Parameter Estimation for Discretely Observed Vasicek Model Driven by Small Lévy Noises

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Abstract—**This paper is concerned with the parameter estimation problem for Vasicek model driven by small Lévy noises from discrete observations. The explicit formula of the least squares estimators are obtained and the estimation error is given. By using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence, the consistency of the least squares estimators are proved when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously. The simulation is made to verify the effectiveness of the estimators.**

Index Terms—**Least squares estimator, Lévy noises, discrete observations, consistency.**

I. INTRODUCTION

Stochastic differential equations are of great importance for studying random phenomena and are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine and finance [4], [16]. However, part or all of the parameters in stochastic differential equations are always unknown. In the case of stochastic models driven by Brownian motion, the popular methods are maximum likelihood estimation and Bayes estimation when the processes can be observed continuously [7], [9], [20], [21]. When the process is observed only at discrete times, the explicit expression of the likelihood function can not be given. Hence, some approximate likelihood methods have been proposed [1], [3], [5], [14], [15]. The least squares estimation is asymptotically equivalent to the maximum likelihood estimation and has been used to estimate the parameters for stochastic differential equations [12], [17]. But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. Lévy noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations with small Lévy noises. Recently, a number of literatures have been devoted to the parameter estimation for the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation has been investigated by some authors [8], [10], [11].

Vasicek model, which was introduced by Oldrich Alfons Vasicek in 1977([19]), is a mathematical model describing the evolution of interest rates. It is a type of one-factor short rate model as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives, and has also been adapted for credit markets. It is known that parameter estimation for Vasicek model driven by Brownian motion has been well developed([13], [18], [22]). However, some features of the financial processes cannot be captured by the Vasicek model, for example, discontinuous sample paths and heavy tailed properties. Therefore, it is natural to replace the Brownian motion by the Lévy process. Recently, the parameter estimation problems for Vasicek model driven by small Lévy noises have been studied by some authors. For example, Davis([6]) used Malliavin calculus and Monte Carlo estimation to study the estimator of the Vasicek model driven by jump process, Bao([2]) developed the approximate bias of the ordinary least squares estimator of the Vasicek model driven by continuous-time Lévy processes. But, in ([2]), only one parameter has been considered, the explicit expression of the estimation error and the consistency of the estimators have not been discussed in both Davis([6]) and ([2]).

In this paper, we consider the parameter estimation for Vasicek model driven by small Lévy noises from discrete observations. The explicit formula of all parameter estimators and the estimation error are derived and the consistency of the estimators are proved. Firstly, the process is discreted based on Euler-Maruyama scheme, the least squares method is used to obtain the explicit formula of the estimators and the estimation errors are given as well. Then, the consistency of the least squares estimators are proved by applying the Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence when the small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously. Finally, the simulation result is provided to verify the effectiveness of the obtained estimators.

This paper is organized as follows. In Section 2, the Vasicek model driven by small Lévy noises is introduced, the contrast function is given and the explicit formula of the least squares estimators are obtained. In Section 3, the estimation errors are derived and the consistency of the estimators are proved. In Section 4, some simulation results are made. The conclusion is given in Section 5.

II. PROBLEM FORMULATION AND PRELIMINARIES

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geq 0})$. Let $(L_t, t \geq 0)$ be an $(\{\mathscr{F}_t\})$-adapted Lévy noises with decomposition

$$L_t = B_t + \int_0^t \int_{|z|>1} zN(ds, dz) + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(ds, dz), \quad (1)$$

where $(B_t, t \geq 0)$ is a standard Brownian motion, $N(ds, dz)$ is a Poisson random measure independent of $(B_t, t \geq$

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0) with characteristic measure $dt\nu(dz)$, and $\widetilde{N}(ds,dz) = N(ds,dz) - \nu(dz)$ is a martingale measure. We assume that $\nu(dz)$ is a Lévy measure on $\mathbb{R}\backslash 0$ satisfying $\int(|z|^2 \wedge 1)\nu(dz) < \infty$.

