Numerical Algorithm to Solve Fractional Integro-Differential Equations Based on Legendre Wavelets Method

Na Guo and Yunpeng Ma*

Abstract—**The purpose of this paper is to study the Legendre wavelets for the solution of linear and nonlinear fractional integro-differential equations. The properties of Legendre wavelets together with the fractional order operational matrix of integration are used to reduce the problem to the solution of a system of algebraic equations. Also a reliable approach for convergence of the Legendre wavelets method is discussed. Further some numerical examples are shown to illustrate the accuracy and reliability of the proposed approach and the results have been compared with the exact solution.**

Index Terms—**Legendre wavelets, Block pulse functions, operational matrix, fractional integro-differential equation, convergence, numerical solution**

I. Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. One of the main advantages of the fractional calculus is that the fractional derivatives provide a superior approach for the description of memory and hereditary properties of various materials and processes [1-3]. Differential equations involving fractional order derivatives are used to model a variety of systems, such as the field of viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion equations and so on [4-6]. Since its tremendous applications in several disciplines, a considerable attention has been given to the exact and the numerical solutions of fractional differential equations and fractional integral equations. Even numerical approximation of fractional differentiation of rough functions is not easy as it is an ill-posed problem.

Other than modeling aspects of these differential equations, the solution techniques and their reliability are rather more important. In order to obtain the goal of highly accurate and reliable solutions, several methods have been proposed to solve the fractional order differential and fractional order integral equations. The most commonly used methods are Variational Iteration Method [7], Adomian Decomposition Method [8-9], Generalized Differential Transform Method [10-11] and Wavelet Method [12-13].

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In this paper, the main objective of the present paper is to introduce the Legendre wavelets method to solve the linear and nonlinear fractional integro-differential equations. The method is based on reducing the equation to a system of algebraic equations by expanding the solution as Legendre wavelets with unknown coefficients. The main characteristic of an operational method is to convert a differential equation into an algebraic one. It not only simplifies the problem but also speeds up the computation.

II. Legendre wavelets and their properties

The Legendre wavelets $\psi_{nm}(x)$ are given by the following [14]

$$\psi_{nm}(x)=\begin{cases}\left(\dfrac{2m+1}{2}\right)^{1/2}2^{\frac{k}{2}}P_m(2^kx-\hat{n}), & \dfrac{\hat{n}-1}{2^k}\le x<\dfrac{\hat{n}+1}{2^k}\\ 0, & otherwise\end{cases}\quad (1)$$

where $k=1,2,\ldots$, $\hat{n}=2n-1$, $n=1,2,\ldots,2^{k-1}$, $m=0,1,\ldots,M-1$ is the degree of the Legendre polynomials and M is a fixed positive integer, $P_m(x)$ are the Legendre polynomials of degree m.

A function $f(x)$ defined over $[0,1)$ may be expanded by the Legendre wavelets as

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x)\quad (2)$$

where $c_{nm}=\langle f(x),\psi_{nm}(x)\rangle$, and $\langle , \rangle$ is the inner product of $f(x)$ and $\psi_{nm}(x)$.

If the infinite series in Eq.(2) is truncated, then Eq.(2) can be written as

$$f(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)=C^T\Psi(x)\quad (3)$$

where C and $\Psi(x)$ are $\hat{m}=2^{k-1}M$ column vectors, given by

$$C=[c_{10},\ldots,c_{1M-1},\ldots,c_{2^{k-1}0},\ldots,c_{2^{k-1}M-1}]^T\quad (4)$$

$$\Psi(x)=[\psi_{10},\ldots,\psi_{1M-1},\ldots,\psi_{2^{k-1}0},\ldots,\psi_{2^{k-1}M-1}]^T\quad (5)$$

For simplicity, we write Eq.(5) as

$$f(x)\approx\sum_{i=1}^{\hat{m}}c_i\psi_i(x)=C^T\Psi(x)\quad (6)$$

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where $c_i = c_{nm}$, $\psi_i = \psi_{nm}$. The index i is determined by the relation $i = M(n-1)+m+1$. Therefore, we have

$$C = [c_1, c_2, \dots, c_M, \dots, c_{M(2^{k-1}-1)+1}, \dots, c_{\hat{m}}]^T \quad (7)$$

$$\Psi(x) = [\psi_1, \psi_2, \dots, \psi_M, \dots, \psi_{M(2^{k-1}-1)+1}, \dots, \psi_{\hat{m}}]^T \quad (8)$$

