Estimation for Parameters in Partially Observed Linear Stochastic System

Chao Wei

Abstract—**This paper is concerned with the estimation problem for partially observed linear stochastic system. The state estimation equation is provided by applying general filter theory, then the state estimator is obtained. The likelihood function is given and estimators of two parameters are derived. The strong consistency and asymptotic normality of parameter estimators are proved by using ergodic theorem, Borel-Cantelli lemma, central limit theorem for stochastic integrals and the strong law of large numbers for Brownian motion. A numerical simulation example is presented to demonstrate the effectiveness of the estimators.**

Index Terms—**linear stochastic system, general filter theory, state and parameter estimation, strong consistency, asymptotic normality.**

I. INTRODUCTION

Stochastic differential equations have been widely used in many application areas such as biology, chemistry and medical science([4], [5]). Recently, stochastic differential equations have been applied to describe the dynamics of a financial asset, asset portfolio and term structure of interest rates, such as the popular Black-Scholes option pricing model ([6]), Vasicek and Cox-Ingersoll-Ross pricing formulas for the zero coupon bond ([8], [9], [27]), Chan-Karloyi-Longstaff-Sanders model ([10]), Constantinides model ([11]), Ait-Sahalia model([1]). Some parameters in pricing formulas describe the related assets dynamic, however, these parameters are always unknown. In the past few decades, some authors studied the parameter estimation problem for economic models. For example, Yu ([33]) used Gaussian approach to study the parameter estimation for continuous-time short-term interest rates model, Overback([23]), Rossi([25]) and Wei([30]) investigated the parameter estimation problem for Cox-Ingersoll-Ross model by applying the maximum likelihood method, least- square method and Gaussian method respectively. Moreover, some popular methods have been used to estimate the parameters in general nonlinear stochastic differential equation. For instance, Bayes estimation([17], [19]), maximum likelihood estimation([3], [31], [32]) and least-square estimation([2], [20]). Some other methods such as minimum contrast estimation([16]), generalized method of moments([13]), M-estimation([26]) and threshold method([7]) have been discussed as well.

A variety of stochastic systems are defined by stochastic differential equations([24]), and sometimes the states of a stochastic system can not be observed completely. Many authors studied the state estimation problem for stochastic systems by using Kalman filtering or extend Kalman filtering([15], [21], [28], [29]). Furthermore, the parameters and states of a stochastic system are always unknown at the same time. Therefore, the parameter and state estimation needed to be solved simultaneously. In recent years, the parameter estimation problem for partially observed linear stochastic systems has been investigated. For example, Deck([12]) used Kalman filtering and Bayes method to study the linear homogenous stochastic systems. Kan([14]) discussed the linear nonhomogenous stochastic systems based on the methods used in ([12]). Mbalawata [22] applied Kalman filtering and maximum likelihood estimation to investigate the parameter and state estimation for linear stochastic systems.

Although the parameter estimation for partially observed linear stochastic systems has been studied by some authors ([12], [14], [22]), the asymptotic property of the parameter estimator has not been discussed in ([22]), and only one parameter has been considered in ([12], [14]). In this paper, the parameter estimation problem for partially observed continuous-time linear stochastic system with two parameters is investigated and the strong consistency and asymptotic normality of two parameter estimators are analyzed. Firstly, the state estimator is obtained by using the general filter theory. Secondly, the likelihood function is given based on the Girsanov theorem, the parameter estimator and the error of estimation are provided. Then, the strong consistency and asymptotic normality of two parameter estimators are proved by applying ergodic theorem, Borel-Cantelli lemma, central limit theorem for stochastic integrals and the strong law of large numbers for Brownian motion. Finally, the simulation is made to verify the effectiveness of the estimators.

This paper is organized as follows. In Section 2, we derive the state estimation equation and obtain the state estimator. In Section 3, the likelihood function is given, the parameter estimators and the error of estimation are obtained, and the strong consistency and asymptotic normality of two parameter estimators are discussed. In Section 4, a numerical simulation example is provided. The conclusion is given in Section 5.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this paper, the estimation problem for partially observed linear stochastic system with two parameters is investigated. The stochastic system is described as follows:

$$\begin{cases} dX_t =(\alpha-\beta X_t)dt + dW_t, \quad X_0 = 0 \\ dY_t =(\alpha_1-\beta_1 X_t)dt + dV_t, \quad Y_0 = 0, \end{cases} \tag{1}$$

where $\alpha_1 > 0$ and $\beta_1 > 0$ are two unknown parameters, $\alpha > 0$ and $\beta > 0$ are constants, $(W_t, t \geq 0)$ and $(V_t, t \geq 0)$ are independent Wiener processes, $\{Y_t\}$ is observable while $\{X_t\}$ is unobservable.