In this paper, we study the parameter estimation for Vasicek model driven by small Lévy noises described by the following stochastic differential equation:

$$\begin{cases} dR_t =(a - bR_t)dt + \varepsilon dL_t, \quad t \in [0,1] \\ R_0 =r_0, \end{cases} \tag{2}$$

where a and b are unknown parameters. Without loss of generality, it is assumed that $\varepsilon \in (0,1]$.

Consider the following contrast function

$$Y_{n,\varepsilon}(a,b) = \sum_{i=1}^{n} |R_{t_i} - R_{t_{i-1}} - (a - bR_{t_{i-1}})\Delta t_{i-1}|^2, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the estimators

$$\begin{cases} \widehat{a}_{n,\varepsilon} = \dfrac{n\sum_{i=1}^n (R_{t_i} - R_{t_{i-1}})R_{t_{i-1}}\sum_{i=1}^n R_{t_{i-1}}}{(\sum_{i=1}^n R_{t_{i-1}})^2 - n\sum_{i=1}^n R_{t_{i-1}}^2} \\ \qquad - \dfrac{n\sum_{i=1}^n (R_{t_i} - R_{t_{i-1}})\sum_{i=1}^n R_{t_{i-1}}^2}{(\sum_{i=1}^n R_{t_{i-1}})^2 - n\sum_{i=1}^n R_{t_{i-1}}^2} \\ \widehat{b}_{n,\varepsilon} = \dfrac{n\sum_{i=1}^n (R_{t_i} - R_{t_{i-1}})\sum_{i=1}^n R_{t_{i-1}}}{(\sum_{i=1}^n R_{t_{i-1}})^2 - n\sum_{i=1}^n R_{t_{i-1}}^2} \\ \qquad - \dfrac{n^2\sum_{i=1}^n (R_{t_i} - R_{t_{i-1}})R_{t_{i-1}}}{(\sum_{i=1}^n R_{t_{i-1}})^2 - n\sum_{i=1}^n R_{t_{i-1}}^2} \end{cases} \tag{4}$$

Before giving the main results, we introduce some assumptions below.

Let $R^0 = (R_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameter:

$$dR_t^0 = (a_0 - b_0 R_t^0)dt, \quad R_0^0 = r_0.$$

Assumption 1: a_0 and b_0 are positive true valves of the parameters.

Assumption 2: $\inf_{0\leq t\leq 1}\{R_t\} > 0$.

In the next sections, the consistency of the least squares estimators are derived and the simulation is made to verify the effectiveness of the estimators.

III. Main Result and Proofs

First of all, we introduce some lemmas which are of great importance for proving the main results.

Lemma 1: Let $N_t^{n,\varepsilon} = R_{[nt]/n}$, in which $[nt]$ denotes the integer part of nt. The sequence $\{N_t^{n,\varepsilon}\}$ converges to the deterministic process $\{R_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Proof: Observe that

$$R_t - R_t^0 = b_0 \int_0^t (R_s - R_s^0)ds + \varepsilon L_t. \tag{5}$$

By using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned} & |R_t - R_t^0|^2 \\ \leq\ & 2b_0^2 |\int_0^t (R_s - R_s^0)ds|^2 + 2\varepsilon^2 |L_t|^2 \\ \leq\ & 2b_0^2 t \int_0^t |R_s - R_s^0|^2 ds + 2\varepsilon^2 \sup_{0\leq s\leq t} |L_s|^2 \end{aligned}$$

.

According to the Gronwall's inequality, we obtain

$$|R_t - R_t^0|^2 \leq 2\varepsilon^2 e^{2b_0^2 t^2} \sup_{0\leq s\leq t} |L_s|^2. \tag{6}$$

Then, it follows that

$$\sup_{0\leq t\leq T} |R_t - R_t^0| \leq \sqrt{2}\varepsilon e^{b_0^2 T^2} \sup_{0\leq t\leq T} |L_t|. \tag{7}$$

Therefore, for each $T > 0$, it is easy to check that

$$\sup_{0\leq t\leq T} |R_t - R_t^0| \xrightarrow{P} 0. \tag{8}$$

As $[nt]/n \to t$ when $n \to \infty$, we conclude that the sequence $\{N_t^{n,\varepsilon}\}$ converges to the deterministic process $\{R_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$. The proof is complete. ■