Similarly, an arbitrary function of two variables $u(x, y)$ defined over $[0,1)\times[0,1)$ may be expanded into the Legendre wavelets basis as

$$u(x, y) \approx \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i(x) \psi_j(y) = \Psi^T(x) U \Psi(y) \quad (9)$$

where $U = [u_{ij}]$ and $u_{ij} = \left\langle \psi_i(x), \left\langle u(x,y), \psi_j(y) \right\rangle \right\rangle$.

We investigate the convergence of the Legendre wavelets expansion in the following theorems.

Theorem 2.1 A function $f(x)$, defined on $[0,1]$, is with bounded second derivative, say $\left| f''(x) \right| \le \tilde{M}$, can be expanded as an infinite sum of Legendre wavelets, and the series converges uniformly to the function $f(x)$, that is

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x),$$

where $c_{nm} = \left\langle f(x), \psi_{nm}(x) \right\rangle$ and $\left\langle , \right\rangle$ is the inner product of $f(x)$ and $\psi_{nm}(x)$.

Proof.

$$c_{nm} = \int_0^1 f(x) \psi_{nm}(x) dx$$

$$= \int_{(\hat{n}-1)/2^k}^{(\hat{n}+1)/2^k} f(x) \left(\frac{2m+1}{2} \right)^{1/2} 2^{\frac{k}{2}} P_m(2^k x - \hat{n}) dx,$$

let $2^k x - \hat{n} = t$, then

$$c_{nm} = \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{2m+1}{2} \right)^{1/2} 2^{\frac{k}{2}} P_m(t) \frac{1}{2^k} dt$$

$$= \left(\frac{2m+1}{2^{k+1}} \right)^{1/2} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k} \right) P_m(t) dt$$

$$= \left(\frac{1}{2^{k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k} \right) d(P_{m+1}(t) - P_{m-1}(t))$$

$$= \left(\frac{1}{2^{k+1}(2m+1)} \right)^{1/2} \left(f\left(\frac{\hat{n}+t}{2^k} \right) (P_{m+1}(t) - P_{m-1}(t)) \right) \Bigg|_{-1}^{1}$$

$$- \left(\frac{1}{2^{k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k} \right) \frac{1}{2^k} (P_{m+1}(t) - P_{m-1}(t)) dt$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k} \right) (P_{m+1}(t) - P_{m-1}(t)) dt$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k} \right) d\left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right)$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)} \right)^{1/2} f'\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right) \Bigg|_{-1}^{1}$$

$$+ \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right) dt$$

$$= \left(\frac{1}{2^{5k+1}(2m+1)} \right)^{1/2} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right) dt$$

Consider

$$\left| \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right) dt \right|^2$$

$$= \left| \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k} \right) \frac{(2m-1)P_{m+2}(t) - (4m+2)P_m(t) + (2m+3)P_{m-2}(t)}{(2m+3)(2m-1)} dt \right|^2$$

$$\le \int_{-1}^{1} \left| f''\left(\frac{\hat{n}+t}{2^k} \right) \right|^2 dt \int_{-1}^{1} \left| \frac{(2m-1)P_{m+2}(t) - (4m+2)P_m(t) + (2m+3)P_{m-2}(t)}{(2m+3)(2m-1)} \right|^2 dt$$

$$< 2\tilde{M}^2 \int_{-1}^{1} \frac{(2m-1)^2 P_{m+2}^2(t) + (4m+2)^2 P_m^2(t) + (2m+3)^2 P_{m-2}^2(t)}{(2m+3)^2 (2m-1)^2} dt$$

$$= \frac{2\tilde{M}^2}{(2m+3)^2(2m-1)^2} \left[(2m-1)^2 \frac{2}{2m+5} + (4m+2)^2 \frac{2}{2m+1} + (2m+3)^2 \frac{2}{2m-3} \right]$$

$$< \frac{2\tilde{M}^2}{(2m+3)^2(2m-1)^2} \frac{12(2m+3)^2}{2m-3}$$

$$= \frac{24\tilde{M}^2}{(2m-1)^2(2m-3)}$$

Thus, we obtain

$$\left| \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k} \right) \left(\frac{P_{m+2}(t) - P_m(t)}{2m+3} - \frac{P_m(t) - P_{m-2}(t)}{2m-1} \right) dt \right| < \frac{\sqrt{24}\tilde{M}}{(2m-1)(2m-3)^{1/2}}.$$

