High-order Hybrid Obreshkov Methods

R. I. Okuonghae*, and I. B. Aiguobasimwin†

***Abstract*—In this paper, a family of high order Obreshkov hybrid formulas are proposed for the numerical solution of first order initial value problems (IVPs) in ordinary differential equations (ODEs). These formulas are stable for step number $k \leq 18$. Results from numerical experiments with the constructed hybrid methods on well-known stiff problems have been reported herein.**

***Index Terms*—Continuous LMM, third derivative LMM, hybrid LMM, stiff problems, boundary locus, $A(\alpha)$-stability.**

I. Introduction

CONSIDER the initial value problems (IVP)

$$y' = f(x, y), \quad x \in [x_0, X], \quad y(x_0) = y_0, \tag{1}$$

where $f : R \times R^m \to R^m$, can be solved using the hybrid linear multistep methods (HLMM)

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) + \sum_{r=1}^{s} h^r \left(\sum_{j=1}^{k} \beta_{i,j}^{(r)} D^r y(x_{n+v_j}, y_{n+v_j})\right), \tag{2}$$

with the hybrid predictor,

$$y_{n+v_i} = \sum_{j=0}^{k} \overline{\alpha}_j y_{n+j} + \sum_{r=1}^{s} h^r \left(\sum_{j=1}^{k} \overline{\beta}_{i,j}^{(r)} D^r y(x_{n+j}, y_{n+j})\right),$$

where k denotes the step number, $h = x_{n+1} - x_n$ is the step length, $y'(x_{n+j}, y_{n+j}) = f_{n+j}$, $D^r y(x_{n+j}, y_{n+j}) = f_{n+j}^{(r-1)}$ and $D^r y(x_{n+v_j}, y_{n+v_j}) = f_{n+v_j}^{(r-1)}$. The v_i in (2) is called the off-step point and is $v = [v_1, v_2, \cdots, v_k]^T$. If the off-step points $v_1 \neq v_2 \neq \cdots \neq v_k$, then the methods have Runge-Kutta's flavour and is regarded as general linear methods (GLM [3]) with k stages, see [28]. If $v_1 = v_2 = \cdots = v_k = v$, then the formula in (2) is a GLM with single stage. The hybrid LMM in (2) is a constituent method of the so-called multiderivative LMM. As noted in [2], the multiderivative LMM was first proposed by Obreshkov in 1940, see [18]. The formulas in (2) offers the opportunity to bypass the Dahlquist order barrier for LMM [6]. The interest herein is to propose a method with an implicit structure. Methods of this kinds gives large stability region, high order and small error constant. These are advantages of hybrid methods over LMM. Several classes of hybrid LMM exist, see examples in [1], [2], [3], [4], [5], [7], [13], [14], [15], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]. Stability issue is one of the main points to be considered when constructing a constituent method in (2). Applying the method in (2) on the Dahlquist test problem:

$$y' = \lambda y, \quad y_0 = 1, \quad Re(\lambda) < 0, \tag{3}$$

yields a stability polynomial

$$\pi(w, z) = w^k - w^{k-1} - z\sum_{j=0}^{k} \beta_j w^j - \sum_{r=1}^{s} z^r \left(\sum_{j=1}^{k} \beta_{i,j}^{(r)} \left(\sum_{j=0}^{k} \overline{\alpha}_j w^j + \sum_{r=1}^{s} z^r \sum_{j=1}^{k} \overline{\beta}_{i,j}^{(r)} w^j\right)\right). \tag{4}$$

Here $z = \lambda h$, and as in [16] and [31], the stability region of the method is defined to be

$$S = \{z \in C; |\pi(w, z)| \leq 1\}.$$

The use of appropriate Taylor expansions of $\{y(x_{n+j}), y'(x_{n+j}), |j = 0(1)k, y'(x_{n+k}), y''(x_{n+k}), y'''(x_{n+k}), y'(x_{n+v}), y''(x_{n+v}), y'''(x_{n+v})\}$ in (2a) and (2b) about the mesh point x_n reduces (2) to the form
L.T.E$_k = C_{p_k+1} h^{p_k+1} y^{(p_k+1)}(x_n) + O(h^{p_k+1})$,
$L.T.E_q = C_{p_q+1}^{(q)} h^{p_q+1} y^{(p_q+1)}(x_n) + O(h^{p_q+1})$,
$q = 1, 2, 3, \cdots, k$, where $L.T.E$ is the local truncation error of the methods, while p_q and p_k are the order of the schemes in (2) alongside its hybrid predictor respectively. The C_{p_k+1} and $C_{p_q+1}^{(q)}$ are the error constants of the methods in (2). The LMM (2) is implicit, hence we solve the arising system of nonlinear algebraic equations in terms of y_{n+k} using the Newton Raphson iterative scheme

$$y_{n+k}^{(s+1)} = y_{n+k}^{(s)} - [F'(y_{n+k}^{(s)})]^{-1} F(y_{n+k}^{(s)}),$$

where $F'(y_{n+k}^{(s)})$ is the Jacobian matrix. The starter for the Newton Raphson scheme is the fourth order Runge-Kutta method (RKM).
In [10], a class of third derivative LMM was considered. The scheme in [10] is an extension of the Enright's second derivative LMM[8]. The LMM were shown to be $A(\alpha)$-stable for step number $k \leq 5$ with α as the angle of stability. In a way, the LMM in [10] is a subclass of the method in (2).
An example of the third derivative hybrid LMM (TDHLMM) in [27] is

$$y_{n+v} = \sum_{j=0}^{k} \overline{\alpha}_j y_{n+j} + h\overline{\beta}_k^{(1)} f_{n+k} + h^2 \overline{\beta}_k^{(2)} f'_{n+k}, \tag{5}$$

with an output scheme,

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v^{(1)} f_{n+v} + h^2 \beta_v^{(2)} f'_{n+v} + h^3 \beta_v^{(3)} f''_{n+v}. \tag{6}$$

The hybrid LMM[27] in (6) is a constituent method of the hybrid multistep multi-derivative methods in (2). The order

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*Department of Mathematics,University of Benin, Benin City. Nigeria. e-mail: (okunoghae01@yahoo.co.uk).
†Department of Mathematics,University of Benin, Benin City. Nigeria. e-mail:(becsin2002@yahoo.co.uk).

of the predictor formula in (5) is $p = k+2$, while the order of the output scheme in (6) is $p = k+4$. The stability plot of the scheme in (5) and (6) shows that the formulas in (2) are A-stable for $k \leq 3$ and $A(\alpha)$-stable for step number $4 \leq k \leq 9$. Their stability characteristics and the error constants are given in Table XI. The $C^{(q)}_{p_q+1}$, and C_{p_k+1}, p_q, $q = 1$, and $p_k(6)$ represent the error constants and the order of the methods in (5) and (6) respectively.

In this paper, a third derivative hybrid LMM

$$y(x_n + vh) = \sum_{j=0}^{k} \overline{\alpha}_j(v) y_{n+j} + h\overline{\beta}^{(1)}_k(v) f_{n+k} + h^2\overline{\beta}^{(2)}_k(v) f'_{n+k} + h^3\overline{\beta}^{(3)}_k(v) f''_{n+k}, \quad (7)$$

$$y(x_n + th) = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j(t,v) f_{n+j} + h\beta^{(1)}_v(t,v) f_{n+v} + h^2\beta^{(2)}_v(t,v) f'_{n+v} + h^3\beta^{(3)}_v(t,v) f''_{n+v}, \quad (8)$$

is proposed. The $v = k - \frac{1}{2}$ in (7) and (8) represents the off-step point while the transformation variable t in (8) is $t = (x - x_n)/h$. The approximations $y(x_n + vh)$ and $\{y(x_n + th)\}^k_{t=1}$ are of order $p = k+3$ and $p = k+4$ respectively. The continuous coefficients $\{\overline{\alpha}_j(v)\}^k_{j=0}$, $\{\overline{\beta}^{(r)}_k(v)\}^3_{r=1}$, $\{\beta_j(t,v)\}^k_{j=0}$, and $\{\beta^{(r)}_v(t,v)\}^3_{r=1}$ are polynomial of order less or equal to p. To implement the formula in (8) compute the solution y_{n+v} in (7) at the hybrid point x_{n+v} and substitute the resulting solution y_{n+v} into the function f_{n+v} in the output method in (8) to obtain y_{n+k}.