The likelihood function can not be given directly due to the unobservability of $\{X_t\}$. Therefore, the estimation problem of $\{X_t\}$ should be solved firstly .

This work was supported in part by the key research projects of universities under Grant 18A110006.

Chao Wei is with the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China.(Email: chaowei0806@aliyun.com..

According to (1), one has

$$X_t = \int_0^t (\alpha - \beta X_s) ds + W_t. \quad (2)$$

Let

$$h_t = X_t, \quad H_t = \alpha - \beta X_t, \quad x_t = W_t, \quad (3)$$

it can be checked that

$$h_t = \int_0^t H_s ds + x_t, \quad (4)$$

where x_t is martingale.

Let

$$m_t = \mathbb{E}[X_t | Y_s, 0 \leq s \leq t]. \quad (5)$$

Then, from the general filter theory [18], we have

$$\begin{aligned} m_t &= m_0 + \int_0^t (\alpha - \beta m_s) ds \\ &- \int_0^t \beta_1 (\mathbb{E}[X_s^2 | Y_u, 0 \leq u \leq s] - m_s^2) d\overline{W}_s, \end{aligned}$$

where

$$\overline{W}_t = \int_0^t dY_s - (\alpha_1 - \beta_1 m_s) ds. \quad (6)$$

Let

$$\gamma_t = \mathbb{E}[X_t^2 | Y_s, 0 \leq s \leq t] - m_t^2. \quad (7)$$

Substituting (7) into (6), we obtain

$$dm_t = (\alpha - \beta m_t) dt - \beta_1 \gamma_t d\overline{W}_t, \quad m_0 = 0. \quad (8)$$

According to (1)and Itô lemma, one has

$$X_t^2 = \int_0^t (2X_s(\alpha - \beta X_s) + 1) ds + 2 \int_0^t X_s dW_s. \quad (9)$$

Let

$$\widetilde{h}_t = X_t^2, \quad \widetilde{H}(t) = 2X_t(\alpha - \beta X_t) + 1, \quad \widetilde{x}_t = 2 \int_0^t X_s dW_s. \quad (10)$$

Therefore,

$$\widetilde{h}_t = \int_0^t \widetilde{H}(s) ds + \widetilde{x}_t, \quad (11)$$

where $\widetilde{x}(t)$ is martingale.

Let

$$n_t = \mathbb{E}[X_t^2 | Y_s, 0 \leq s \leq t]. \quad (12)$$

Then, it can be checked that

$$\begin{aligned} n_t &= n_0 + \int_0^t (2\alpha m_s - 2\beta n_s + 1) ds \\ &- \beta_1 \int_0^t (\mathbb{E}[X_s^3 | Y_u, 0 \leq u \leq s] - n_s m_s) d\overline{W}_s. \end{aligned}$$

From (8), we have

$$m_t^2 = m_0^2 + \int_0^t (2m_s(\alpha - \beta m_s) + \beta_1^2 \gamma_s^2) ds - \int_0^t 2\beta_1 m_s \gamma_s d\overline{W}_s. \quad (13)$$

Hence,

$$\begin{aligned} &\gamma_t \\ =\ & n_t - m_t^2 \\ =\ & \gamma_0 + \int_0^t (-2\beta\gamma_s + \sigma^2 - \beta_1^2 \gamma_s^2) ds \\ -\ & \beta_1 \int_0^t (\mathbb{E}[X_s^3 | Y_u, 0 \leq u \leq s] - n_s m_s - 2m_s \gamma_s) d\overline{W}_s. \end{aligned}$$

