Positive Periodic Solutions in Shifts Delta(+/-) for a Neutral Dynamic Equation on Time Scales

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***Abstract*—In this paper, based on the theory of calculus on time scales, by using Krasnoselskii's fixed point theorem and contraction mapping principle, sufficient conditions are established for the existence of positive periodic solutions in shifts $\delta_{\pm}$ for a neutral functional dynamic equation on time scales of the following form:**

$$x^{\Delta}(t) = -a(t)x(t) + f^{\Delta}(t, x(t)) + b(t)g(t, x(\tau(t))), t \in \mathbb{T},$$

where $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. Finally, some numerical examples are presented to illustrate the feasibility and effectiveness of the results.

***Index Terms*—periodic solution; neutral dynamic equation; shift operator; time scale.**

I. Introduction

A time scale is a nonempty arbitrary closed subset of reals. Stefan Hilger [1] introduced the notion of time scale in 1988 in order to unify the theory of continuous and discrete calculus. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors [2,3] powerfully. Nowadays the theory on time scales has been widely applied to several scientific fields such as biology, heat transfer, stock market, wound healing and epidemic models.

The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are a few results concerning periodic dynamic equations on time scales; see, for example, [4,5]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition. To overcome such difficulties, Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_{\pm}$ which are first defined in [6] and then generalized in [7].

Recently, based on a fixed-point theorem in cones, E. Çetin *et al.* studied the existence of positive periodic solutions in shifts $\delta_{\pm}$ for some nonlinear first-order functional dynamic equation on time scales; see [8,9]. However, to the best of our knowledge, there are few papers published on the existence of positive periodic solutions in shifts $\delta_{\pm}$ for a neutral functional dynamic equation. As we know, neutral functional dynamic equation on time scales plays an important role in applications; see, for example, [10,11].

Motivated by the above, in the present paper, we consider the following neutral functional dynamic equation:

$$x^{\Delta}(t) = -a(t)x(t) + f^{\Delta}(t, x(t)) + b(t)g(t, x(\tau(t))), \quad (1)$$

where $t \in \mathbb{T}, \mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $a, b \in C_{rd}(\mathbb{T}, (0, \infty))$ is Δ-periodic in shifts $\delta_{\pm}$ with period ω and $-a \in \mathcal{R}^+$; $f, g \in C_{rd}(\mathbb{T} \times (0, \infty), (0, \infty))$ is periodic in shifts $\delta_{\pm}$ with period ω with respect to the first variable; $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ is periodic in shifts $\delta_{\pm}$ with period ω.

The main purpose of this paper is to establish some sufficient conditions for the existence of at least one positive periodic solutions in shifts $\delta_{\pm}$ of equation (1) using Krasnoselskii's fixed point theorem and contraction mapping principle.

II. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of all right-dense continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

Manuscript received June 5, 2017; revised September 12, 2017. This work was supported in part by the Key Project of Scientific Research in Colleges and Universities in Henan Province (Nos. 18A110005, 16A110008).

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(Advance online publication: 28 May 2018)

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Lemma 1. [12] *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$*;*
(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*
(iv) $(e_p(t,s))^\Delta = p(t)e_p(t,s)$*;*
(v) $(e_p(c,\cdot))^\Delta = -p(e_p(c,\cdot))^\sigma$ *and* $\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b)$*;*
(vi) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t.$

For more details about the calculus on time scales, see [12].

The following definitions, lemmas about the shift operators and the new periodicity concept for time scales which can be found in [9,13].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_\mathbb{T}$ in $\delta_\pm(s,t)$ is called the shift size. The value $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s,t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 1. [13] *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 2. [13] *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P,\infty)_{\mathbb{T}^*}$ such that $(\omega,t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega,t)$. The smallest number $\omega \in [P,\infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 3. [13] *(Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P,\infty)_{\mathbb{T}^*}$ such that $(\omega,t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega,t)$. The smallest number $\omega \in [P,\infty)_{\mathbb{T}^*}$ is called the period of f.*

Lemma 2. [13] *$\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ and $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 3. [9] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P,\infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

(i) $e_p(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_p(t,t_0)$ *for* $t, t_0 \in \mathbb{T}^*$*;*
(ii) $e_p(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_p(t,\sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ *for* $t, s \in \mathbb{T}^*$.