In Section II we discuss the derivation of the hybrid predictor formula in (7) using collocation and interpolation techniques, see [21]. While Section III summarize the derivation of the output method in (8). In Section IV, an $A(\alpha)$-stability analysis is provided. Four numerical experiments will be given in Section V to validate the aims of this paper.

II. Derivation of the Hybrid Predictor Formula in (7)

To derive the hybrid predictor formula, the solution of the IVPs is assumed to be the polynomial

$$y(x) = \sum_{j=0}^{N} a_j x^j, \quad (9)$$

where $\{x^j\}^N_{j=0}$ is the monomial polynomial basis function and $\{a_j\}^N_{j=0}$ are the real parameter constants to be determined. Setting $N = k+3$, we have

$$\begin{cases} y'(x) = \sum_{j=1}^{N=k+3} j a_j x^{j-1}, \\ y''(x) = \sum_{j=2}^{N=k+3} j(j-1) a_j x^{j-2}, \\ y'''(x) = \sum_{j=3}^{N=k+3} j(j-1)(j-2) a_j x^{j-3}, \end{cases} \quad (10)$$

where $y'(x) = f(x,y)$, $y''(x) = f'(x,y)$ and $y'''(x) = f''(x,y)$. Collocating (10) at $\{x = x_{n+j}\}^k_{j=0}$, and interpolating (9) at $x = x_{n+v}$, we obtain the linear system of equations

$$XA = B, \quad (11)$$

where

$$X = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \cdots & \cdots & x_n^{k+3} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \cdots & \cdots & x_{n+1}^{k+3} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_{n+k} & x_{n+k}^2 & x_{n+k}^3 & \cdots & \cdots & x_{n+k}^{k+3} \\ 0 & 1 & 2x_{n+k} & 3x_{n+k}^2 & \cdots & \cdots & (k+3)x_{n+k}^{k+2} \\ 0 & 0 & 2 & 6x_{n+k} & \cdots & \cdots & (k+2)(k+3)x_{n+k}^{k+1} \\ 0 & 0 & 0 & 12 & \cdots & \cdots & (k+1)(k+2)(k+3)x_{n+k}^{k} \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ a_{k+2} \\ a_{k+3} \end{pmatrix}, \quad B = \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k} \\ f_{n+k} \\ f'_{n+k} \\ f''_{n+k} \end{pmatrix}.$$

Solving (11) the values of $\{a_j\}^{k+3}_{j=0}$ are determined and substituting the resulting values $a'_j s$ into the polynomial representation (9) with further simplification yield the coefficients $\{\overline{\alpha}_j(v)\}^k_{j=0}$, $\overline{\beta}^{(1)}_1(v)$, $\overline{\beta}^{(2)}_1(v)$, $\overline{\beta}^{(3)}_1(v)$ in equation9, with $v = (x - x_n)/h$ and $\alpha_v(v) = 1$ on the left hand side of equation9 for a fixed value of k. For example, setting $k = 1$ in (7) is the hybrid predictor formula

$$y_{n+v} = \overline{\alpha}_0(v) y_n + \overline{\alpha}_1(v) y_{n+1} + h\overline{\beta}^{(1)}_1(v) f_{n+1} + h^2\overline{\beta}^{(2)}_1(v) f'_{n+1} + h^3\overline{\beta}^{(3)}_1(v) f''_{n+1}. \quad (12)$$

Again, for the method in (7), fix $k = 1$ in (11) to obtain

$$X = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 \\ 0 & 0 & 0 & 6 & 24x_{n+1} \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad B = \begin{pmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f'_{n+1} \\ f''_{n+1} \end{pmatrix}. \quad (13)$$

Solving equation (13) for the values of a_0, a_1, a_2, a_3, a_4 and substituting the resulting values $a'_j s$ and the transformation $t = v = (x - x_n)/h$ into the polynomial representation (9) gives the method in (12) with continuous coefficients,

$$\overline{\alpha}_0(v) = \left(1 - 4v + 6v^2 - 4v^3 + v^4\right),$$
$$\overline{\alpha}_1(v) = \left(4v - 6v^2 + 4v^3 - v^4\right),$$
$$\overline{\beta}^{(1)}_1(v) = \left(-3v + 6v^2 - 4v^3 + v^4\right),$$
$$\overline{\beta}^{(2)}_1(v) = \left(v - \frac{5v^2}{2} + 2v^3 - \frac{v^4}{2}\right),$$
$$\overline{\beta}^{(3)}_1(v) = \left(-\frac{v}{6} + \frac{v^2}{2} - \frac{v^3}{2} + \frac{v^4}{6}\right).$$

Setting $v = \frac{1}{2}$ in the continuous coefficients and substituting the resulting expressions into (12) gives

$$y_{n+\frac{1}{2}} = \frac{1}{16}\left(y_n + 15y_{n+1}\right) - \frac{7}{16}hf_{n+1} + \frac{3}{32}h^2 f'_{n+1} - \frac{1}{96}h^3 f''_{n+1}, \quad p = 4. \tag{14}$$

Following these procedures we obtained the discrete coefficients of the method in (7) for step number $k \leq 18$, see Tables IX, X, and XV.

III. The hybrid LMM in (8)

Using the ideas in section 2, we obtain a system of equation

$$X=\begin{pmatrix} 1 & x_{n+k-1} & x^2_{n+k-1} & x^3_{n+k-1} & \cdots & \cdots & x^{k+4}_{n+k-1} \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & \cdots & (k+4)x_n^{k+3} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & \cdots & \cdots & (k+4)x^{k+3}_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+k} & 3x^2_{n+k} & \cdots & \cdots & (k+4)x^{k+3}_{n+k} \\ 0 & 1 & 2x_{n+v} & 3x^2_{n+v} & \cdots & \cdots & (k+4)x^{k+3}_{n+v} \\ 0 & 0 & 2 & 6x_{n+v} & \cdots & \cdots & (k+3)(k+4)x^{k+2}_{n+v} \\ 0 & 0 & 0 & 12 & \cdots & \cdots & (k+2)(k+3)(k+4)x^{k+1}_{n+v} \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k+1} \\ a_{k+2} \\ a_{k+3} \\ a_{k+4} \end{pmatrix}, \quad B = \begin{pmatrix} y_{n+k-1} \\ f_n \\ f_{n+1} \\ \vdots \\ f_{n+k} \\ f_{n+v} \\ f'_{n+v} \\ f''_{n+v} \end{pmatrix}. \tag{15}$$

Now, consider a case for $k = 1$ in (8) yields

$$XA = B, \tag{16}$$

where

$$X=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} \\ 0 & 1 & 2x_{n+v} & 3x^2_{n+v} & 4x^3_{n+v} & 5x^4_{n+v} \\ 0 & 0 & 2 & 6x_{n+v} & 12x^2_{n+v} & 20x^3_{n+v} \\ 0 & 0 & 0 & 12 & 24x_{n+v} & 60x^2_{n+v} \end{pmatrix},$$

$$A=\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}, \quad B = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+v} \\ f'_{n+v} \\ f''_{n+v} \end{pmatrix}.$$

Solving (17) by **MATHEMATICA** software package we obtain the values of $\{a_j\}_{j=0}^{5}$. Substituting the resulting values $a_j's$ and $t = (x - x_n)/h$ into (9) yields

$$y(x_n + th) = y_n + h\beta_0(t, v)f_n + h\beta_1(t, v)f_{n+1} + h\beta_v^{(1)}(t, v)f_{n+v} + h^2\beta_v^{(2)}(t, v)f'_{n+v} + h^3\beta_v^{(3)}(t, v)f''_{n+v}, \tag{17}$$

where,
$\beta_0(t, v) = t\left(4t^4 + 20v^3 + 20t^2v(1 + v) - 10tv^2(3 + v) -5t^3(1 + 3v)\right)/20v^3,$
$\beta_1(t, v) = t^2\left(-4t^3 + 15t^2v - 20tv^2 + 10v^3\right)/\Psi_1,$
$\Psi_1 = 20(-1 + v)^3,$
$\beta_v^{(3)}(t, v) = t^2\left(12t^3 - 30v^2 + 20tv(2 + v) -15t^2(1 + 2v)\right)/120(-1 + v)v,$
$\beta_v^{(1)}(t, v) = t^2\left(4t^3(1 + 3(-1 + v)v) - 10v^2(3 - 8v + 6v^2) +20tv\left(1 - 2v + 2v^3\right) - 5t^2\left(1 - 6v^2 + 8v^3\right)\right)/\Psi_2,$
$\Psi_2 = 20(-1 + v)^3v^3,$
$\beta_v^{(2)}(t, v) = -t^2\left(10(2 - 3v)v^2 + t^3(-4 + 8v) +5t^2(1 + v - 5v^2) + 20tv(-1 + v + v^2)\right)/\Psi_3,$
$\Psi_3 = 20(-1 + v)^2v^2.$
Inserting $t = 1$ and $v = \frac{1}{2}$ into the continuous coefficients of the formula in (17) gives

$$y_{n+1} = y_n + h\left(\tfrac{1}{10}f_n + \tfrac{4}{5}f_{n+\frac{1}{2}} + \tfrac{1}{10}f_{n+1}\right) + \tfrac{1}{60}h^3 f''_{n+\frac{1}{2}}, \quad p = 6. \tag{18}$$

Following this approach we obtain the discrete methods of the hybrid LMM in (8) of order $k + 5$ for step number $k \leq 18$, see Tables XVI, XVII, and XVIII for their discrete coefficients. The composition of (14) and (18) produces a family of stable methods.