Therefore, we have

$$|c_{nm}| < \frac{\sqrt{12}\tilde{M}}{2^{\frac{5k}{2}}} \frac{1}{(2m+1)^{\frac{1}{2}}} \frac{1}{(2m-1)(2m-3)^{\frac{1}{2}}}$$

$$< \frac{\sqrt{12}}{2^{\frac{5k}{2}}} \frac{1}{(2m-3)^2}$$

$$< \frac{\sqrt{12}}{(2n)^{\frac{5}{2}}} \frac{1}{(2m-3)^2}.$$

Hence, the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}$ is absolute convergent, it follows that $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x)$ converges to the functions $f(x)$ uniformly.

Theorem 2.2 If a continuous function $u(x, y)$ defined on $[0,1)\times[0,1)$ has bounded mixed fourth partial derivative $\left| \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} \right| \le \hat{M}$, then the Legendre wavelets expansion of $u(x, y)$ converges uniformly to it.

Proof. Let $u(x, y)$ be a function defined on $[0,1]\times[0,1]$ and $\left| \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} \right| \le \hat{M}$, where $\hat{M}$ is a positive constant. The Legendre wavelet coefficients of function $u(x, y)$ are defined as

$$u_{ij} = \int_0^1 \int_0^1 u(x,y) \psi_i(x) \psi_j(y) dx dy$$

$$= \int_0^1 \int_{I_{nk}} u(x,y) \left(\frac{2m+1}{2} \right)^{1/2} 2^{k/2} P_m(2^k x - \hat{n}) \psi_j(y) dx dy,$$

by change of $2^k x - \hat{n} = t$, and $dx = \frac{1}{2^k} dt$, we get

$$u_{ij}=\left(\frac{2m+1}{2}\right)^{1/2}2^{-k/2}\int_0^1\psi_j(y)\int_{-1}^1u\left(\frac{\hat{n}+t}{2^k},y\right)P_m(t)dtdy$$
$$=\left(\frac{1}{2^{k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\int_{-1}^1u\left(\frac{\hat{n}+t}{2^k},y\right)d(P_{m+1}(t)-P_{m-1}(t))dy$$
$$=\left(\frac{1}{2^{k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\left[u\left(\frac{\hat{n}+t}{2^k},y\right)(P_{m+1}(t)-P_{m-1}(t))\Big|_{-1}^1\right]dy$$
$$-\left(\frac{1}{2^{k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\int_{-1}^1\frac{\partial u\left(\frac{\hat{n}+t}{2^k},y\right)}{\partial t}\frac{1}{2^k}(P_{m+1}(t)-P_{m-1}(t))dtdy$$
$$=-\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\int_{-1}^1\frac{\partial u\left(\frac{\hat{n}+t}{2^k},y\right)}{\partial t}(P_{m+1}(t)-P_{m-1}(t))dtdy$$
$$=-\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\int_{-1}^1\frac{\partial u\left(\frac{\hat{n}+t}{2^k},y\right)}{\partial t}d\left(\frac{P_{m+2}(t)-P_m(t)}{2m+3}-\frac{P_m(t)-P_{m-2}(t)}{2m-1}\right)dy$$
$$=\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{1/2}\int_0^1\psi_j(y)\int_{-1}^1\frac{\partial^2 u\left(\frac{\hat{n}+t}{2^k},y\right)}{\partial t^2}\left(\frac{P_{m+2}(t)-P_m(t)}{2m+3}-\frac{P_m(t)-P_{m-2}(t)}{2m-1}\right)dtdy$$