Since

$$\begin{aligned} & \mathbb{E}[X_s^3 | Y_u, 0 \leq u \leq s] \\ =\ & 3n_s m_s - 2m_s^3 \\ =\ & n_s m_s + 2m_s \gamma_s, \end{aligned}$$

we obtain

$$\gamma_t = \gamma_0 + \int_0^t (-2\beta\gamma_s + 1 - \beta_1^2 \gamma_s^2) ds. \quad (14)$$

Namely,

$$d\gamma_t = -\beta_1^2 \gamma_t^2 dt - 2\beta\gamma_t dt + 1. \quad (15)$$

It is easy to check that

$$\gamma_t \to \gamma = \frac{-\beta + \sqrt{\beta^2 + \beta_1^2}}{\beta_1^2}. \quad (16)$$

We assume that system (8) has reached the steady state, i.e.,

$$dm_t = (\alpha - \beta m_t) dt - \beta_1 \gamma d\overline{W}_t, \quad m_0 = 0. \quad (17)$$

Therefore, on has

$$m_t = \frac{\alpha}{\beta}(1 - e^{-\beta t}) - \beta_1 \gamma e^{-\beta t} \int_0^t e^{\beta s} d\overline{W}_s. \quad (18)$$

From (6, we have

$$dY_t = (\alpha_1 - \beta_1 m_t) dt + d\overline{W}_t. \quad (19)$$

Remark 1: System (8) reaches the steady state means that the Riccatti equation satisfies $\frac{d\gamma_t}{dt} = 0$. Hence, we obtain γ_t=γ. The details can be found in [5].

III. Main Result and Proofs

In the following theorem, the maximum likelihood estimators are obtained and the strong consistency of the maximum likelihood estimators are proved by applying ergodic theorem, maximal inequality for martingale, Borel-Cantelli lemma and the strong law of large numbers for Brownian motion.

Theorem 1: The maximum likelihood estimators $\widehat{\alpha}_1$ and $\widehat{\beta}_1$ have the following expressions:

$$\begin{cases} \widehat{\alpha}_1 = \dfrac{(Y_t - Y_0) \int_0^t m_s^2 ds - \int_0^t m_s dY_s \int_0^t m_s ds}{t \int_0^t m_s^2 ds - (\int_0^t m_s ds)^2} \\ \widehat{\beta}_1 = \dfrac{(Y_t - Y_0) \int_0^t m_s ds - t \int_0^t m_s dY_s}{t \int_0^t m_s^2 ds - (\int_0^t m_s ds)^2}. \end{cases}$$

Moreover, when $T \to \infty$, $\widehat{\alpha}_1$ and $\widehat{\beta}_1$ are strong consistent, i.e.

$$\widehat{\alpha}_1 \overset{a.s.}{\to} \alpha_1, \quad \widehat{\beta}_1 \overset{a.s.}{\to} \beta_1.$$

Proof:

According to (19), the likelihood function has the following expression:

$$\ell_t(\alpha_1,\beta_1)=\int_0^t(\alpha_1-\beta_1 m_s)dY_s-\frac{1}{2}\int_0^t(\alpha_1-\beta_1 m_s)^2 ds. \tag{20}$$

Solving the equation set

$$\begin{cases}\dfrac{\partial \ell_t(\alpha_1,\beta_1)}{\partial \alpha_1}=0\\ \dfrac{\partial \ell_t(\alpha_1,\beta_1)}{\partial \beta_1}=0,\end{cases} \tag{21}$$

we obtain the maximum likelihood estimators

$$\begin{cases}\widehat{\alpha}_1=\dfrac{(Y_t-Y_0)\int_0^t m_s^2 ds-\int_0^t m_s dY_s\int_0^t m_s ds}{t\int_0^t m_s^2 ds-(\int_0^t m_s ds)^2}\\ \widehat{\beta}_1=\dfrac{(Y_t-Y_0)\int_0^t m_s ds-t\int_0^t m_s dY_s}{t\int_0^t m_s^2 ds-(\int_0^t m_s ds)^2}.\end{cases} \tag{22}$$