IV. Stability of the methods

Substituting the hybrid solutions y_{n+v} in (7) at hybrid points x_{n+v} into the output method in (8) for a corresponding k, and applying the resultant method to the scalar test problem (4) yield the stability polynomial

$$\pi(w, z) = w^k - w^{k-1} - z\sum_{j=0}^{k}\beta_j w^j -z\beta_v^{(1)}\left(\sum_{j=0}^{k}\overline{\alpha}_j w^j + z\overline{\beta}_k^{(1)}w^k + z^2\overline{\beta}_k^{(2)}w^k + z^3\overline{\beta}_k^{(3)}w^k\right) -z^2\beta_v^{(2)}\left(\sum_{j=0}^{k}\overline{\alpha}_j w^j + z\overline{\beta}_k^{(1)}w^k + z^2\overline{\beta}_k^{(2)}w^k + z^3\overline{\beta}_k^{(3)}w^k\right) -z^3\beta_v^{(3)}\left(\sum_{j=0}^{k}\overline{\alpha}_j w^j + z\overline{\beta}_k^{(1)}w^k + z^2\overline{\beta}_k^{(2)}w^k + z^3\overline{\beta}_k^{(3)}w^k\right).$$

where, $w = e^{i\theta}, \theta \in [0, 2\pi]$. Solving the stability polynomial equation $\pi(r, z) = 0$ for the values of z, we obtain the roots of $\pi(r, z) = 0$ to be z_1, z_2 and z_3. The boundary loci of the roots of the stability polynomial $\pi(r, z)$ of the hybrid method in (8) reveals that the new algorithm in (8) are A-stable for $k \leq 3$ and $A(\alpha)$-stable for $4 \leq k \leq 18$. For $k \geq 19$, the hybrid LMM (8) is not stable. The exterior of the graphs in Fig. 1 are the stability regions of the composite methods in (8). The advantages of the hybrid LMM(8) over the hybrid LMM(6) are:

- The stable hybrid LMM (6) have step number $k \leq 9$, while the step number of the hybrid LMM(8) is $k \leq 18$. The hybrid LMM (8) thus have more stable methods than the hybrid LMM in (6).
- Also, the order of the hybrid predictor in (6) is one less than that of the hybrid LMM(8) for a given k.

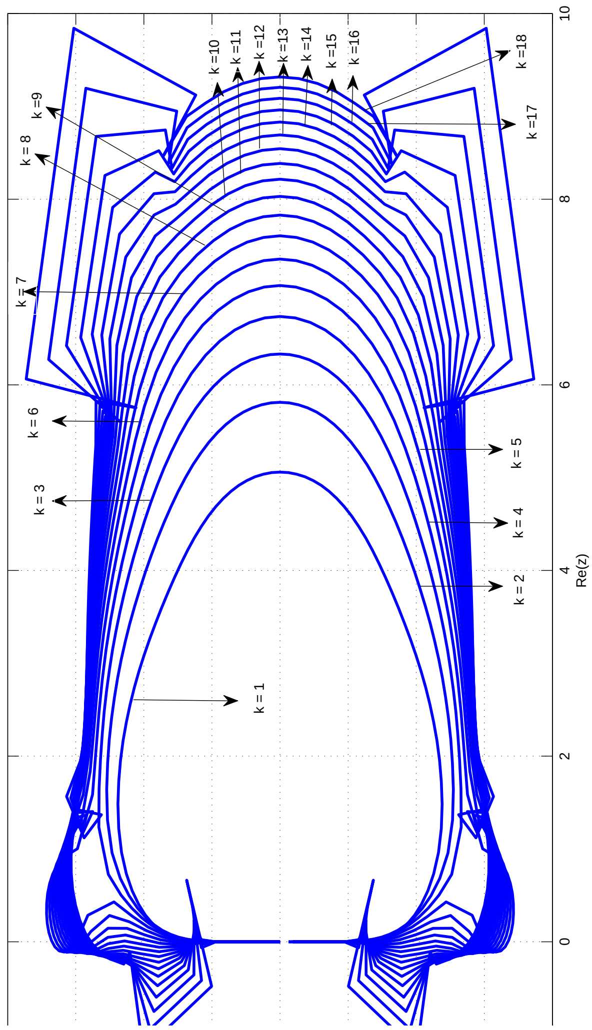

Fig. 1 *The stability regions (exterior of the closed curves) of the hybrid method in (8);* $k \leq 18$.

• The inclusion of the third derivative function (i.e $y'''_{n+k} = f''_{n+k}$) into the hybrid predictor formula in (6) to form formula (8) have accounted for these advantages, see Tables VIII, XII, XIII, XIV, and XV.

V. Numerical examples

Consider some numerical experiments to evaluate the accuracy of the hybrid LMM in (6) and (7). Methods considered for comparison are

• **TDHLMM:** The new proposed third derivative hybrid LMM of order $p = 6$ in (8).

• **TDHLMM:** The third derivative hybrid LMM of order $p = 6$ in [29] and is

$$\begin{cases} y_{n+\frac{1}{2}} = \frac{1}{8}\big(y_n + 7y_{n+1}\big) - \frac{3h}{8}f_{n+1} + \frac{h^2}{16}f'_{n+1}, \quad p = 3, \\ y_{n+1} = y_n + h\left(\frac{1}{10}f_n + \frac{4}{5}f_{n+\frac{1}{2}} + \frac{1}{10}f_{n+1}\right) \\ +\frac{1}{60}h^3 f''_{n+\frac{1}{2}}, \quad p = 6. \end{cases}$$

• **TDBDF:** The third derivative backward differentiation formula in [27],

$$y_{n+4} = \frac{1}{5845}\big(-27y_n + 256y_{n+1} - 1296y_{n+2} + 6912y_{n+3}\big) + \frac{996h}{1169}f_{n+4} - \frac{360h^2}{16}f'_{n+4} + \frac{288h^3}{5845}f''_{n+4}, \quad p = 6.$$

•**TDMM:** The third derivative multistep method of order $p = 6$ in [10],

$$y_{n+3} = y_2 + h\left(\frac{1}{810}f_n - \frac{7}{480}f_{n+1} + \frac{1}{3}f_{n+2} + \frac{8813}{12960}f_{n+3}\right) - \frac{83h^2}{432}f'_{n+3} + \frac{17h^3}{720}f''_{n+3}, \quad p = 6.$$

Tables III, IV, V and VI shows the results of our numerical experiments for the stiff IVPs:

Example 1 A stiff equation

$$\begin{cases} y' = -10^4(y - sin(x)) + cos(x), \\ y(0) = 0, \quad x \in [0, 1], \\ y(x) = sin(x). \end{cases}$$

Example 2 A stiff system of equations

$$\begin{cases} y_1' = -8y_1 + 7y_2, \\ y_2' = 42y_1 - 43y_2, \\ y(0) = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \\ x \in [0, 15], \\ y_1(x) = 2e^{-x} - e^{-50x}, \\ y_2(x) = 2e^{-x} + 6e^{-50x}. \end{cases}$$

Example 3 A stiff system of equations

$$\begin{cases} y_1' = -0.1y_2, \quad y_1(0) = 1, \\ y_2' = -10y_2, \quad y_2(0) = 1, \\ x \in [0, 15], \\ y_1(x) = e^{-0.1x}, \\ y_2(x) = e^{-10x}. \end{cases}$$

Example 4 A stiff system of equations

$$\begin{cases} y_1' = y_2, \quad y_1(0) = 1.01, \\ y_2' = -100y_1 - 101y_2, \quad y_2(0) = -2, \\ x \in [0, 15] \\ y_1(x) = 0.01e^{-100x} + e^{-x}, \\ y_2(x) = e^{-100x} - e^{-10x}. \end{cases}$$

We will integrate these problems using constant step size $h = 0.0001$. The results from the hybrid LMM in (6), (8), TDMM [10] and the TDBDF in [29] are given in Tables I, II, III, IV, V, VI, VII, IX, X, XV, and XIX. Tables I, II shows the absolute error (Error = $|y(x_n) - y_n|$) while Tables III, IV, V, VI, VII, IX, X, XV, XIX depict the maximum error ($\{\|y(x_n){-}y_n\|\}_{n=1}^2$). The $y(x_n)$ and y_n denote the exact and the approximate solutions respectively. The notations $Feval$ and CPU time stand for function evaluations (including derivatives), and time (measured in seconds) respectively.