Lemma 2: As $\varepsilon \to 0$ and $n \to \infty$,

$$\sum_{i=1}^n R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 R_s^0 dL_s.$$

Proof: Note that

$$\sum_{i=1}^n R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) = \int_0^1 N_s^{n,\varepsilon} dL_s. \tag{9}$$

Then, it is elementary to see that

$$\begin{aligned} & |\int_0^1 N_s^{n,\varepsilon} dL_s - \int_0^1 R_s^0 dL_s| \\ =\ & |\int_0^1 (N_s^{n,\varepsilon} - R_s^0) dB_s \\ +\ & \int_0^1 \int_{|z|>1} (N_s^{n,\varepsilon} - R_s^0) z N(ds,dz) \\ +\ & \int_0^1 \int_{|z|\leq 1} (N_s^{n,\varepsilon} - R_s^0) z \widetilde{N}(ds,dz)| \\ \leq\ & |\int_0^1 (N_s^{n,\varepsilon} - R_s^0) dB_s| \\ +\ & |\int_0^1 \int_{|z|>1} (N_s^{n,\varepsilon} - R_s^0) z N(ds,dz)| \\ +\ & |\int_0^1 \int_{|z|\leq 1} (N_s^{n,\varepsilon} - R_s^0) z \widetilde{N}(ds,dz)|. \end{aligned}$$

It can be easily to check that

$$\begin{aligned} & |\int_0^1 \int_{|z|>1} (N_s^{n,\varepsilon} - R_s^0) z N(ds,dz)| \\ \leq\ & \int_0^1 \int_{|z|>1} |N_s^{n,\varepsilon} - R_s^0||z| N(ds,dz) \\ \leq\ & \sup_{0\leq s\leq 1} |N_s^{n,\varepsilon} - R_s^0| \int_0^1 \int_{|z|>1} |z| N(ds,dz) \\ \xrightarrow{P}\ & 0, \end{aligned}$$

as $\varepsilon \to 0$ and $n \to \infty$.

By using the Markov inequality and dominated convergence, we have $|\int_0^1 (N_s^{n,\varepsilon} - R_s^0) dB_s| \xrightarrow{P} 0$ and $|\int_0^1 \int_{|z|\leq 1} (N_s^{n,\varepsilon} - R_s^0) z \widetilde{N}(ds,dz)| \xrightarrow{P} 0$.

Thus, combining the previous results, it follows that

$$\sum_{i=1}^n R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 R_s^0 dL_s. \tag{10}$$

The proof is complete. ■

In the following theorem, the consistency in probability of the least squares estimators are proved by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence.

Theorem 1: When $\varepsilon \to 0$ and $n \to \infty$, the least squares estimators $\widehat{a}$ and $\widehat{b}$ are consistent in probability, namely

$$\widehat{a}_{n,\varepsilon} \xrightarrow{P} a_0, \quad \widehat{b}_{n,\varepsilon} \xrightarrow{P} b_0.$$

Proof: By using the Euler-Maruyama scheme, from (2), we have

$$R_{t_i} - R_{t_{i-1}} = (a_0 - b_0 R_{t_{i-1}})\Delta t_{i-1} + \varepsilon(L_{t_i} - L_{t_{i-1}}). \quad (11)$$

Then, it is easy to see that

$$\sum_{i=1}^{n}(R_{t_i} - R_{t_{i-1}}) = a_0 - \frac{1}{n}b_0\sum_{i=1}^{n}R_{t_{i-1}} + \varepsilon\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}}), \quad (12)$$

and

$$\begin{aligned} &\sum_{i=1}^{n}(R_{t_i} - R_{t_{i-1}})R_{t_{i-1}} \quad (13)\\ = &\ \frac{1}{n}a_0\sum_{i=1}^{n}R_{t_{i-1}} - \frac{1}{n}b_0\sum_{i=1}^{n}R_{t_{i-1}}^2\\ + &\ \varepsilon\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})R_{t_{i-1}}. \end{aligned}$$