From (19), one has

$$\int_0^t m_s dY_s=\alpha_1\int_0^t m_s ds-\beta_1\int_0^t m_s^2 ds+\int_0^t m_s d\overline{W}_s, \tag{23}$$

$$Y_t-Y_0=\alpha_1 t-\beta_1\int_0^t m_s ds+\overline{W}_t. \tag{24}$$

Substituting (23) and (24) into the expression of $\widehat{\alpha_1}$, we obtain

$$\begin{aligned}&\widehat{\alpha}_1-\alpha_1\\ =&\ \frac{\overline{W}_t\int_0^t m_s^2 ds-\int_0^t m_s d\overline{W}_s\int_0^t m_s ds}{t\int_0^t m_s^2 ds-(\int_0^t m_s ds)^2}\\ =&\ \frac{\frac{1}{t}\overline{W}_t\frac{1}{t}\int_0^t m_s^2 ds-\frac{1}{t}\int_0^t m_s d\overline{W}_s\frac{1}{t}\int_0^t m_s ds}{\frac{1}{t}\int_0^t m_s^2 ds-(\frac{1}{t}\int_0^t m_s ds)^2}.\end{aligned}$$

By using ergodic theorem, it can be checked that

$$\frac{1}{t}\int_0^t m_s ds\overset{a.s.}{\to}\frac{\alpha}{\beta}, \tag{25}$$

$$\frac{1}{t}\int_0^t m_s^2 ds\overset{a.s.}{\to}\frac{\alpha^2}{\beta^2}+\frac{\beta_1^2\gamma^2}{2\beta}. \tag{26}$$

According to the strong law of large numbers for Brownian motion, we have

$$\frac{\overline{W}_t}{t}\overset{a.s.}{\to}0. \tag{27}$$

From the maximal inequality for martingale and Borel-Cantelli lemma, one has

$$\frac{1}{t}\int_0^t m_s d\overline{W}_s\overset{a.s.}{\to}0. \tag{28}$$

From above results, we obtain

$$\widehat{\alpha}_1\overset{a.s.}{\to}\alpha_1. \tag{29}$$

From (24) and $\frac{\partial \ell_t(\alpha_1,\beta_1)}{\partial \alpha_1}=0$, it is easy to check that

$$\begin{aligned}&\widehat{\beta}_1-\beta_1\\ =&\ \frac{(\widehat{\alpha}_1-\alpha_1)t-\overline{W}_t}{\int_0^t m_s ds}\\ =&\ \frac{(\widehat{\alpha}_1-\alpha_1)-\frac{1}{t}\overline{W}_t}{\frac{1}{t}\int_0^t m_s ds}.\end{aligned}$$

Since $\widehat{\alpha}_1\overset{a.s.}{\to}\alpha_1$, $\frac{\overline{W}_t}{t}\overset{a.s.}{\to}0$, $\frac{1}{t}\int_0^t m_s ds\overset{a.s.}{\to}\frac{\alpha}{\beta}$, one has

$$\widehat{\beta}_1\overset{a.s.}{\to}\beta_1. \tag{30}$$

Therefore, the maximum likelihood estimators $\widehat{\alpha}_1$ and $\widehat{\beta}_1$ are strong consistent. The proof is complete.

■

In the following theorem, the asymptotic normality of the error of estimation is proved by using the central limit theorem for stochastic integrals and the ergodic theorem.

Theorem 2: When $t\to\infty$,

$$\sqrt{t}(\widehat{\alpha_1}-\alpha_1)\overset{d}{\to}N(0,\frac{2\alpha+\beta_1^2\gamma^2}{\beta\beta_1^2\gamma^2}).$$

and

$$\sqrt{t}(\widehat{\beta_1}-\beta_1)\overset{d}{\to}N(0,\frac{2\beta}{\beta_1^2\gamma^2}).$$

Proof: Since

$$\mathbb{E}\int_0^t m_s d\overline{W}_s=0,$$

and

$$\begin{aligned}&\mathbb{E}[\int_0^t m_s d\overline{W}_s]^2\\ =&\ \int_0^t \mathbb{E}[m_s]^2 ds\\ =&\ \frac{\alpha^2}{\beta^2}(1-e^{-\beta t})^2 t+\frac{\beta_1^2\gamma^2[1-e^{-2\beta t}]}{2\beta}t.\end{aligned}$$

According to the central limit theorem for stochastic integrals, it follows that

$$\frac{1}{\sqrt{t}}\int_0^t m_s d\overline{W}_s\overset{d}{\to}N(0,\frac{\alpha^2}{\beta^2}+\frac{\beta_1^2\gamma^2}{2\beta}).$$