TABLE I
Results for Example 1

x	Method	Err$(y(x_n) - y_n)$
0.2	TDHLMM(8)	2.1375×10^{-4}
	TDHLMM(6)	1.9670×10^{-4}
	TDBDF[29]	2.2792×10^{-3}
	TDMM[10]	5.1312×10^{-3}
0.4	TDHLMM(8)	2.0088×10^{-4}
	TDHLMM(6)	1.8487×10^{-4}
	TDBDF[29]	4.5544×10^{-3}
	TDMM[10]	9.9471×10^{-3}
0.6	TDHLMM(8)	1.8001×10^{-4}
	TDHLMM(6)	1.6563×10^{-4}
	TDBDF[29]	6.6480×10^{-3}
	TDMM[10]	1.4366×10^{-2}
0.8	TDHLMM(8)	1.5196×10^{-4}
	TDHLMM(6)	1.3981×10^{-4}
	TDBDF[29]	8.4767×10^{-3}
	TDMM[10]	1.8213×10^{-2}
1.0	TDHLMM(8)	1.7860×10^{-4}
	TDHLMM(6)	1.0846×10^{-4}
	TDBDF[29]	9.9673×10^{-3}
	TDMM[10]	1.6691×10^{-2}

The results in Tables I, II shows that the new hybrid method in (8) perform better than the existing TDMM and TDBDF in [10] and [29] respectively in terms of accuracy and computational cost but compares favorably with the hybrid LMM in (6). This shows that there are potential usefulness of the new algorithm (8) when applied to oscillatory stiff problems for a fixed step size.

Again, the small error values recorded in Tables III, IV, V and VI, shows that the proposed hybrid LMM (8) compares with the hybrid LMM in (6) but outperform the TDBDF [29] and TDMM in [10] in terms of accuracy and computational cost. The numerical results given in Tables VII and XIX confirm the outperformance of the proposed TDHLMM in (8) over the existing TDBDF and the TDMM in terms of accuracy and computational cost but compares with the TDHLMM in (6).

VI. Conclusion

In this paper, a family of stable high order third derivative hybrid LMM for the numerical integration of stiff IVPs in ODEs (1) have been presented. The plot of the boundary locus of the roots of the stability polynomials in section IV shows that the high order hybrid formula (8) is A-stable for $k \leq 3$ and $A(\alpha)$-stable for $4 \leq k \leq 18$. At $k \geq 19$ instability sets in. To attest the accuracy of the methods in (6) and (8) we compared their errors (see, Tables III-VI) and found that the hybrid LMM in (8) compares with the TDHLMM in (6) but is more accurate than the TDMM and TDBDF in [10] and [29] respectively on the stiff ODEs solved. Further more the hybrid LMM (8) are shown to contains more stable methods than the method in (6), TDBDF and TDMM, (see, [10], and [29]), and in a way these are advantages which hybrid LMM (8) have over the hybrid LMM (6), TDBDF [29] and TDMM [10].

TABLE II
CONTINUATION OF TABLE I

x	Method	Feval	$CPU - time$
0.2	TDHLMM(8)	7996	1.0755
	TDHLMM(6)	7996	0.4893
	TDBDF[29]	5997	0.3388
	TDMM[10]	11994	0.4718
0.4	TDHLMM(8)	15996	2.2317
	TDHLMM(6)	15996	0.6695
	TDBDF[29]	11997	0.6750
	TDMM[10]	23994	0.7227
0.6	TDHLMM(8)	23996	3.2989
	TDHLMM(6)	23996	0.6336
	TDBDF[29]	17997	0.9341
	TDMM[10]	35994	1.0863
0.8	TDHLMM(8)	31996	3.0464
	TDHLMM(6)	31996	1.2324
	TDBDF[29]	23997	1.2198
	TDMM[10]	47994	1.2566
1.0	TDHLMM(8)	39996	2.5181
	TDHLMM(6)	39996	1.5431
	TDBDF[29]	29997	1.6004
	TDMM[10]	59994	1.6691

TABLE III
RESULTS FOR EXAMPLE 2

x	Method	MaxErr$(y(x_n) - y_n)$
5	TDHLMM(8)	8.7794×10^{-15}
	TDHLMM(6)	8.1410×10^{-15}
	TDBDF[29]	1.3472×10^{-2}
	TDMM[10]	1.3476×10^{-2}
10	TDHLMM(8)	1.1942×10^{-16}
	TDHLMM(6)	1.1159×10^{-16}
	TDBDF[29]	9.0808×10^{-5}
	TDMM[10]	9.0808×10^{-5}
15	TDHLMM(8)	1.2093×10^{-18}
	TDHLMM(6)	1.1269×10^{-18}
	TDBDF[29]	6.1186×10^{-7}
	TDMM[10]	6.1186×10^{-7}

TABLE IV
CONTINUATION OF TABLE III

x	Method	Feval	$CPU - time$
5	TDHLMM(8)	199996	20.5349
	TDHLMM(6)	199996	20.2701
	TDBDF[29]	149997	19.7722
	TDMM[10]	299994	16.2736
10	TDHLMM(8)	399996	44.5201
	TDHLMM(6)	399996	55.0782
	TDBDF[29]	299997	72.8636
	TDMM[10]	599994	57.9373
15	TDHLMM(8)	599996	135.0380
	TDHLMM(6)	599996	140.5443
	TDBDF[29]	499996	129.2100
	TDMM[10]	899994	188.3261

TABLE V
RESULTS FOR EXAMPLE 3

x	Method	MaxErr$(y(x_n) - y_n)$
5	TDHLMM(8)	5.1370×10^{-13}
	TDHLMM(6)	5.1370×10^{-13}
	TDBDF[29]	5.5118×10^{-1}
	TDMM[10]	4.6293×10^{-1}
10	TDHLMM(8)	6.1251×10^{-13}
	TDHLMM(6)	6.1251×10^{-13}
	TDBDF[29]	4.7687×10^{-1}
	TDMM[10]	4.8877×10^{-1}
15	TDHLMM(8)	5.5719×10^{-13}
	TDHLMM(6)	5.5719×10^{-13}
	TDBDF[29]	5.5354×10^{-1}
	TDMM[10]	4.8877×10^{-1}

TABLE VI
CONTINUATION OF TABLE V

x	Method	Feval	$CPU - time$
5	TDHLMM(8)	199996	21.9943
	TDHLMM(6)	199996	23.8431
	TDBDF[29]	149997	18.8889
	TDMM[10]	299994	21.3296
10	TDHLMM(8)	399996	71.0465
	TDHLMM(6)	399996	77.4558
	TDBDF[29]	299997	71.5335
	TDMM[10]	599994	71.7773
15	TDHLMM(8)	599996	208.4335
	TDHLMM(6)	599996	204.9257
	TDBDF[29]	499996	193.5967
	TDMM[10]	899994	185.6705

TABLE VII
RESULTS FOR EXAMPLE 4

x	Method	MaxErr$(y(x_n) - y_n)$
5	TDHLMM(8)	1.4321×10^{-10}
	TDHLMM(6)	1.3479×10^{-10}
	TDBDF[29]	6.7386×10^{-3}
	TDMM[10]	6.7383×10^{-3}
10	TDHLMM(8)	1.9299×10^{-12}
	TDHLMM(6)	1.8164×10^{-12}
	TDBDF[29]	4.5404×10^{-5}
	TDMM[10]	4.5404×10^{-5}
15	TDHLMM(8)	1.9506×10^{-14}
	TDHLMM(6)	1.8358×10^{-14}
	TDBDF[29]	3.0593×10^{-7}
	TDMM[10]	3.0593×10^{-7}