Lemma 4. [13] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P,\infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^t f(s)\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\Delta s.$$

Lemma 5. [12] *Suppose that r is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\Delta = r(t)y + f(t), \ y(t_0) = y_0$$

is given by

$$y(t) = e_r(t,t_0)y_0 + \int_{t_0}^t e_r(t,\sigma(\tau))f(\tau)\Delta\tau.$$

Set

$$X = \{x : x \in C_{rd}(\mathbb{T},\mathbb{R}), x(\delta_+^\omega(t)) = x(t)\}$$

with the norm $\|x\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} |x(t)|$, then X is a Banach space.

Lemma 6. *The function $x(t) \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of equation (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of*

$$\begin{aligned} x(t) &= f(t,x(t)) + \int_t^{\delta_+^\omega(t)} G(t,s)[-a(s)f(s,x(s)) \\ &\quad + b(s)g(s,x(\tau(s)))]\Delta s, \end{aligned} \tag{2}$$

where

$$G(t,s) = \frac{e_{-a}(t,\sigma(s))}{e_{-a}(t_0,\delta_+^\omega(t_0)) - 1}.$$

Proof: If $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of equation (1). By Lemmas 1 and 5, for $s \in [t, \delta_+^\omega(t)]_\mathbb{T}$, we have

$$\begin{aligned} x(s) &= e_{-a}(s,t)x(t) + \int_t^s e_{-a}(s,\sigma(\theta))[f^\Delta(\theta,x(\theta)) \\ &\quad + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta \\ &= e_{-a}(s,t)x(t) + e_{-a}(s,s)f(s,x(s)) \\ &\quad - e_{-a}(s,t)f(t,x(t)) \\ &\quad - \int_t^s [e_{-a}(s,\theta)]^\Delta f(\theta,x(\theta))\Delta\theta \\ &\quad + \int_t^s e_{-a}(s,\sigma(\theta))b(\theta)g(\theta,x(\tau(\theta)))\Delta\theta \\ &= e_{-a}(s,t)x(t) + f(s,x(s)) - e_{-a}(s,t)f(t,x(t)) \\ &\quad - \int_t^s a(\theta)e_{-a}(s,\sigma(\theta))f(\theta,x(\theta))\Delta\theta \\ &\quad + \int_t^s e_{-a}(s,\sigma(\theta))b(\theta)g(\theta,x(\tau(\theta)))\Delta\theta \end{aligned}$$

$$\begin{aligned} &= e_{-a}(s,t)x(t) + f(s,x(s)) - e_{-a}(s,t)f(t,x(t)) \\ &\quad + \int_t^s e_{-a}(s,\sigma(\theta))[-a(\theta)f(\theta,x(\theta)) \\ &\quad + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta. \end{aligned} \tag{3}$$

Let $s = \delta_+^\omega(t)$ in (3), we have

$$\begin{aligned} & x(\delta_+^\omega(t)) \\ = \;& e_{-a}(\delta_+^\omega(t),t)x(t) + f(\delta_+^\omega(t), x(\delta_+^\omega(t))) \\ & -e_{-a}(\delta_+^\omega(t),t)f(t,x(t)) \\ & + \int_t^{\delta_+^\omega(t)} e_{-a}(\delta_+^\omega(t),\sigma(\theta))[-a(\theta)f(\theta,x(\theta)) \\ & + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta. \end{aligned}$$