As $\frac{1}{\sqrt{t}}\overline{W}_t\overset{d}{\to}N(0,1)$, together with (25) and (26), one has

$$\sqrt{t}(\widehat{\alpha_1}-\alpha_1)\overset{d}{\to}N(0,\frac{2\alpha+\beta_1^2\gamma^2}{\beta\beta_1^2\gamma^2}).$$

By applying the same method, it can be checked that

$$\sqrt{t}(\widehat{\beta_1}-\beta_1)\overset{d}{\to}N(0,\frac{2\beta}{\beta_1^2\gamma^2}).$$

The proof is complete.

■

Remark 2: When the parameters α_1 and β_1 are unknown, we have obtained the estimators and proved the consistency and asymptotic normality of the estimators. However, when α and β are unknown, the likelihood function and the proof will be different. We will give the specific steps below.

According to (17), the likelihood function has the following expression:

$$\ell_t=\int_0^t\frac{(\alpha-\beta m_s)}{\beta_1^2\gamma^2}dm_s-\frac{1}{2}\int_0^t\frac{(\alpha-\beta m_s)^2}{\beta_1^2\gamma^2}ds.$$

Since β is in the expression of γ, it is difficult to obtain the explicit formula of the estimator of β. Thus, we discuss the parameter estimation problem of α and β separately.

When α is unknown, it is easy to check that

$$\widehat{\alpha}=\frac{1}{t}m_t+\beta\frac{1}{t}\int_0^t m_s ds.$$

Since $\frac{1}{t}m_t \stackrel{a.s.}{\to} 0$ and $\frac{1}{t}\int_0^t m_s ds \stackrel{a.s.}{\to} \frac{\alpha}{\beta}$, it follows that

$$\widehat{\alpha} \stackrel{a.s.}{\to} \alpha.$$

When β is unknown, it is difficult to obtain the explicit formula of the estimator. We assume that β_0 is the true parameter. According to (17), it can be checked that

$$\begin{aligned}\ell_t(\beta) &= \int_0^t \frac{(\alpha-\beta m_s)}{\beta_1^2\gamma^2}dm_s - \frac{1}{2}\int_0^t \frac{(\alpha-\beta m_s)^2}{\beta_1^2\gamma^2}ds \\ &= \int_0^t \frac{(\alpha-\beta m_s)}{\beta_1^2\gamma^2}(\alpha-\beta_0 m_s)ds - \beta_1\gamma^* d\overline{W}_s) \\ &- \frac{1}{2}\int_0^t \frac{(\alpha-\beta m_s)^2}{\beta_1^2\gamma^2}ds \\ &= \int_0^t \frac{(\alpha-\beta m_s)(\alpha-\beta_0 m_s)}{\beta_1^2\gamma^2}ds \\ &- \int_0^t \frac{(\alpha-\beta m_s)\gamma^*}{\beta_1\gamma^2}d\overline{W}_s \\ &- \frac{1}{2}\int_0^t \frac{(\alpha-\beta m_s)^2}{\beta_1^2\gamma^2}ds \\ &= \frac{\alpha^2 t}{2\beta_1^2\gamma^2} - \frac{\alpha\beta_0}{\beta_1^2\gamma^2}\int_0^t m_s ds + \frac{2\beta\beta_0-\beta^2}{2\beta_1^2\gamma^2}\int_0^t m_s^2 ds \\ &- \int_0^t \frac{(\alpha-\beta m_s)\gamma^*}{\beta_1\gamma^2}d\overline{W}_s.\end{aligned}$$

It is obviously that $\frac{1}{t}\int_0^t \frac{(\alpha-\beta m_s)\gamma^*}{\beta_1\gamma^2}d\overline{W}_s \stackrel{a.s.}{\to} 0$, together with (25) and (26), it follows that

$$\frac{1}{t}\ell_t(\beta) \stackrel{a.s.}{\to} \frac{(2\alpha^2\beta+\beta_1^2\beta_0\beta\gamma^{*2})(2\beta_0-\beta)}{4\beta_1^2\beta_0^2}.$$

Since $\frac{(2\alpha^2\beta+\beta_1^2\beta_0\beta\gamma^{*2})(2\beta_0-\beta)}{4\beta_1^2\beta_0^2}$ reaches the maximum value when $\beta=\beta_0$, we have that $\widehat{\beta} \stackrel{a.s.}{\to} \beta_0$.