TABLE VIII
CONTINUATION OF XII

k	18
p_1 (7)	21
p_k (8)	22
$C^{(1)}_{p_1+1}$ (7)	$\frac{103385}{2199023255552}$
C_{p_k+1} (8)	$\frac{-28075623577881641}{6434279721678038630400000}$
α	53^0

TABLE IX

THE DISCRETE COEFFICIENTS OF THE HYBRID PREDICTOR IN (7), $v = k - \frac{1}{2}$

k	1	2	3	4	5	6	7	8
$\overline{\alpha}_0$	$\frac{1}{16}$	$\frac{-1}{512}$	$\frac{1}{3456}$	$\frac{-5}{65536}$	$\frac{7}{256000}$	$\frac{-7}{589824}$	$\frac{33}{5619712}$	$\frac{-429}{134217728}$
$\overline{\alpha}_1$	$\frac{15}{16}$	$\frac{3}{32}$	$\frac{-5}{1024}$	$\frac{7}{6912}$	$\frac{-45}{131072}$	$\frac{77}{512000}$	$\frac{-91}{1179648}$	$\frac{495}{11239424}$
$\overline{\alpha}_2$	0	$\frac{465}{512}$	$\frac{15}{128}$	$\frac{-35}{4096}$	$\frac{7}{3072}$	$\frac{-495}{524288}$	$\frac{1001}{2048000}$	$\frac{-455}{1572864}$
$\overline{\alpha}_3$	0	0	$\frac{24535}{27648}$	$\frac{35}{256}$	$\frac{-105}{8192}$	$\frac{77}{18432}$	$\frac{-2145}{1048576}$	$\frac{1001}{819200}$
$\overline{\alpha}_4$	0	0	0	$\frac{1541015}{1769472}$	$\frac{315}{2048}$	$\frac{-1155}{65536}$	$\frac{1001}{147456}$	$\frac{-32175}{8388608}$
$\overline{\alpha}_5$	0	0	0	0	$\frac{42125531}{49152000}$	$\frac{693}{4096}$	$\frac{-3003}{131072}$	$\frac{1001}{98304}$
$\overline{\alpha}_6$	0	0	0	0	0	$\frac{55382019}{65536000}$	$\frac{3003}{16384}$	$\frac{-15015}{524288}$
$\overline{\alpha}_7$	0	0	0	0	0	0	$\frac{112546488769}{134873088000}$	$\frac{6435}{32768}$
$\overline{\alpha}_8$	0	0	0	0	0	0	0	$\frac{949434718947}{1150917017600}$
$\overline{\beta}_k^{(1)}$	$\frac{-7}{16}$	$\frac{-105}{256}$	$\frac{-1805}{4608}$	$\frac{-55685}{147456}$	$\frac{-300013}{819200}$	$\frac{-3505733}{9830400}$	$\frac{-335572523}{963379200}$	$\frac{-1401794537}{4110417920}$
$\overline{\beta}_k^{(2)}$	$\frac{3}{32}$	$\frac{21}{256}$	$\frac{115}{1536}$	$\frac{1715}{24576}$	$\frac{5397}{81920}$	$\frac{20559}{327680}$	$\frac{275847}{4587520}$	$\frac{1700127}{29360128}$
$\overline{\beta}_k^{(3)}$	$\frac{-1}{96}$	$\frac{-1}{128}$	$\frac{-5}{768}$	$\frac{-35}{6144}$	$\frac{-21}{4096}$	$\frac{-77}{16384}$	$\frac{-143}{32768}$	$\frac{-2145}{524288}$

TABLE X

CONTINUATION OF TABLE IX

k	9	10	11	12	13
$\overline{\alpha}_0$	$\frac{715}{382205952}$	$\frac{-2431}{2097152000}$	$\frac{4199}{5582618624}$	$\frac{-29393}{57982058496}$	$\frac{52003}{147438174208}$
$\overline{\alpha}_1$	$\frac{-7293}{268435456}$	$\frac{13585}{764411904}$	$\frac{-51051}{4194304000}$	$\frac{96577}{11165237248}$	$\frac{-734825}{115964116992}$
$\overline{\alpha}_2$	$\frac{8415}{44957696}$	$\frac{-138567}{1073741824}$	$\frac{95095}{1019215872}$	$\frac{-1174173}{16777216000}$	$\frac{2414425}{44660948992}$
$\overline{\alpha}_3$	$\frac{-7735}{9437184}$	$\frac{53295}{89915392}$	$\frac{-969969}{2147483648}$	$\frac{2187185}{6115295232}$	$\frac{-391391}{1342177280}$
$\overline{\alpha}_4$	$\frac{17017}{6553600}$	$\frac{-146965}{75497472}$	$\frac{159885}{102760448}$	$\frac{-22309287}{17179869184}$	$\frac{54679625}{48922361856}$
$\overline{\alpha}_5$	$\frac{-109395}{16777216}$	$\frac{323323}{65536000}$	$\frac{-205751}{50331648}$	$\frac{735471}{205520896}$	$\frac{-111546435}{34359738368}$
$\overline{\alpha}_6$	$\frac{17017}{1179648}$	$\frac{-692835}{67108864}$	$\frac{2263261}{262144000}$	$\frac{-4732273}{603979776}$	$\frac{6128925}{822083584}$
$\overline{\alpha}_7$	$\frac{-36465}{1048576}$	$\frac{46189}{2359296}$	$\frac{-2078505}{134217728}$	$\frac{7436429}{524288000}$	$\frac{-16900975}{1207959552}$
$\overline{\alpha}_8$	$\frac{109395}{524288}$	$\frac{-692835}{16777216}$	$\frac{323323}{12582912}$	$\frac{-47805615}{2147483648}$	$\frac{7436429}{335544320}$
$\overline{\alpha}_9$	$\frac{1369743780532699}{1678037011660800}$	$\frac{230945}{1048576}$	$\frac{-1616615}{33554432}$	$\frac{7436429}{226492416}$	$\frac{-132793375}{4294967296}$
$\overline{\alpha}_{10}$	0	$\frac{5425625817417377}{6712148046643200}$	$\frac{969969}{4194304}$	$\frac{-7436429}{134217728}$	$\frac{37182145}{905969664}$
$\overline{\alpha}_{11}$	0	0	$\frac{27260350600663193}{34033786857455616}$	$\frac{2028117}{8388608}$	$\frac{-16900975}{268435456}$
$\overline{\alpha}_{12}$	0	0	0	$\frac{16216534692600892117}{20420272114473369600}$	$\frac{16900975}{67108864}$
$\overline{\alpha}_{13}$	0	0	0	0	$\frac{2827251526953679405285}{3589067026839839440896}$
$\overline{\beta}_k^{(1)}$	$\frac{-222757759081}{665887703040}$	$\frac{-4376973241927}{13317754060800}$	$\frac{-49619129184677}{153471261081600}$	$\frac{-1172798911730641}{3683310265958400}$	$\frac{-15630801570008773}{49798354795757568}$
$\overline{\beta}_k^{(2)}$	$\frac{29582839}{528482304}$	$\frac{573713569}{10569646080}$	$\frac{584295049}{11072962560}$	$\frac{13661878997}{265751101440}$	$\frac{69339054385}{1381905727488}$
$\overline{\beta}_k^{(3)}$	$\frac{-12155}{3145728}$	$\frac{-46189}{12582912}$	$\frac{-29393}{8388608}$	$\frac{-676039}{201326592}$	$\frac{-1300075}{402653184}$

TABLE XI

ANGLE OF STABILITY AND ERROR CONSTANTS OF THE HYBRID METHOD IN (5) AND (6).

k	1	2	3	4	5	6	7	8	9
p_1 (5)	3	4	5	6	7	8	9	10	11
p_k (6)	6	6	7	8	9	10	11	12	13
$C_{p_1+1}^{(1)}$ (5)	$\frac{-1}{384}$	$\frac{-1}{1280}$	$\frac{-1}{3072}$	$\frac{-1}{6144}$	$\frac{-3}{32768}$	$\frac{-11}{196608}$	$\frac{-143}{3932160}$	$\frac{-13}{524288}$	$\frac{-221}{12582912}$
C_{p_k+1} (6)	$\frac{-1}{806400}$	$\frac{-1}{806400}$	$\frac{-1}{1411200}$	$\frac{-23}{58060800}$	$\frac{-71}{304819200}$	$\frac{-16601}{114960384000}$	$\frac{-16}{650496000}$	$\frac{-2915333}{46030137753600}$	$\frac{-3307771}{74798973849600}$
α	90^0	90^0	90^0	78^0	76^0	75^0	73^0	69^0	67^0