Substituting (12) and (13) into the expression of $\widehat{a}_{n,\varepsilon}$, it follows that

$$\begin{aligned} &\widehat{a}_{n,\varepsilon} - a_0\\ = &\ \frac{n\varepsilon\sum_{i=1}^{n}R_{t_{i-1}}\sum_{i=1}^{n}R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})}{(\sum_{i=1}^{n}R_{t_{i-1}})^2 - n\sum_{i=1}^{n}R_{t_{i-1}}^2}\\ - &\ \frac{n\varepsilon\sum_{i=1}^{n}R_{t_{i-1}}^2\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})}{(\sum_{i=1}^{n}R_{t_{i-1}})^2 - n\sum_{i=1}^{n}R_{t_{i-1}}^2}\\ = &\ \frac{\varepsilon\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}\sum_{i=1}^{n}R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})}{(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2}\\ - &\ \frac{\varepsilon\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})}{(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2} \end{aligned}$$

Let

$$R_N = \inf_{0 \le t_{i-1} \le 1}\{R_{t_{i-1}}\}, \quad (14)$$

and

$$R_M = \sup_{0 \le t_{i-1} \le 1}\{R_{t_{i-1}}\}. \quad (15)$$

.

We make an assumption that $R_N \neq R_M$.

From (15), it follows that

$$\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}} \le R_M < \infty,$$

and

$$\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2 \le R_M^2 < \infty,$$

Therefore, from Lemma 1 and Lemma 2, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\varepsilon\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}\sum_{i=1}^{n}R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0, \quad (16)$$

and

$$\varepsilon\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0. \quad (17)$$

Under the assumption that $R_N \neq R_M$, it is obviously that

$$(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2 < 0. \quad (18)$$

From (14) and (15), it follows that

$$(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2 > R_N^2 - R_M^2. \quad (19)$$

Then, we have

$$\frac{1}{(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2} < \frac{1}{R_N^2 - R_M^2} < \infty. \quad (20)$$

Combining the previous arguments, when $\varepsilon \to 0$, $n \to \infty$, we have

$$\widehat{a}_{n,\varepsilon} \xrightarrow{P} a_0. \quad (21)$$

Since

$$\begin{aligned} &\widehat{b}_{n,\varepsilon} - b_0\\ = &\ \frac{(\widehat{a}_{n,\varepsilon} - a_0)\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}}{\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2}\\ - &\ \frac{\varepsilon\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})R_{t_{i-1}}}{\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2}. \end{aligned}$$

As $\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2 \ge R_N^2 > 0$, we get that $\frac{1}{\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2} \le \frac{1}{R_N^2} < \infty$.

Together with the results that $\widehat{a}_{n,\varepsilon} - a_0 \xrightarrow{P} 0$ and $\varepsilon\sum_{i=1}^{n}R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0$, it follows that

$$\widehat{b}_{n,\varepsilon} - b_0 \xrightarrow{P} 0, \quad (22)$$

as $\varepsilon \to 0$ and $n \to \infty$.

Therefore, $\widehat{a}_{n,\varepsilon}$ and $\widehat{b}_{n,\varepsilon}$ are consistent in probability. The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\widehat{a}_{n,\varepsilon} - a_0) \xrightarrow{P} \frac{\int_0^1 R_s^0 ds\int_0^1 R_s^0 dL_s - L_1\int_0^1 (R_s^0)^2 ds}{(\int_0^1 R_s^0 ds)^2 - \int_0^1 (R_s^0)^2 ds},$$

and

$$\varepsilon^{-1}(\widehat{b}_{n,\varepsilon} - b_0) \xrightarrow{P} \frac{\int_0^1 R_s^0 dL_s - L_1\int_0^1 R_s^0 ds}{(\int_0^1 R_s^0 ds)^2 - \int_0^1 (R_s^0)^2 ds}.$$

Proof: Since

$$\begin{aligned} &\varepsilon^{-1}(\widehat{a}_{n,\varepsilon} - a_0)\\ = &\ \frac{\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}\sum_{i=1}^{n}R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}})}{(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2}\\ - &\ \frac{\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})}{(\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}}^2}. \end{aligned}$$

According to the Lemma 2, it is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n}R_{t_{i-1}} \xrightarrow{P} \int_0^1 R_s^0 ds,$$

and

$$\frac{1}{n}\sum_{i=1}^{n} R_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (R_s^0)^2 ds.$$