Noticing that $f(\delta_+^\omega(t), x(\delta_+^\omega(t))) = f(t,x(t))$, $x(\delta_+^\omega(t)) = x(t)$, then

$$\begin{aligned} & x(t) \\ = \;& e_{-a}(\delta_+^\omega(t),t)x(t) + f(t,x(t)) \\ & -e_{-a}(\delta_+^\omega(t),t)f(t,x(t)) \\ & + \int_t^{\delta_+^\omega(t)} e_{-a}(\delta_+^\omega(t),\sigma(\theta))[-a(\theta)f(\theta,x(\theta)) \\ & + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta. \end{aligned} \tag{4}$$

Multipling on both sides of (4) by $e_{-a}(t,\delta_+^\omega(t))$, then

$$\begin{aligned} & e_{-a}(t,\delta_+^\omega(t))x(t) \\ = \;& x(t) + (e_{-a}(t,\delta_+^\omega(t)) - 1)f(t,x(t)) \\ & + \int_t^{\delta_+^\omega(t)} e_{-a}(t,\sigma(\theta))[-a(\theta)f(\theta,x(\theta)) \\ & + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta, \end{aligned}$$

that is

$$\begin{aligned} x(t) &= f(t,x(t)) + \int_t^{\delta_+^\omega(t)} \frac{e_{-a}(t,\sigma(\theta))}{e_{-a}(t,\delta_+^\omega(t)) - 1} \\ &\quad \times[-a(\theta)f(\theta,x(\theta)) + b(\theta)g(\theta,x(\tau(\theta)))]\Delta\theta. \end{aligned}$$

Since $e_{-a}(t,\delta_+^\omega(t)) = e_{-a}(t_0,\delta_+^\omega(t_0))$, then x satisfies (2).

Let x be an ω-periodic solution in shifts $\delta_\pm$ of (2). By (2) and Lemmas 1 and 2, we have

$$\begin{aligned} & x^\Delta(t) \\ = \;& f^\Delta(t,x(t)) - a(t)x(t) + a(t)f(t,x(t)) \\ & + G(\sigma(t),\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) \\ & \times[-a(\delta_+^\omega(t))f(\delta_+^\omega(t),x(\delta_+^\omega(t))) \\ & + b(\delta_+^\omega(t))g(\delta_+^\omega(t),x(\tau(\delta_+^\omega(t))))] \\ & - G(\sigma(t),t)[-a(t)f(t,x(t)) + b(t)g(t,x(\tau(t)))] \\ = \;& -a(t)x(t) + f^\Delta(t,x(t)) + b(t)g(t,x(\tau(t))). \end{aligned}$$

So, x is an ω-periodic solution in shifts $\delta_\pm$ of equation (1). This completes the proof.

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$0 < \frac{1}{\xi - 1} \le G(t,s) \le \frac{\xi}{\xi - 1}, \ \forall s \in [t, \delta_+^\omega(t)]_{\mathbb{T}},$$

where $\xi = e_{-a}(t_0,\delta_+^\omega(t_0))$. By Lemma 3, we have

$$G(\delta_+^\omega(t),\delta_+^\omega(s)) = G(t,s), \ \forall t \in \mathbb{T}^*, s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}. \tag{5}$$

Define an operator $H : X \to X$ by

$$\begin{aligned} (Hx)(t) &= f(t,x(t)) + \int_t^{\delta_+^\omega(t)} G(t,s)[-a(s)f(s,x(s)) \\ &\quad + b(s)g(s,x(\tau(s)))]\Delta s. \end{aligned} \tag{6}$$

For convenience, we introduce the notation

$$\varphi^u = \sup_{t\in[t_0,\delta_+^\omega(t_0))_{\mathbb{T}}} \varphi(t),$$
$$\varphi^l = \inf_{t\in[t_0,\delta_+^\omega(t_0))_{\mathbb{T}}} \varphi(t),$$
$$\hat{\varphi} = \int_{t_0}^{\delta_+^\omega(t_0)} \varphi(s)\Delta s,$$

where φ is a positive and bounded function.

III. Main Results

In this section, we shall state and prove our main results about the existence of at least one positive periodic solution in shifts $\delta_\pm$ of equation (1).