TABLE XII

ANGLE OF STABILITY AND ERROR CONSTANTS OF THE METHOD IN (8).

k	1	2	3	4	5	6	7	8
p_1 (7)	4	5	6	7	8	9	10	11
p_k (8)	6	6	7	8	9	10	11	12
$C_{p_1+1}^{(1)}$ (7)	$\frac{1}{3840}$	$\frac{1}{15360}$	$\frac{1}{43008}$	$\frac{1}{98304}$	$\frac{1}{196608}$	$\frac{11}{3932160}$	$\frac{13}{7864320}$	$\frac{13}{12582912}$
C_{p_k+1} (8)	$\frac{-1}{806400}$	$\frac{-1}{806400}$	$\frac{-1}{1411200}$	$\frac{-23}{58060800}$	$\frac{-71}{304819200}$	$\frac{-16601}{114960384000}$	$\frac{-16}{650496000}$	$\frac{-2915333}{46030137753600}$
α	90^0	90^0	90^0	89^0	88^0	88^0	84^0	84^0

For $k = 18$ in (7), the coefficients are:

$$\overline{\alpha}_0 = -\frac{21607465}{267181325549568}, \quad \overline{\alpha}_1 = \frac{618759225}{337618789203968}, \quad \overline{\alpha}_2 = -\frac{11197545975}{562949953421312},$$

TABLE XIII
CONTINUATION OF XII

k	9	10	11	12	13
p_1 (7)	12	13	14	15	16
p_k (8)	13	14	15	16	17
$C^{(1)}_{p_1+1}$ (7)	$\frac{17}{25165824}$	$\frac{323}{704643072}$	$\frac{323}{1006632960}$	$\frac{7429}{32212254720}$	$\frac{2185}{12884901888}$
C_{p_k+1} (8)	$\frac{-3307771}{74798973849600}$	$\frac{-14810423}{466279317504000}$	$\frac{-133516991}{5711921639424000}$	$\frac{-125975989493}{7170658550415360000}$	$\frac{-6505755121}{483734902210560000}$
α	83^0	78^0	77^0	76^0	73^0

TABLE XIV
CONTINUATION OF XII

k	14	15	16	17
p_1 (7)	17	18	19	20
p_k (8)	18	19	20	21
$C^{(1)}_{p_1+1}$ (7)	$\frac{2185}{17179869184}$	$\frac{3335}{34359738368}$	$\frac{20677}{274877906944}$	$\frac{227447}{3848290697216}$
C_{p_k+1} (8)	$\frac{-872467295136263}{8338041762422980608000 0}$	$\frac{-116819976368363}{1414489227553898496000 0}$	$\frac{-33841462074966061}{5125443318665891020800000}$	$\frac{-1330405884897877}{249153494657369702400000}$
α	69^0	64^0	62^0	57^0

TABLE XV
CONTINUATION OF TABLE IX

k	14	15	16	17
$\overline{\alpha}_0$	$\frac{-185725}{736586891264}$	$\frac{7429}{40265318400}$	$\frac{-9694845}{70368744177664}$	$\frac{17678835}{168809394601984}$
$\overline{\alpha}_1$	$\frac{1404081}{294876348416}$	$\frac{-5386025}{1473173782528}$	$\frac{230299}{80530636800}$	$\frac{-319929885}{140737488355328}$
$\overline{\alpha}_2$	$\frac{-734825}{17179869184}$	$\frac{40718349}{1179505393664}$	$\frac{-166966775}{5892695130112}$	$\frac{1836634525}{11785390260224}$
$\overline{\alpha}_3$	$\frac{21729825}{89321897984}$	$\frac{-21309925}{103079215104}$	$\frac{420756273}{2359010787328}$	$\frac{-1836634525}{11785390260224}$
$\overline{\alpha}_4$	$\frac{-10567557}{10737418240}$	$\frac{630164925}{714575183872}$	$\frac{-660607675}{824633720832}$	$\frac{13884957009}{18872086298624}$
$\overline{\alpha}_5$	$\frac{10935925}{3623878656}$	$\frac{-306459153}{107374182400}$	$\frac{3907022535}{1429150367744}$	$\frac{-1453336885}{549755813888}$
$\overline{\alpha}_6$	$\frac{-1003917915}{137438953472}$	$\frac{317141825}{43486543872}$	$\frac{-3166744581}{429496729600}$	$\frac{3907022535}{519691042816}$
$\overline{\alpha}_7$	$\frac{165480975}{11509170176}$	$\frac{-4159088505}{274877906944}$	$\frac{1404485225}{86973087744}$	$\frac{-14928938739}{858993459200}$
$\overline{\alpha}_8$	$\frac{-50702925}{2147483648}$	$\frac{4798948275}{184146722816}$	$\frac{-128931743655}{4398046511104}$	$\frac{15449337475}{463856467968}$
$\overline{\alpha}_9$	$\frac{22309287}{671088640}$	$\frac{-490128275}{12884901888}$	$\frac{16529710725}{368293445632}$	$\frac{-472749726735}{8796093022208}$
$\overline{\alpha}_{10}$	$\frac{-717084225}{17179869184}$	$\frac{646969323}{13421772800}$	$\frac{-3038795305}{51539607552}$	$\frac{109096090785}{1473173782528}$
$\overline{\alpha}_{11}$	$\frac{3380195}{67108864}$	$\frac{-1890494775}{34359738368}$	$\frac{1823277183}{26843545600}$	$\frac{-3038795305}{34359738368}$
$\overline{\alpha}_{12}$	$\frac{-152108775}{2147483648}$	$\frac{98025655}{1610612736}$	$\frac{-19535112675}{274877906944}$	$\frac{20056049013}{214748364800}$
$\overline{\alpha}_{13}$	$\frac{35102025}{134217728}$	$\frac{-339319575}{4294967296}$	$\frac{233753485}{3221225472}$	$\frac{-49589132175}{549755813888}$
$\overline{\alpha}_{14}$	$\frac{2909581849195234436929}{3721995435241314975744}$	$\frac{145422675}{536870912}$	$\frac{-1502700975}{17179869184}$	$\frac{367326905}{4294967296}$
$\overline{\alpha}_{15}$	0	$\frac{1733078817154584281611 3}{2233197261144788985446 4}$	$\frac{300540195}{1073741824}$	$\frac{-3305942145}{34359738368}$
$\overline{\alpha}_{16}$	0	0	$\frac{4405974680902572131447189}{5716984988530659802742784}$	$\frac{9917826435}{34359738368}$
$\overline{\alpha}_{17}$	0	0	0	$\frac{1303230166670878624419129605}{1702275590827341309750018048}$
$\overline{\beta}^{(1)}_k$	$\frac{-2284668726871879}{7377534043815936}$	$\frac{-13534484747796343}{44265204262895616}$	$\frac{-1711609253590902955}{5665946145650638848}$	$\frac{-29636518134678098123}{99239905217759674368}$
$\overline{\beta}^{(2)}_k$	$\frac{210885500055}{4299262263296}$	$\frac{412887721357}{8598524526592}$	$\frac{25899879459329}{550305569701888}$	$\frac{78549071042067}{1700944488169472}$
$\overline{\beta}^{(3)}_k$	$\frac{-1671525}{536870912}$	$\frac{-3231615}{1073741824}$	$\frac{-100180065}{34359738368}$	$\frac{-194467185}{68719476736}$

$$\overline{\alpha}_3 = \frac{17733023}{128849018880}, \quad \overline{\alpha}_4 = -\frac{9183172625}{13469017440256}, \quad \overline{\alpha}_5 = \frac{97194699063}{37744172597248},$$
$$\overline{\alpha}_6 = -\frac{50866790975}{6597069766656}, \quad \overline{\alpha}_7 = \frac{19535112675}{1039382085632}, \quad \overline{\alpha}_8 = -\frac{104502571173}{2748779069440},$$
$$\overline{\alpha}_9 = \frac{540726811625}{8349416423424}, \quad \overline{\alpha}_{10} = -\frac{3309248087145}{35184372088832}, \quad \overline{\alpha}_{11} = \frac{49589132175}{420906795008},$$
$$\overline{\alpha}_{12} = \frac{-106357835675}{824633720832}, \quad \overline{\alpha}_{13} = \frac{10799411007}{85899345920}, \quad \overline{\alpha}_{14} = -\frac{247945660875}{2199023255552},$$
$$\overline{\alpha}_{15} = \frac{2571288335}{25769803776}, \quad \overline{\alpha}_{16} = -\frac{115707975075}{1099511627776}, \quad \overline{\alpha}_{17} = \frac{20419054425}{68719476736},$$
$$\overline{\alpha}_{18} = \frac{22199375868987874934950384 75}{29181867271325851024286023 68}, \quad \overline{\beta}^{(1)}_{18} = -\frac{1507546312321810105 75}{5103766554056211824 64}, \quad \overline{\beta}^{(2)}_{18} = \frac{44084062779215}{971968278953984},$$
$$\overline{\beta}^{(3)}_{18} = -\frac{756261275}{274877906944}.$$