Together with the results that $\sum_{i=1}^{n} R_{t_{i-1}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 R_s^0 dL_s$ and $\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}}) = L_1$, it follows that

$$\varepsilon^{-1}(\widehat{a}_{n,\varepsilon} - a_0) \xrightarrow{P} \frac{\int_0^1 R_s^0 ds \int_0^1 R_s^0 dL_s - L_1 \int_0^1 (R_s^0)^2 ds}{(\int_0^1 R_s^0 ds)^2 - \int_0^1 (R_s^0)^2 ds}.$$

Since

$$\varepsilon^{-1}(\widehat{b}_{n,\varepsilon} - b_0) = \frac{\varepsilon^{-1}(\widehat{a}_{n,\varepsilon} - a_0)\frac{1}{n}\sum_{i=1}^{n} R_{t_{i-1}}}{\frac{1}{n}\sum_{i=1}^{n} R_{t_{i-1}}^2} - \frac{\sum_{i=1}^{n}(L_{t_i} - L_{t_{i-1}})R_{t_{i-1}}}{\frac{1}{n}\sum_{i=1}^{n} R_{t_{i-1}}^2}.$$

From above results, we obtain that

$$\varepsilon^{-1}(\widehat{b}_{n,\varepsilon} - b_0) \xrightarrow{P} \frac{\int_0^1 R_s^0 dL_s - L_1 \int_0^1 R_s^0 ds}{(\int_0^1 R_s^0 ds)^2 - \int_0^1 (R_s^0)^2 ds}.$$

The proof is complete. ■

IV. Simulation

In this experiment, we generate a discrete sample $(R_{t_{i-1}})_{i=1,\ldots,n}$ and compute $\widehat{a}_{n,\varepsilon}$ and $\widehat{b}_{n,\varepsilon}$ from the sample. We let $r_0 = 0.05$. For every given true value of the parameters-(a_0, b_0), the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $\varepsilon = 0.01$, the size is increasing from 500 to 5000. In Table 2, $\varepsilon = 0.001$, the size is increasing from 5000 to 50000. The tables list the value of "$a_0 - LSE$","$b - LSE$"and the absolute errors (AE) of LSE, LSE means least squares estimator.

Two tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.

TABLE I
LSE simulation results of a_0 and b_0

True		Aver		AE	
(a_0, b_0)	Size n	$a_0 - LSE$	$b_0 - LSE$	a_0	b_0
(1,1)	500	0.9651	0.9563	0.0349	0.0437
	1000	0.9732	0.9674	0.0268	0.0326
	5000	0.9868	0.9782	0.0132	0.0218
(2,3)	500	1.9685	2.9637	0.0315	0.0363
	1000	2.0246	3.0212	0.0246	0.0212
	5000	2.0124	3.0108	0.0124	0.0108
(4,5)	500	3.9686	4.9684	0.0314	0.0316
	1000	3.9782	4.9764	0.0218	0.0236
	5000	3.9831	4.9875	0.0169	0.0125

TABLE II
LSE simulation results of a_0 and b_0

True		Aver		AE	
(a_0, b_0)	Size n	$a_0 - LSE$	$b_0 - LSE$	a_0	b_0
(1,1)	5000	0.9963	0.9980	0.0037	0.0020
	10000	0.9985	0.9989	0.0015	0.0011
	50000	0.9996	0.9997	0.0004	0.0003
(2,3)	5000	1.9974	2.9972	0.0026	0.0028
	10000	2.0012	3.0012	0.0012	0.0012
	50000	2.0003	3.0004	0.0003	0.0004
(4,5)	5000	3.9978	4.9972	0.0022	0.0028
	10000	4.0012	4.0016	0.0012	0.0016
	50000	4.0005	4.0006	0.0005	0.0006

V. Conclusion

In this paper, the parameter estimation for Vasicek model driven by small Lévy noises has been studied from discrete observations. The least squares method has been used to obtain the estimators. The explicit formula of the estimation error has been given and the consistency of the least squares estimators has been proved. Further research topics will include the parameter estimation for general nonlinear stochastic differential equations driven by lévy noises.

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