Lemma 7. [14] *(Krasnoselskii's fixed point theorem) Let M be a closed convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that B and C map M into X, such that*
(1) $x, y \in M$, implies $Bx + Cy \in M$,
(2) C is continuous and $C(M)$ is contained in a compact set,
(3) B is a contraction mapping.
Then there exists $z \in M$ with $z = Bz + Cz$.

In order to apply Krasnoselskii's fixed point theorem, we need to construct two mappings, one is a contraction and the other is compact. Let

$$(Hx)(t) = (Bx)(t) + (Cx)(t),$$

where $B, C : X \to X$ are given by

$$(Bx)(t) = f(t,x(t)), \tag{7}$$

$$\begin{aligned} (Cx)(t) &= \int_t^{\delta_+^\omega(t)} G(t,s)[-a(s)f(s,x(s)) \\ &\quad + b(s)g(s,x(\tau(s)))]\Delta s. \end{aligned} \tag{8}$$

Hereafter, we make the following assumption:

(H_1) There exist positive numbers L_f, L_g such that

$$|f(t,u) - f(t,v)| \le L_f|u - v|,$$
$$|g(t,u) - g(t,v)| \le L_g|u - v|,$$

for all $t \in \mathbb{T}$, $u, v \in X$.

Lemma 8. [15] *The operator B is a contraction provided $L_f < 1$.*

Lemma 9. *The operator C is continuous and the image $C(M)$ is contained in a compact set, where $M = \{x \in X : \|x\| \le \eta\}$, η is a fixed constant.*

Proof: Firstly, we show that C is continuous. Because of the continuity of f and g, for any $\eta > 0$ and $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that

$$\{\phi, \psi \in C(\mathbb{T},(0,\infty)), \|\phi\| \le \eta, \|\psi\| \le \eta, \|\phi - \psi\| < \delta_0\}$$

imply

$$|f(s,\phi(s)) - f(s,\psi(s))| < \frac{(\xi - 1)\varepsilon}{2\xi\hat{a}},$$

and

$$|g(s,\phi(\tau(s)))-g(s,\psi(\tau(s)))|<\frac{(\xi-1)\varepsilon}{2\xi\hat{b}}.$$

Therefore, if $x,y,\in X$ with $\|x\|\le\eta,\|y\|\le\eta,\|x-y\|<\delta_0$, then

$$\begin{aligned}
&|(Cx)(t)-(Cy)(t)|\\
\le&\ \bigg|\int_t^{\delta_+^\omega(t)}G(t,s)[-a(s)f(s,x(s))+b(s)g(s,x(\tau(s)))]\\
&-\int_t^{\delta_+^\omega(t)}G(t,s)[-a(s)f(s,y(s))+g(s,y(\tau(s)))]\bigg|\\
\le&\ \frac{\xi}{\xi-1}\int_{t_0}^{\delta_+^\omega(t_0)}a(s)|f(s,x(s))-f(s,y(s))|\\
&+b(s)|g(s,x(\tau(s)))-g(s,y(\tau(s)))|\Delta s\\
<&\ \varepsilon,
\end{aligned}$$

which yields $\|Cx-Cy\|=\sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}|(Cx)(t)-(Cy)(t)|\le\varepsilon$, that is, C is continuous.

Next, we show that C maps any bounded sets in X into relatively compact sets. We firstly prove that C maps bounded sets into bounded sets. Let $\varepsilon=1$, for any $\eta>0$, there exists a $\delta_0>0$ such that $\{x,y\in X,\|x\|\le\eta,\|y\|\le\eta,\|x-y\|<\delta_0,s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}\}$ imply

$$\begin{aligned}
&|f(s,x(s))-f(s,y(s))|<1,\\
&|g(s,x(\tau(s)))-g(s,y(\tau(s)))|<1.
\end{aligned}$$