Now, let

$\tau_m(t)=(2m-1)P_{m+2}(t)-2(2m+1)P_m(t)+(2m+3)P_{m-2}(t)$,

then we have

$$u_{ij}=\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{1/2}\times$$
$$\frac{1}{(2m-1)(2m+3)}\int_0^1\psi_j(y)\int_{-1}^1\frac{\partial^2 u\left(\frac{\hat{n}+t}{2^k},y\right)}{\partial t^2}\tau_m(t)dtdy.$$

By solving this equation, we have

$$u_{ij}=A(k,m)\int_{-1}^1\int_{-1}^1\frac{\partial^4 u\left(\frac{\hat{n}+t}{2^k},\frac{\hat{n}+s}{2^k}\right)}{\partial t^2\partial s^2}\tau_m(t)\tau_m(s)dtds,$$

where $A(k,m)=\frac{1}{2^{5k+1}(2m+1)}\frac{1}{(2m-1)^2(2m+3)^2}$.

So we have

$$|u_{ij}|\le A(k,m)\int_{-1}^1\int_{-1}^1\left|\frac{\partial^4 u\left(\frac{\hat{n}+t}{2^k},\frac{\hat{n}+s}{2^k}\right)}{\partial t^2\partial s^2}\right||\tau_m(t)||\tau_m(s)|dtds\cdot$$

Thus, we have

$$|u_{ij}|\le A(k,m)\frac{24\hat{M}(2m+3)^2}{2m-3}$$
$$\le\frac{1}{2^{5k}(2m+1)}\frac{1}{(2m-3)(2m-1)^2}.$$
$$\le\frac{12\hat{M}}{(2n)^5(2m-3)^4}$$

This means that the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{ij}$ is absolutely convergence.

III. Operational matrix of the integration for Legendre wavelets

A. Fractional calculus

Before we introduce the Legendre wavelets operational matrix of the fractional integration, we first review some basic definitions of fractional calculus, which have been given in [15].

Definition 1. The Riemann-Liouville fractional integral operator J^α of order α is given by

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}f(\tau)d\tau,\quad \alpha>0 \tag{10}$$

$$J^0 f(x)=f(x) \tag{11}$$

Definition 2. The Caputo definition of fractional differential operator is given by

$$D_*^\alpha f(x)=\begin{cases}\dfrac{d^r f(x)}{dx^r}, & \alpha=r\in N;\\ \dfrac{1}{\Gamma(r-\alpha)}\displaystyle\int_0^x\frac{f^{(r)}(\tau)}{(x-\tau)^{\alpha-r+1}}d\tau, & 0\le r-1<\alpha<r.\end{cases} \tag{12}$$

The Caputo fractional derivatives of order α is also defined as $D_*^\alpha f(x)=J^{r-\alpha}D^r f(x)$. The relation between the Riemann-Liouville operator and Caputo operator are :

$$D_*^\alpha J^\alpha f(x)=f(x) \tag{13}$$

$$J^\alpha D_*^\alpha f(x)=f(x)-\sum_{k=0}^{r-1}f^{(k)}(0^+)\frac{x^k}{k!},\qquad x>0 \tag{14}$$

B. Fractional order Legendre wavelets operational matrix of integration.

In this part, we may simply introduce the operational matrix of fractional integration of Legendre wavelets, more detailed introduction can be found in the Ref. [14].

Apart from the Legendre wavelets, we consider another basis set of block pulse functions. The set of these functions, over the interval $[0,1)$, is defined as [16]

$$b_i(x)=\begin{cases}1, & ih\le x<(i+1)h\\ 0, & otherwise,\end{cases}\qquad i=0,1,2,\ldots,\hat{m}-1 \tag{15}$$

with a positive integer value for $\hat{m}$ and $h=\frac{1}{\hat{m}}$.