IV. Simulation

In system (1), let $\alpha=8, \beta=7$, step size $\Delta=0.005$. For every given true value of the parameters-α_1, β_1, the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, the size is increasing from 1000 to 10000. This table lists the value of "$\alpha_1 - MLE, \beta_1 - MLE$" and the Absolute Errors (AE). The table illustrates that the Absolute Errors of α_1 and β_1 depend on the size of given value of α_1 and β_1. But under the hypothesis of normal distribution, there is no obvious difference between estimator and true value, estimators-$\widehat{\alpha}_1$ and $\widehat{\beta}_1$ are good.

V. Conclusion

The aim of this paper is to estimate two parameters for partially observed linear stochastic system. The state estimator has been obtained by using general filter theory. The likelihood function has been given and the parameter estimators and the error of estimation have been derived. The strong consistency and asymptotic normality of two parameter estimators are proved by applying ergodic theorem, Borel-Cantelli lemma and the strong law of large numbers for Brownian motion. Further research topics will include the parameter estimation for partially observed nonlinear stochastic systems.

TABLE I
MLE SIMULATION RESULTS OF α_1 AND β_1 $\quad\Delta=0.005, \alpha=8, \beta=7$

True		Aver		AE	
(α_1,β_1)	Size n	$\widehat{\alpha_1}$	$\widehat{\beta_1}$	α_1	β_1
(1.5,1)	1000	1.4494	1.0651	0.0506	0.0651
	5000	1.4568	0.9429	0.0432	0.0571
	10000	1.5050	1.0230	0.0050	0.0230
(2,1)	1000	1.9536	1.0667	0.0464	0.0667
	5000	2.0355	1.0471	0.0355	0.0471
	10000	2.0080	1.0112	0.0080	0.0112
(2.5,1.5)	1000	2.4517	1.4617	0.0483	0.0383
	5000	2.5254	1.5249	0.0254	0.0249
	10000	2.4813	1.4846	0.0187	0.0154
(3,2)	1000	2.9657	1.9756	0.0343	0.0244
	5000	2.9763	1.9834	0.0237	0.0166
	10000	2.9780	1.9783	0.0220	0.0217

References

[1] Y Ait-Sahalia, "Testing continuous-time models of the spot interest rate," *Review of Financial Studies*, vol. 9, no. 2, pp. 385-426, 1996.
[2] J Bai, "Least squares estimation of a shift in linear processes," *The Review of Financial Studies*, 1994, vol. 9, no. 1, pp. 453-472, 1994.
[3] M Barczy, G Pap, "Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes," *Journal of Statistical Planning and Inference*, vol. 140, no. 6, pp. 1576-1593, 2010.
[4] B Øksendal, *Stochastic Differential Equations: an Introduction with Applications*, Springer-Verlag, Berlin, 2003.
[5] J P N Bishwal, *Parameter estimation in stochastic differential equations*, Springer-Verlag, London, 2008.
[6] F Black, M Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637-654, 1973.
[7] J W Cai, P Chen, X Mei, "Range-Based Threshold Spot Volatility Estimation for Jump Diffusion Models," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 43-48, 2017.
[8] J Cox, J Ingersoll, S Ross, "An intertemporal general equilibrium model of asset prices," *Econometrica*, vol. 53, no. 2, pp. 363-384, 1985.
[9] J Cox, J Ingersoll, S Ross, "A theory of the term structure of interest rates," *Econometrica*, vol. 53, no. 2, pp. 385-408, 1985.
[10] K C Chan, "An empirical comparision of alternative models of the short-term interest rate," *The Journal of Finance*, vol. 47, no. 3, pp. 1209-1227, 1992.
[11] G M Constantinides, "A theory of the nominal term structure of interest rates," *Review of Financial Studies*, vol. 5, no. 4, pp. 531-552, 1992.
[12] T Deck, T G Theting, "Robust parameter estimation for stochastic differential equations," *Acta Applicandae Mathematicae*, vol. 84, no. 3, pp. 279-314, 2004.
[13] L P Hansen, "Large sample properties of generalized method of moments estimators," *Econometrica*, vol. 50, no. 4, pp. 1029-1054, 1982.
[14] X Kan, H S Shu, Y Che, "Asymptotic parameter estimation for a class of linear stochastic systems using Kalman-Bucy filtering," *Mathematical Problems in Engineering*, Article ID: 342705, DOI: 10.1155/2012/342705, 2012.
[15] R Kannan, "Orientation estimation based on LKF using differential state equation," *IEEE Sensors Journal*, vol. 15, no. 11, pp. 6156-6163, 2015.
[16] M Kessler, "Estimation of an ergodic diffusion from discrete observations," *Journal of Statistics*, vol. 24, no. 2, pp. 211-229, 1997.
[17] Y A Kutoyants, *Statistical Inference for Ergodie Diffusion Processes*, Springer-Verlag, London, 2004.