Choose a positive integer N such that $\frac{\eta}{N}<\delta_0$. Let $x\in X$ and define $x^k(\cdot)=\frac{x(\cdot)k}{N}$, $k=0,1,2,\cdots,N$. If $\|x\|<\eta$, then

$$\begin{aligned}
\|x^k-x^{k-1}\|&=\sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}\left|\frac{x(\cdot)k}{N}-\frac{x(\cdot)(k-1)}{N}\right|\\
&\le\|x\|\frac{1}{N}\le\frac{\eta}{N}<\delta_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|f(s,x^k(s))-f(s,x^{k-1}(s))|<1,\\
&|g(s,x^k(\tau(s)))-g(s,x^{k-1}(\tau(s)))|<1,
\end{aligned}$$

and these yield

$$\begin{aligned}
f(s,x(s))&=f(s,x^N(s))\\
&\le\sum_{k=1}^N|f(s,x^k(s))-f(s,x^{k-1}(s))|+f(s,0)\\
&<N+\sup_{s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}f(s,0)=:Q. \qquad (9)
\end{aligned}$$

Similarly, we have

$$g(s,x(\tau(s)))<N+\sup_{s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}g(s,0)=:U. \qquad (10)$$

It follows from (8)-(10) that for $t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned}
\|Cx\|&=\sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}|(Cx)(t)|\\
&\le\frac{\xi}{\xi-1}\bigg(\int_{t_0}^{\delta_+^\omega(t_0)}|-a(s)f(s,x(s))\\
&\quad+b(s)g(s,x(\tau(s)))|\Delta s\bigg)\\
&<\frac{\xi}{\xi-1}(\hat{a}Q+\hat{b}U):=D.
\end{aligned}$$

Furthermore, for $t\in\mathbb{T}$, we have

$$\begin{aligned}
&(Cx)^\Delta(t)\\
=&\ -a(t)(Cx)(t)-a(t)f(t,x(t))+b(t)g(t,x(\tau(t))),
\end{aligned}$$

and

$$\begin{aligned}
&\|(Cx)^\Delta\|\\
=&\ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}|-a(t)(Cx)(t)-a(t)f(t,x(t))\\
&+b(t)g(t,x(\tau(t)))|\\
\le&\ a^u(D+Q)+b^uU.
\end{aligned}$$

To sum up, $\{Cx: x\in X,\|x\|\le\eta\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, the functional C is completely continuous, that is $C(M)$ is compact. This completes the proof.

Theorem 1. *Assume that (H_1) holds. Let $\alpha=\|f(\cdot,0)\|,\beta=\|g(\cdot,0)\|$. Let R_0 be a positive constant satisfies*

$$\begin{aligned}
&L_fR_0+\alpha+\frac{\xi}{\xi-1}[\hat{a}(L_fR_0+\alpha)\\
&+\hat{b}(L_gR_0+\beta)]\le R_0. \qquad (11)
\end{aligned}$$

Then equation (1) has a positive periodic solution in shifts $\delta_\pm$ in $M=\{x\in X:\|x\|\le R_0\}$.

Proof: Define $M=\{x\in X:\|x\|\le R_0\}$. By Lemma 9, the mapping C defined by (8) is continuous and $C(M)$ is contained in a compact set. By Lemma 8, the mapping B defined by (7) is a contraction and it is clear that $B:X\to X$.

Next, we show that if $x,y\in M$, we have $\|Bx+Cy\|\le R_0$. In fact, let $x,y\in M$ with $\|x\|,\|y\|\le R_0$. Then

$$\begin{aligned}
&\|Bx+Cy\|\\
\le&\ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}\bigg|f(t,x(t))\\
&+\int_t^{\delta_+^\omega(t)}G(t,s)[-a(s)f(s,y(s))\\
&+b(s)g(s,y(\tau(s)))]\Delta s\bigg|\\
\le&\ \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}\bigg[|f(t,x(t))-f(t,0)|+|f(t,0)|\\
&+\int_t^{\delta_+^\omega(t)}G(t,s)|-a(s)f(s,y(s))\\
&+b(s)g(s,y(\tau(s)))|\Delta s\bigg]\\
\le&\ L_f\|x\|+\alpha+\frac{\xi}{\xi-1}[\hat{a}(L_f\|y\|+\alpha)\\
&+\hat{b}(L_g\|y\|+\beta)]\\
\le&\ L_fR_0+\alpha+\frac{\xi}{\xi-1}[\hat{a}(L_fR_0+\alpha)+\hat{b}(L_gR_0+\beta)]\\
\le&\ R_0.
\end{aligned}$$