The following properties of block pulse functions will be used in this paper

$$b_i(x)b_j(x)=\begin{cases}0, & i\ne j\\ b_i(x), & i=j\end{cases} \tag{16}$$

$$\int_0^1 b_i(x)b_j(x)dx=\begin{cases}0, & i\ne j\\ \dfrac{1}{\hat{m}}, & i=j\end{cases} \tag{17}$$

Let $B(x)=[b_0(x),b_1(x),\ldots,b_{\hat{m}-1}(x)]^T$. We suppose

$$J^\alpha\left(B(x)\right)\approx F^\alpha B(x) \tag{18}$$

where F^α is called the block pulse operational matrix of fractional integration [16], here

$$F^\alpha=h^\alpha\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix}1 & \xi_1 & \xi_2 & \cdots & \xi_{\hat{m}-1}\\ 0 & 1 & \xi_1 & \cdots & \xi_{\hat{m}-2}\\ 0 & 0 & 1 & \cdots & \xi_{\hat{m}-3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}\text{ and}$$

$$\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1},k=1,2,\ldots,\hat{m}-1.$$

There is a relation between the block pulse functions and Legendre wavelets, namely

$$\Psi(x) = \Phi B(x) \tag{19}$$

where $\Phi = [\Psi(x_0), \Psi(x_1), \ldots, \Psi_{\hat{m}-1}]$, $x_i = \dfrac{i}{\hat{m}}$, $i = 0,1,\ldots,\hat{m}-1$.

If J^{α} is fractional integration operator of Legendre wavelets, we can get:

$$J^{\alpha}\Psi(x) \approx P^{\alpha}\Psi(x) \tag{20}$$

where P^{α} is called the Legendre wavelets operational matrix of fractional integration. Using Eq.(18) and Eq.(19), we have

$$J^{\alpha}\Psi(x) \approx J^{\alpha}\Phi B(t) = \Phi J^{\alpha} B(t) \approx \Phi F^{\alpha} B(t) \tag{21}$$

From Eq.(20) and Eq.(21), we can obtain

$$P^{\alpha}\Psi(x) = P^{\alpha}\Phi B(x) = \Phi F^{\alpha} B(x) \tag{22}$$

Then, the matrix P^{α} is given by

$$P^{\alpha} = \Phi F^{\alpha} \Phi^{-1} \tag{23}$$

IV. THE ALGORITHM FOR FINDING NUMERICAL SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

A. Linear multi-order fractional integro-differential equations

Consider the linear multi-order fractional integro-differential equations

$$\sum_{i=1}^{r} a_i(x) D_*^{\alpha_i} y(x) = D_*^{\alpha} y(x) + \lambda_1 \int_0^x k_1(x,t) y(t) dt + \lambda_2 \int_0^1 k_2(x,t) y(t) dt + f(x) \tag{24}$$

subject to initial conditions

$$y^{(s)}(0) = 0, \quad s = 0,1,\ldots,\lceil \alpha \rceil - 1 \tag{25}$$

where $\alpha > \alpha_1 > \alpha_2 > \cdots \alpha_r$, D_*^{α} denote the Caputo fractional order derivative of order α, $a_i(x)$ is known function, $\lceil \alpha \rceil$ is the ceiling function, $f(x)$ is input term and $y(x)$ is the output response. $k_1(x,t), k_2(x,t)$ are given functions. λ_1, λ_2 are real constants.

Now we approximate $D_*^{\alpha} y(x), k_1(x,t), k_2(x,t)$ and $f(x)$ in terms of Legendre wavelets as follows

$$\begin{aligned} &D_*^{\alpha} y(x) \approx C^T \Psi(x), \\ &k_1(x,t) \approx \Psi^T(x) K_1 \Psi(t), \\ &k_2(x,t) \approx \Psi^T(x) K_2 \Psi(t) \end{aligned} \tag{26}$$

and

$$f(x) \approx F^T \Psi(x) \tag{27}$$

where $K_1 = [k_{ij}^1]_{\hat{m}\times\hat{m}}, K_2 = [k_{ij}^2]_{\hat{m}\times\hat{m}}$ and $F = [f_1, f_2, \ldots f_{\hat{m}}]^T$. A similar approximation scheme is follow for variable coefficient $a_i(x)$ as well.

Let

$$a_i(x) \approx \left(A_{\hat{m}}^i\right)^T \Psi(x) \tag{28}$$

where $A_{\hat{m}}^i$ is known $\hat{m}\times 1$ column vector.