The following are the coefficients of the output method in (8) for $k = 16$, $k = 17$, and $k = 18$.
(i). For $k = 16$,

$$\beta_0 = \frac{116819976368363}{9059822965698785280000}, \quad \beta_1 = -\frac{13956304648392919}{5191904119274256384 0000}, \quad \beta_2 = \frac{3363331121369}{1256588962320384000},$$
$$\beta_3 = -\frac{644150257220081}{380140938777600000 00}, \quad \beta_4 = \frac{56698655109648907}{7400279683371294720 00}, \quad \beta_5 = -\frac{27489914255435927}{1043106736005734400 00},$$

TABLE XVI
THE DISCRETE COEFFICIENTS OF THE HYBRID LMM IN (8), $v = k - \frac{1}{2}$, α_{k-1}

k	1	2	3	4	5	6	7	8
β_0	$\frac{1}{10}$	0	$\frac{-1}{105000}$	$\frac{1}{288120}$	$\frac{-23}{15746400}$	$\frac{71}{100623600}$	$\frac{-1277}{3372969600}$	$\frac{61}{277200000}$
β_1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{7560}$	$\frac{-1}{21000}$	$\frac{197}{10372320}$	$\frac{-1013}{110224800}$	$\frac{134951}{26564630400}$	$\frac{-27037}{8769720960}$
β_2	0	$\frac{1}{10}$	$\frac{27}{280}$	$\frac{1}{2520}$	$\frac{-251}{1890000}$	$\frac{89}{1481760}$	$\frac{-53593}{1616630400}$	$\frac{109999}{5312926080}$
β_3	0	0	$\frac{83}{840}$	$\frac{11}{120}$	$\frac{323}{408240}$	$\frac{-107}{378000}$	$\frac{28349}{195592320}$	$\frac{-5833}{64665216}$
β_4	0	0	0	$\frac{41}{420}$	$\frac{373}{4320}$	$\frac{67}{51030}$	$\frac{-257447}{498960000}$	$\frac{1657}{5588352}$
β_5	0	0	0	0	$\frac{14599}{151200}$	$\frac{12203}{151200}$	$\frac{117349}{59875200}$	$\frac{-2117323}{2494800000}$
β_6	0	0	0	0	0	$\frac{4819}{50400}$	$\frac{1494589}{19958400}$	$\frac{32693}{11975040}$
β_7	0	0	0	0	0	0	$\frac{1891723}{19958400}$	$\frac{275201}{3991680}$
β_8	0	0	0	0	0	0	0	$\frac{18769}{199584}$
β_9	0	0	0	0	0	0	0	0
β_{10}	0	0	0	0	0	0	0	0
$\beta_v^{(1)}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{95048}{118125}$	$\frac{2188888}{2701125}$	$\frac{24114649792}{29536801875}$	$\frac{923974447552}{1123242379875}$	$\frac{2884271941216384}{3480179306979375}$	$\frac{26980916685133952}{32315950707665625}$
$\beta_v^{(2)}$	0	0	$\frac{-8}{7875}$	$\frac{-64}{25725}$	$\frac{-392992}{93767625}$	$\frac{-1951232}{324168075}$	$\frac{-611554688}{77260057875}$	$\frac{-7064897536}{717414823125}$
$\beta_v^{(3)}$	$\frac{1}{60}$	$\frac{1}{60}$	$\frac{3}{175}$	$\frac{13}{735}$	$\frac{5426}{297675}$	$\frac{1754}{93555}$	$\frac{33008}{1715175}$	$\frac{314032}{15926625}$

TABLE XVII
CONTINUATION OF TABLE XVI

k	9	10	11	12
β_0	$\frac{-2915333}{21415347233280}$	$\frac{3307771}{37372243708800}$	$\frac{-14810423}{247150455552000}$	$\frac{133516991}{3182094807091200}$
β_1	$\frac{16375189}{8172964800000}$	$\frac{-29377501}{21415347233280}$	$\frac{8776683037}{8969338490112000}$	$\frac{-8736730187}{12110372322048000}$
β_2	$\frac{-5603183}{399022303680}$	$\frac{3930029}{389188800000}$	$\frac{-749651389}{98840064153600}$	$\frac{52837290067}{8969338490112000}$
β_3	$\frac{5173}{82223856}$	$\frac{-6278089}{133007434560}$	$\frac{1564845703}{42032390400000}$	$\frac{-620939779}{20395568793600}$
β_4	$\frac{-5190203}{25219434240}$	$\frac{498221}{3139456320}$	$\frac{-12503042077}{95765352883200}$	$\frac{8246055859}{73556683200000}$
β_5	$\frac{19301089}{35597802240}$	$\frac{-1941229}{4670265600}$	$\frac{1323575927}{3767347584000}$	$\frac{-2317483453}{7366565606400}$
β_6	$\frac{751819}{579150000}$	$\frac{32532173}{35597802240}$	$\frac{-17417190937}{22697490816000}$	$\frac{241704167}{342486144000}$
β_7	$\frac{5920261}{1634592960}$	$\frac{-8896331}{4729725000}$	$\frac{3092332951}{2135868134400}$	$\frac{-210041550619}{158882435712000}$
β_8	$\frac{18285433}{290594304}$	$\frac{5049431}{1089728640}$	$\frac{-28479964621}{10897286400000}$	$\frac{65322964829}{29902153881600}$
β_9	$\frac{406999627}{4358914560}$	$\frac{35402761}{622702080}$	$\frac{40703130877}{7061441587200}$	$\frac{-114813631883}{32691859200000}$
β_{10}	0	$\frac{2021767051}{21794572800}$	$\frac{9480425729}{186810624000}$	$\frac{7073607469}{1008777369600}$
β_{11}	0	0	$\frac{9275408641}{100590336000}$	$\frac{58351960643}{1307674368000}$
β_{12}	0	0	0	$\frac{19985454563}{217945728000}$
$\beta_v^{(1)}$	$\frac{25516248402766339291289 6}{30340615599494068305937 5}$	$\frac{5934957111166001015951872}{7006945535250162104728125}$	$\frac{1452322809448761532933213184}{1702687765065789391448934375}$	$\frac{870946527423481017196690045952}{1014099400439777390231768656 25}$
$\beta_v^{(2)}$	$\frac{-4673778456310784}{396213141100651875}$	$\frac{-6621134890442752}{481592403829411875}$	$\frac{-1836932372218502144}{117026954130547085625}$	$\frac{-53443468107108573184}{3030425252765145860625}$
$\beta_v^{(3)}$	$\frac{10435184224}{517408266375}$	$\frac{2045479264}{99300576375}$	$\frac{169001355136}{8043346686375}$	$\frac{193859604352}{9055795919625}$

$$\beta_6 = \frac{1496775485341589891}{2085909359260446720000}, \quad \beta_7 = -\frac{27949043222289253}{1757771700907622400 0}, \quad \beta_8 = \frac{692811308413337}{236532139683840000},$$

$$\beta_9 = -\frac{328813048682276341}{71953076891824128000}, \quad \beta_{10} = \frac{226573220046911879}{36797642873671680000}, \quad \beta_{11} = -\frac{60360531225279253}{82110442775961600000},$$

$$\beta_{12} = \frac{169386146453044553}{20862134720114688000}, \quad \beta_{13} = -\frac{2762410962625387}{304112751022080000}, \quad \beta_{14} = \frac{800235952475431}{60822550204416000},$$

$$\beta_{15} = \frac{42669914521521979}{21287892571545600000}, \quad \beta_{16} = \frac{23947293120218047}{2660986571443200000}, \quad \beta_v^{(3)} = \frac{116653376406728704}{5104507653402782625},$$

$$\beta_v^{(1)} = \frac{9164725701105666110092033581351262492363980 8}{10394644958926775076351856471533470037726562 5}, \quad \beta_v^{(2)} = -\frac{58152753075899509696022393651 2}{2303465752887296052545657980312 5}.$$