Thus $Bx+Cy\in M$. Hence all the conditions of Krasnoselskii's theorem are satisfied, that is, there exists a fixed point $z\in M$, such that $z=Bz+Cz$. By Lemma 7, equation (1) has a positive periodic solution in shifts $\delta_\pm$. The proof is completed.

Theorem 2. *Assume that* (H_1) *holds. If*

$$L_f + \frac{\xi}{\xi - 1}(\hat{a}L_f + \hat{b}L_g) < 1, \tag{12}$$

then equation (1) has a unique positive periodic solution in shifts $\delta_\pm$.

Proof: Let the mapping H be given by (6). For $x, y \in X$, we have

$$\begin{aligned} & \|Hx - Hy\| \\ = & \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t) - (Hy)(t)| \\ = & \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \Bigg| f(t, x(t)) \\ & + \int_t^{\delta_+^\omega(t)} G(t,s)[-a(s)f(s,x(s)) \\ & + b(s)g(s, x(\tau(s)))] \\ & - f(t, y(t)) - \int_t^{\delta_+^\omega(t)} G(t,s)[-a(s)f(s,y(s)) \\ & + b(s)g(s, y(\tau(s)))] \Bigg| \\ \le & L_f\|x-y\| + \frac{\xi}{\xi-1}(\hat{a}L_f\|x-y\| + \hat{b}L_g\|x-y\|) \\ = & [L_f + \frac{\xi}{\xi-1}(\hat{a}L_f + \hat{b}L_g)]\|x-y\|. \end{aligned}$$

This completes the proof by invoking the contraction mapping principle.

IV. Numerical examples

Example 1. Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\omega = 2\pi$, then $\delta_+^\omega(t) = t + 2\pi$. For small positive ε_1 and ε_2, we consider the following perturbed dynamic equation

$$\begin{aligned} x^\Delta(t) &= -x(t) + \varepsilon_1(x^2(t))^\Delta \\ &\quad + \varepsilon_2(1 - 0.5\cos t + x^2(\tau(t))), \end{aligned} \tag{13}$$

that is, $a(t) = b(t) = 1$, $f(t, x(t)) = \varepsilon_1(x^2(t))$, $g(t, x(\tau(t))) = \varepsilon_2(1 - 0.5\cos t + x^2(\tau(t)))$.

It is easy to verify $a(t)$, $b(t)$, $f(t,x)$, $g(t,x)$ satisfy

$$\begin{aligned} & a(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = a(t), b(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b(t), \\ & f(\delta_+^\omega(t), x) = f(t,x), g(\delta_+^\omega(t), x) = g(t,x), \forall t \in \mathbb{T}^*, \end{aligned}$$

and $-a \in \mathcal{R}^+$.

By a direct calculation, we can get

$$\xi = e^{2\pi}, \hat{a} = 2\pi, \hat{b} = 2\pi.$$

Define $M = \{x \in X : \|x\| \le R_0\}$, where R_0 is a positive constant.