Now using Eq.(23) and (26) together with above approximation of $a_i(x)$, we obtain

$$\begin{aligned} D_*^{\alpha_i} y(x) &= J^{\alpha-\alpha_i}\left(D_*^{\alpha} y(x)\right) \\ &\approx J^{\alpha-\alpha_i}\left(C^T \Psi(x)\right) \\ &= C^T P^{\alpha-\alpha_i} \Psi(x) \end{aligned} \tag{29}$$

and

$$y(x) \approx C^T P^{\alpha} \Psi(x) = C^T P^{\alpha} \Phi B(x) \tag{30}$$

Let $E = [e_0, e_1, \ldots, e_{\hat{m}-1}] = C^T P^{\alpha} \Phi$, then

$$\begin{aligned} &\int_0^x k_1(x,t) y(t) dt \\ &= \int_0^x \Psi^T(x) K_1 \Psi(t) \Psi^T(t) [C^T P^{\alpha}]^T dt \\ &= \Psi^T(x) K_1 \Phi \int_0^x B(t) B^T(t) E^T dt \\ &= \Psi^T(x) K_1 \Phi \int_0^x diag(E) B(t) dt \\ &= B^T(x) \Phi^T K_1 \Phi diag(E) F^1 B(x) \\ &= \tilde{Q}^T B(x) \end{aligned} \tag{31}$$

where $\tilde{Q}$ is a $\hat{m}$ -vector with elements equal to the diagonal entries of the following matrix

$$Q = \Phi^T K_1 \Phi diag(E) F^1 \tag{32}$$

and

$$\begin{aligned} &\int_0^1 k_2(x,t) y(t) dt \\ &= \int_0^1 \Psi^T(x) K_2 \Psi(t) \Psi^T(t) [C^T P^{\alpha}]^T dt \\ &= \frac{1}{\hat{m}} \Psi^T(x) K_2 [C^T P^{\alpha}]^T \\ &= \frac{1}{\hat{m}} C^T P^{\alpha} K_2^T \Phi B(x) \end{aligned} \tag{33}$$

Substituting the above equations into Eq.(24), we have

$$\begin{aligned} &\sum_{i=1}^{r} \left(A_{\hat{m}}^i\right)^T \Phi B(x) B^T(x) \Phi^T [P^{\alpha-\alpha_i}]^T C = \\ &C^T \Phi B(x) + \lambda_1 \tilde{Q}^T B(x) \\ &+ \frac{\lambda_2}{\hat{m}} C^T P^{\alpha} K_2^T \Phi B(x) + F^T \Phi B(x) \end{aligned} \tag{34}$$

Define $\Phi^T [P^{\alpha-\alpha_i}]^T C = [v_0^i, v_1^i, \ldots, v_{\hat{m}-1}^i]^T = [V_{\hat{m}}^i]^T$, then Eq.(34) becomes

$$\begin{aligned} &\sum_{i=1}^{r} \left(A_{\hat{m}}^i\right)^T \Phi diag(V_{\hat{m}}^i) B(x) = \\ &C^T \Phi B(x) + \lambda_1 \tilde{Q}^T B(x) + \\ &\frac{\lambda_2}{\hat{m}} C^T P^{\alpha} K_2^T \Phi B(x) + F^T \Phi B(x) \end{aligned} \tag{35}$$

Dispersing Eq.(35), we get

$$\sum_{i=1}^{r}\left(A_{\hat{m}}^{i}\right)^{T}\Phi diag(V_{\hat{m}}^{i})=C^{T}\Phi+ \lambda_1\tilde{Q}^{T}+\frac{\lambda_2}{\hat{m}}C^{T}P^{\alpha}K_2^{T}\Phi+F^{T}\Phi \quad (36)$$

which is a linear system of algebraic equations. By solving this system we can obtain the approximation of Eq.(30).

