problem B3 with $c = \frac{1}{2}$, $b = -1$, $\mu_0 = 1/4$, $\Delta t = 0.01$ and $\Delta x = 0.01$ at $t = 1$.

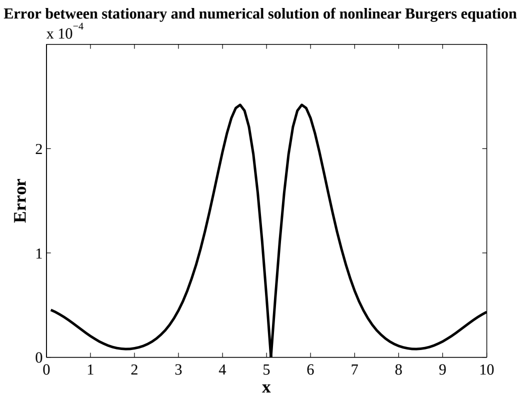

Fig. 14. The plot of absolute error between stationary and numerical solutions for Problem B3 at $t = 1$

IV. Conclusion

In this work, we have investigated the modified Burger's equation in the form,

$$u_t + (c + bu)u_x = (\mu_0 + \mu_1 u)u_{xx}, \tag{41}$$

where the parameters c, b, μ_0 and μ_1 are given. An example of the linear Burger' s equation (21) and the solutions were computed using the MATLAB program. We have found that the numerical solutions in the FTCS implicit scheme converge to related exact solutions is agreed with the theoretical convergence results.

Three cases of the numerical solutions for the nonlinear Burger's equation were obtained by the FTCS implicit scheme as follows.

Case B1 : Nonlinear modified Burger's equation in conservative form with

$$c = 0,\ b = 1,\ \mu_0 = \frac{1}{4} \text{ and } \mu_1 = 0.$$

In the case, we have found that the numerical solutions obtained by the FTCS implicit scheme converge to the exact solution and the maximum of absolute error tends to zero when the grid sizes Δx and Δt close to zero.

Case B2 : Nonlinear modified Burger's equation with

$$c = 0,\ b = 1,\ \mu_0 = \frac{1}{4} \text{ and } 0 \leq \mu_1 \leq 1.$$

In the case, we have fixed time and varied the values of μ_1 in $0 \leq \mu_1 \leq 1$. The results showed that the numerical solutions are slightly decreased when μ_1 is increasing. Moreover, when μ_1 is fixed, we found that the numerical solutions are increased and they are converged to 1.00 for all x as t is increasing.

Case B3 : Nonlinear modified Burger's equation for stationary solution with

$$c = \frac{1}{2},\ b = -1,\ \mu_0 = \frac{1}{4} \text{ and } \mu_1 = 0.$$

In this case, we obtained the numerical solutions by the FTCS implicit scheme. All solutions are converged to the stationary exact solution.

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