For $x, y \in M$, we have

$$|f(t,x) - f(t,y)| \le 2\varepsilon_1 R_0|x - y|,$$

and

$$\begin{aligned} |g(t, x(\cdot)) - g(t, y(\cdot))| &= \varepsilon_2|x^2(\cdot) - y^2(\cdot)| \\ &\le 2\varepsilon_2 R_0|x(\cdot) - y(\cdot)|. \end{aligned}$$

Let $L_f = 2\varepsilon_1 R_0, L_g = 2\varepsilon_2 R_0$, $\alpha = \|f(\cdot, 0)\| = 0, \beta = \|g(\cdot, 0)\| = 1.5\varepsilon_2$. Thus, inequality (11) becomes

$$2\varepsilon_1 R_0^2 + \frac{e^{2\pi}}{e^{2\pi}-1}[4\pi\varepsilon_1 R_0^2 + 2\pi(2\varepsilon_2 R_0^2 + 1.5\varepsilon_2)] \le R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 1, (13) has a positive periodic solution in shifts $\delta_\pm$ with period $\omega = 2\pi$.

Moreover,

$$2\varepsilon_1 R_0 + \frac{e^{2\pi}}{e^{2\pi}-1}(4\pi\varepsilon_1 R_0 + 4\pi\varepsilon_2 R_0) < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 2, (13) has a unique positive periodic solution in shifts $\delta_\pm$ with period $\omega = 2\pi$.

Example 2. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\omega = 4$, $t_0 = 1$, then $\delta_+^\omega(t) = 4t$. For small positive ε_1 and ε_2, we consider the following perturbed dynamic equation

$$x^\Delta(t) = -\frac{1}{5t}x(t) + \varepsilon_1(x^2(t))^\Delta + \varepsilon_2\frac{1}{5t}(1 + x^2(\tau(t))), \tag{14}$$

that is, $a(t) = b(t) = \frac{1}{5t}$, $g(t, x(\tau(t))) = \varepsilon_2(1 + x^2(\tau(t)))$, $f(t, x(t)) = \varepsilon_1(x^2(t))$.

It is easy to verify $a(t)$, $b(t)$, $f(t,x)$, $g(t,x)$ satisfy

$$\begin{aligned} & a(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = a(t), b(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b(t), \\ & f(\delta_+^\omega(t), x) = f(t,x), g(\delta_+^\omega(t), x) = g(t,x), \forall t \in \mathbb{T}^*, \end{aligned}$$

and $-a \in \mathcal{R}^+$.

By a direct calculation, we can get

$$\xi = \Pi_{t\in[1,4)}(1 + \frac{t}{5-t}) = 2.0833, \hat{a} = \hat{b} = \int_1^4 \frac{1}{5t}\Delta t.$$

Define $M = \{x \in X : \|x\| \le R_0\}$, where R_0 is a positive constant.

For $x, y \in M$, we have

$$|f(t,x) - f(t,y)| \le 2\varepsilon_1 R_0|x - y|,$$

and

$$\begin{aligned} & |g(t, x(\cdot)) - g(t, y(\cdot))| \\ = & \varepsilon_2|x^2(\cdot) - y^2(\cdot)| \le 2\varepsilon_2 R_0|x(\cdot) - y(\cdot)|. \end{aligned}$$

Let $L_f = 2\varepsilon_1 R_0, L_g = 2\varepsilon_2 R_0$, $\alpha = \|f(\cdot, 0)\| = 0, \beta = \|g(\cdot, 0)\| = \varepsilon_2$. Thus, inequality (11) becomes

$$2\varepsilon_1 R_0^2 + \frac{\xi}{\xi-1}[\hat{a}(2\varepsilon_1 R_0^2) + \hat{b}(2\varepsilon_2 R_0^2 + \varepsilon_2)] \le R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 1, (13) has a positive periodic solution in shifts $\delta_\pm$ with period $\omega = 4$.

Moreover,

$$2\varepsilon_1 R_0 + \frac{\xi}{\xi-1}(\hat{a}2\varepsilon_1 R_0 + \hat{b}2\varepsilon_2 R_0) < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 2, (13) has a unique positive periodic solution in shifts $\delta_\pm$ with period $\omega = 4$.