[18] R S Liptser, A N Shiryayev, *Statistics of Rondom Processes*, Springer, New York, 1977.

[19] C P Li, H B Hao, "Likelihood and Bayesian Estimation in Stress Strength Model from Generalized Exponential Distribution Containing Outliers," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp. 155-159, 2016.

[20] H Long, Y Shimizu, W Sun, "Least squares estimators for discretely observed stochastic processes driven by small Lévy noises," *Journal of Multivariate Analysis*, vol. 116, no. 1, pp. 422-439, 2013.

[21] X Lu, L Xie, H Zhang, W Wang, "Robust Kalman filtering for discrete-time systems with measurement delay," *IEEE Transactions on Circuits and Systems -II: Express Briefs*, vol. 54, no. 6, pp. 522-526, 2007.

[22] I S Mbalawata, S Särkkä, H Haario, "Parameter estimation in stochastic differential equations with Markov chain Monte Carlo and non-linear Kalman filtering," *Computational and Applied Statistics*, vol. 28, no. 3, pp. 1195-1223, 2013.

[23] L Overback, T Rydén, "Estimation in Cox-Ingersoll-Ross model," *Econometric Theory*, vol. 13, no. 3, pp. 430-461, 1997.

[24] P E Protter, "Stochastic Integration and Differential Equations: S-tochastic Modelling and Applied Probability," 2nd ed, Applications of Mathematics(New York) 21, Springer, Berlin, 2004.

[25] G D Rossi, "Maximum likelihood estimation of the Cox-Ingersoll-Ross model using particle filters," *Computational Economics*, vol. 36, no. 1, pp. 1-16, 2010.

[26] Y Shimizu, "Estimation of parameters for discretely observed diffusion processes with a variety of rates for information," *Annals of the Institute of Statistical Mathematics*, vol. 64, no. 3, pp. 545-575, 2012.

[27] O Vasicek, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, vol. 5, no. 2, pp. 177-188, 1977.

[28] Z Wang, D W C Ho, X Liu, "Variance-constrained filtering for uncertain stochastic systems with missing measurements," *IEEE Trans. on Automatic Control*, vol. 48, no. 7, pp. 1254-1258, 2003.

[29] Z Wang, J Lam, X H Liu, "Filtering for a class of nonlinear discrete-time stochastic systems with state delays," *Journal of Computational and Applied Mathematics*, vol. 201, no. 1, pp. 153-163, 2007.

[30] C Wei, H S Shu, Y R Liu, "Gaussian estimation for discretely observed Cox-Ingersoll-Ross model," *International Journal of General Systems*, vol. 45, no. 5, pp. 561-574, 2016.

[31] C Wei, H S Shu, "Maximum likelihood estimation for the drift parameter in diffusion processes," *Stochastics: An International Journal of Probability and Stochastic Processes*, vol. 88, no. 6, pp. 699-710, 2016.

[32] J H Wen, X J Wang, S H Mao, X P Xiao, "Maximum likelihood estimation of McKeanCVlasov stochastic differential equation and its application," *Applied Mathematics and Computation*, vol. 274, no. 1, pp. 237-246, 2015.

[33] J YU, P C B Phillips, "A Gaussian approach for continuous time models of the short-term interest rate," *Econometrics Journal*, vol. 4, no. 2, pp. 210-224, 2001.