(ii). For $k = 17$,

$$\beta_0 = -\frac{33841462074966061}{336610117951199059968 0000}, \quad \beta_1 = \frac{203761802665292947}{93014182447840862208 0000}, \quad \beta_2 = -\frac{434709687227056061}{19036981770672273408 0000},$$

$$\beta_3 = \frac{5703429870536141}{376253508925963468800}, \quad \beta_4 = -\frac{1119711571443371}{155563628220000000000}, \quad \beta_5 = \frac{1766231524933031561}{678358970975702016000 0},$$

$$\beta_6 = -\frac{2312243077099946928 1}{309802700593703116800 00}, \quad \beta_7 = \frac{8478451544842042733}{4867121838274375680000}, \quad \beta_8 = -\frac{2916581095040731}{863466800445849600},$$

$$\beta_9 = \frac{1589662051578954403 1}{28738654971586560000 00}, \quad \beta_{10} = -\frac{133079759712391799}{17131684974243840000}, \quad \beta_{11} = \frac{2017775517276730301}{212024513700679680000},$$

$$\beta_{12} = -\frac{12703020446237718811}{1219340075223029760000}, \quad \beta_{13} = \frac{5080317638037319}{478090587335961600}, \quad \beta_{14} = -\frac{113478084449170457}{10270474486272000000},$$

$$\beta_{15} = \frac{728389418194688563}{48634646874992640000}, \quad \beta_{16} = \frac{434057127890848013}{31222242438266880000}, \quad \beta_{17} = \frac{399790748392466023}{4460320348323840000},$$

$$\beta_v^{(1)} = \frac{7532424483990211083325457297861835258586430308352}{8489826270203443543610378773124961653313169921875}, \quad \beta_v^{(2)} = -\frac{5099522002865932946231524620029132 8}{1881355653670699050916666155420234375},$$

$$\beta_v^{(3)} = \frac{966627914667862788505 6}{416910662591672270896875}.$$

TABLE XVIII
CONTINUATION OF TABLE XVI

k	13	14	15
β_0	$\frac{-125975989493}{4168212048000000000}$	$-\frac{46580217381683}{21608011256832000}$	$\frac{-872467295136263}{519190411927425638400000}$
β_1	$\frac{1773875194229}{3245736703233024000}$	$\frac{-20769720941}{490377888000000000}$	$\frac{934766406174959}{2793397263238213632000}$
β_2	$\frac{-276390955681}{58821808421376000}$	$\frac{12430302082427}{3245736703233024000}$	$\frac{-77433822493471}{243680088960000000000}$
β_3	$\frac{23403970084333}{914872525991424000}$	$\frac{-1291288754557}{58821808421376000}$	$\frac{567047348462591}{29601118733485178880}$
β_4	$\frac{-25994884537207}{262123850135347200}$	$\frac{41007158984711}{457436262995712000}$	$\frac{-15463868541317363}{187759212481032192000}$
β_5	$\frac{29229202465739}{100037089152000000}$	$\frac{-364415814226391}{1310619250676736000}$	$\frac{187093704861246329}{695303119753482240000}$
β_6	$\frac{-2567652445453}{3756948459264000}$	$\frac{13660553544493}{20007417830400000}$	$\frac{-207849628848102757}{298821189154295808000}$
β_7	$\frac{3536032391027}{2689886174976000}$	$\frac{-395687036413}{288996035328000}$	$\frac{233774040489539749}{159659194286592000000}$
β_8	$\frac{-46580217381683}{21608011256832000}$	$\frac{12400278226153}{5379772349952000}$	$\frac{-164347411374625}{63958290570510336}$
β_9	$\frac{8284602996121}{2614302596505600}$	$\frac{-72629260886167}{21608011256832000}$	$\frac{28305655450148723}{7359528574734336000}$
β_{10}	$\frac{-76569518757829}{16672848192000000}$	$\frac{58168828662059}{13071512982528000}$	$\frac{-373154895435176749}{73899398498365440000}$
β_{11}	$\frac{10055280751519}{1200445069824000}$	$\frac{-3915294901369}{666913927680000}$	$\frac{42287891684986807}{6954044906704896000}$
β_{12}	$\frac{10266112107763}{266765571072000}$	$\frac{369729746861}{37513908432000}$	$\frac{-55976436712378019}{7602818775552000000}$
β_{13}	$\frac{2212383672329}{24251415552000}$	$\frac{8626661817271}{266765571072000}$	$\frac{250702550453279}{21896118073589760}$
β_{14}	0	$\frac{3459873829063}{38109367296000}$	$\frac{3716916791651909}{141919283810304000}$
β_{15}	0	0	$\frac{192396084796322917}{2128789257154560000}$
$\beta_v^{(1)}$	$\frac{14640058650460163202362705799233536}{16931838203771289035476256595703125}$	$\frac{6121638099042544020782341946392576}{7033225100028073907043983508984375}$	$\frac{11692091240740871878283098362997227398004736}{13346150504479512157042681012257233021484375}$
$\beta_v^{(2)}$	$\frac{-197955841629713069682688}{10119455754769326356015625}$	$\frac{-30087328117404475457536}{1401155412198829803140625}$	$\frac{-2142466599860288487262486298624}{91683210247595290332852671953125}$
$\beta_v^{(3)}$	$\frac{131774082428672}{6047977989178125}$	$\frac{6184371392768}{279137445654375}$	$\frac{14177427097769996288}{629830379815020759375}$

TABLE XIX
CONTINUATION OF TABLE VII

x	Method	Feval	$CPU-time$
5	TDHLMM(8)	199996	16.7258
	TDHLMM(6)	199996	22.0396
	TDBDF[29]	149997	86.7256
	TDMM[10]	299994	105.9252
10	TDHLMM(8)	399996	86.0563
	TDHLMM(6)	399996	77.3734
	TDBDF[29]	299997	182.7800
	TDMM[10]	599994	136.5398
15	TDHLMM(8)	599996	198.86123
	TDHLMM(6)	599996	210.7978
	TDBDF[29]	449997	221.5977
	TDMM[10]	899994	191.1616

(iii). For $k = 18$,

$$\beta_0 = \frac{1330405884897877}{1673317055675865600000000}, \quad \beta_1 = -\frac{24343072174033957}{134644047180479623987200}, \quad \beta_2 = \frac{28187363165850683}{14309874222744748032000},$$
$$\beta_3 = -\frac{104237283655215653}{7614792708268909363200}, \quad \beta_4 = \frac{6282590727626358889}{92182109686861049856000}, \quad \beta_5 = -\frac{21687070030031627}{837650305800000000000},$$
$$\beta_6 = \frac{1058893586336440273}{1356717941951404032000}, \quad \beta_7 = -\frac{83178007336587302171}{43372378083118436352000}, \quad \beta_8 = \frac{200664850158837391}{51232861455519744000},$$
$$\beta_9 = -\frac{3619605412337437}{536139516616704000}, \quad \beta_{10} = \frac{142991357900432776627}{14369327485793280000000}, \quad \beta_{11} = -\frac{3352161742062834863}{263827948603355136000},$$
$$\beta_{12} = \frac{4236180286887090343}{296834319180951552000}, \quad \beta_{13} = -\frac{3517654171116544631}{243868015044605952000}, \quad \beta_{14} = \frac{914796569714599841}{66932682227034624000},$$
$$\beta_{15} = -\frac{12951640426824528679}{975695076195840000000}, \quad \beta_{16} = \frac{2138075007773440481}{126450081874980864000}, \quad \beta_{17} = \frac{9699147218622149}{1248889697530675200},$$
$$\beta_{18} = \frac{557578398333615829}{6244448487653376000}, \quad \beta_v^{(1)} = \frac{821260448868871592798480340859013392989565878272}{919950215037524842186883148790630519708857421875},$$
$$\beta_v^{(2)} = -\frac{5901262231296627471263170306244608}{203862067735215067436790450277734375}, \quad \beta_v^{(3)} = \frac{1062006198320978354176}{45176077989809688796875}.$$

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