V. Conclusion

This paper studied the existence results of positive periodic solutions in shifts $\delta_\pm$ for a neutral functional dynamic equation on time scales. The application of the main results have been done on a perturbed Van Der Pol equation and so on. The examples explain that the results obtained in this paper are more general and extend the previous work. Besides, it is important to notice that the methods used in this paper can be extended to other types of biological models [16-18]. Future work will include biological dynamic systems modeling and analysis on time scales.

References

[1] S. Hilger, *Ein Masskettenkalkl mit Anwendug auf Zentrumsmanning-faltigkeiten [Ph.D. thesis]*, Universität Würzburg, 1988.

[2] B. Jain and A. Sheng, "An exploration of the approximation of derivative functions via finite differences," *Rose-Hulman Undergraduate Math J.*, Vol. 8, pp. 1-19, 2007.

[3] F. Atici and D. Biles, "Lebedinsky A: An application of time scales to economics," *Math. Comput. Model.*, Vol. 43, pp. 718-726, 2006.

[4] E. Kaufmann and Y. Raffoul, "Periodic solutions for a neutral nonlinear dynamical equation on a time scale," *J. Math. Anal. Appl.*, Vol. 319, No. 1, pp. 315-325, 2006.

[5] X. Liu and W. Li, "Periodic solutions for dynamic equations on time scales," *Nonlinear Anal. TMA.*, Vol. 67, No. 5, pp. 1457-1463, 2007.

[6] M. Adıvar, "Function bounds for solutions of Volterra integro dynamic equations on the time scales," *Electron. J. Qual. Theo.*, Vol. 7, pp. 1-22, 2010.

[7] M. Adıvar, Y. Raffoul, "Existence of resolvent for Volterra integral equations on time scales," *B. Aust. Math. Soc.*, Vol. 82, No. 1, pp. 139-155, 2010.

[8] E. Çetin, "Positive periodic solutions in shifts $\delta_{\pm}$ for a nonlinear first-order functional dynamic equation on time scales," *Adv. Differ. Equ.*, Vol. 2014, 2014:76.

[9] E. Çetin and F.S. Topal, "Periodic solutions in shifts $\delta_{\pm}$ for a nonlinear dynamic equation on time scales," *Abstr. Appl. Anal.*, Vol. 2012, Article ID 707319.

[10] M. Hu, L. Wang and Z. Wang, "Positive periodic solutions in shifts $\delta_{\pm}$ for a class of higher-dimensional functional dynamic equations with impulses on time scales," *Abstr. Appl. Anal.*, Vol. 2014, Article ID 509052.

[11] M. Hu and L. Wang, "Multiple periodic solutions in shifts $\delta_{\pm}$ for an impulsive functional dynamic equation on time scales," *Adv. Differ. Equ.*, Vol. 2014, 2014:152.

[12] M. Bohner and A. Peterson, *Dynamic equations on time scales: An Introduction with Applications*. Birkhauser, Boston, 2001.

[13] M. Adıvar, "A new periodicity concept for time scales," *Math. Slovaca*, Vol. 63, No. 4, pp. 817-828, 2013.

[14] D. Smart, *Fixed points Theorem*. Cambridge Univ. Press, Cambridge, UK, 1980.

[15] S. Abbas and D. Bahuguna, "Almost periodic solutions of neutral functional differential equations," *Comput. Math. Appl.*, Vol. 55, No. 11, pp. 2593-2601, 2008.

[16] Y. Yang and T. Zhang, "Dynamics of a harvesting schoener's competition model with time-varying delays and impulsive effects," IAENG International Journal of Applied Mathematics, vol. 45, no. 4, pp263-272, 2015.

[17] Y. Guo, "Existence and exponential stability of pseudo almost periodic solutions for Mackey-Glass equation with time-varying delay," IAENG International Journal of Applied Mathematics, vol. 46, no. 1, pp71-75, 2016.

[18] L. Li and B. Du, "Global asymptotical stability in a stochastic predator-prey system with variable delays," IAENG International Journal of Applied Mathematics, vol. 46, no. 2, pp241-246, 2016.