B. Nonlinear multi-order fractional integro-differential equations

In this section we deal with nonlinear multi-order fractional integro-differential equation of the form

$$\sum_{i=1}^{r}a_i(x)D_*^{\alpha_i}y(x)=D_*^{\alpha}y(x)+\lambda_1\int_0^x k_1(x,t)[y(t)]^p dt +\lambda_2\int_0^1 k_2(x,t)[y(t)]^q dt+f(x) \quad (37)$$

subject to initial conditions

$$y^{(s)}(0)=0,\quad s=0,1,\ldots,\lceil\alpha\rceil-1,$$

where $p,q\in N$, and the other parameters and variables are the same as the section 4.1. While dealing with such a situation, the same procedure of expansion of fractional order derivatives via block pulse functions is adopted with exception at the term containing $[y(t)]^p,[y(t)]^q$.

From Eq.(30), we have $y(x)\approx EB(x)$ and hence

$$[y(t)]^p\approx[EB(t)]^p =[e_0^p,e_1^p,\ldots,e_{\hat{m}-1}^p]B(t)=E_pB(t) \quad (38)$$

and

$$[y(t)]^q\approx[EB(t)]^q =[e_0^q,e_1^q,\ldots,e_{\hat{m}-1}^q]B(t)=E_qB(t) \quad (39)$$

Following the procedure of section 4.1and using the Eq.(38) and Eq.(39), the Eq.(37) is transformed into a nonlinear system of algebraic equations

$$\sum_{i=1}^{r}\left(A_{\hat{m}}^{i}\right)^{T}\Phi diag(V_{\hat{m}}^{i})=C^{T}\Phi +\lambda_1\tilde{W}^{T}+\frac{\lambda_2}{\hat{m}}E_q\Phi^{T}K_2^{T}\Phi+F^{T}\Phi \quad (40)$$

where $\tilde{W}$ is a $\hat{m}$ -vector with elements equal to the diagonal entries of the following matrix

$$W=\Phi^{T}K_1\Phi diag(E_p)F^{1} \quad (41)$$

Solving the system of equations given by Eq.(40), the approximate numerical solution $y(x)$ is obtained. The Eq.(40) can be solved by iterative numerical technique such as Newton's method.

V. Numerical Examples

In order to illustrate the effectiveness of the proposed method, we consider numerical examples of linear and nonlinear nature.

Example 5.1. Consider the following linear equation:

$$x^2D^{1.5}y(x)+xD^{0.5}y(x)=D^{1.7}y(x)+ \int_0^x(x-t)y(t)dt+\int_0^1(x+t)y(t)dt+f(x) \quad (42)$$

with this condition $y'(0)=y(0)=0$ and

$$f(x)=\left(\frac{\Gamma(3)}{\Gamma(1.5)}+\frac{\Gamma(3)}{\Gamma(2.5)}\right)x^{2.5}+ \left(\frac{\Gamma(4)}{\Gamma(2.5)}+\frac{\Gamma(4)}{\Gamma(3.5)}\right)x^{3.5}-\frac{\Gamma(3)}{\Gamma(1.3)}x^{0.3}-\frac{\Gamma(4)}{\Gamma(2.3)}x^{1.3}-\frac{x^4}{12}-\frac{x^5}{20}-\frac{7x}{12}-\frac{9}{20}.$$

The exact solution of this problem is $y(x)=x^2+x^3$. Table I shows the approximate solutions and exact solutions for different k, $M=2$.

TABLE I
THE APPROXIMATE SOLUTION AND EXACT SOLUTION FOR DIFFERENT k, $M=2$.

x	$k=4$	$k=5$	$k=6$	$k=7$	Exact solution
0	0.000024	0.000012	0.000007	0.000000	0.00000
1/8	0.015822	0.016551	0.017198	0.017566	0.017578
2/8	0.075531	0.077098	0.077920	0.078115	0.078125
3/8	0.192880	0.193159	0.193317	0.193351	0.193359
4/8	0.361498	0.368505	0.373205	0.374988	0.375000
5/8	0.622950	0.626293	0.631080	0.634693	0.634765
6/8	0.930901	0.963194	0.981423	0.984162	0.984375
7/8	1.391340	1.408939	1.422946	1.434649	1.435546

From the Table I, we can see clearly that the numerical solutions are more and more close to the exact solution when k increases.

Example 5.2. Consider this equation:

$$(x^2-1)D^{1.6}y(x)+(x^2+1)D^{1.2}y(x)+x^2D^{0.75}y(x)= D^{2.3}y(x)+\frac{1}{4}\int_0^x(x-t)y(t)dt+\frac{1}{2}\int_0^1 xty(t)dt+f(x) \quad (43)$$

where

$$f(x)=\frac{\Gamma(4.5)}{\Gamma(2.9)}(x^{3.9}-x^{1.9})+\frac{\Gamma(4.5)}{\Gamma(3.3)}(x^{4.3}+x^{2.3}) +\frac{\Gamma(4.5)}{\Gamma(3.75)}x^{4.75}-\frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2}-\frac{x^{5.5}}{99}-\frac{x}{11},$$

such that $y''(0)=y'(0)=y(0)=0$, the exact solution is $y(x)=x^{\frac{7}{2}}$. The numerical results for $k=3,4,5,6$, $M=2$ are shown in Figs.1-4. From the Figs.1-4, we can find easily that the numerical solutions are in good agreement with the exact solutions. The absolute errors for different values of k are shown in Table II. Through Table II, we can also see that the errors are smaller and smaller when k increases.

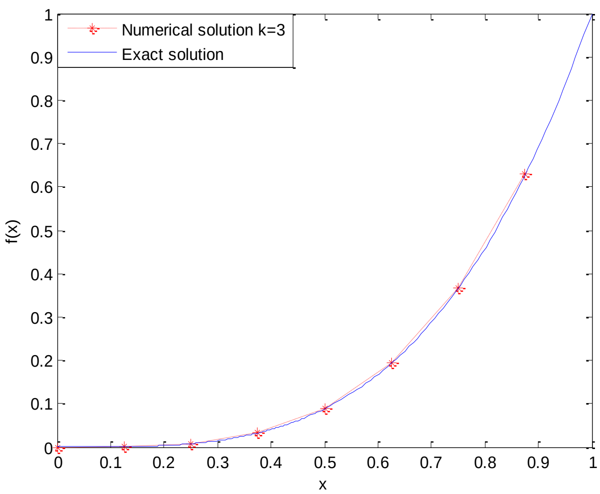

Fig. 1 Comparison of Num. sol. and Exa. Sol. of $k=3$, $M=2$.

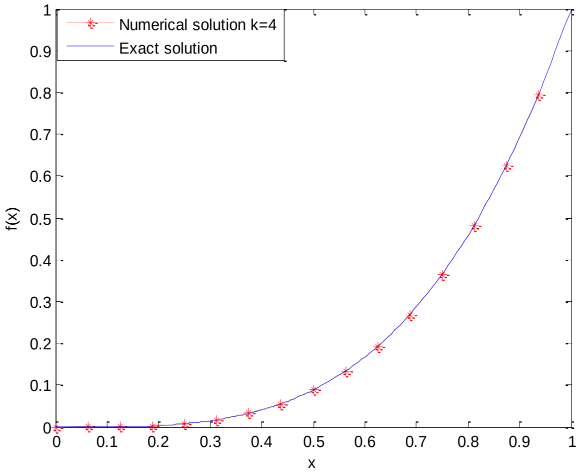

Fig. 2 Comparison of Num. sol. and Exa. Sol. of $k = 4, M = 2$.

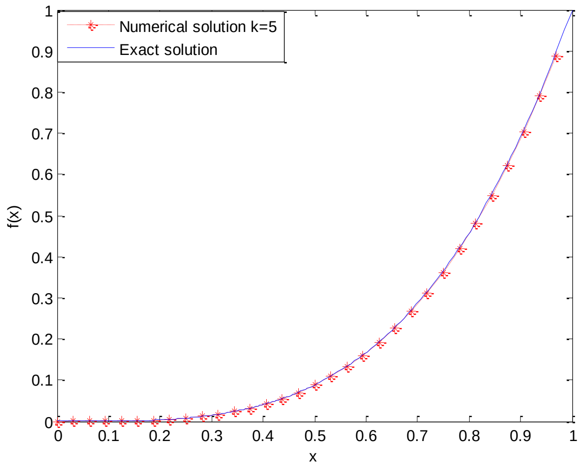

Fig. 3 Comparison of Num. sol. and Exa. Sol. of $k = 5, M = 2$.

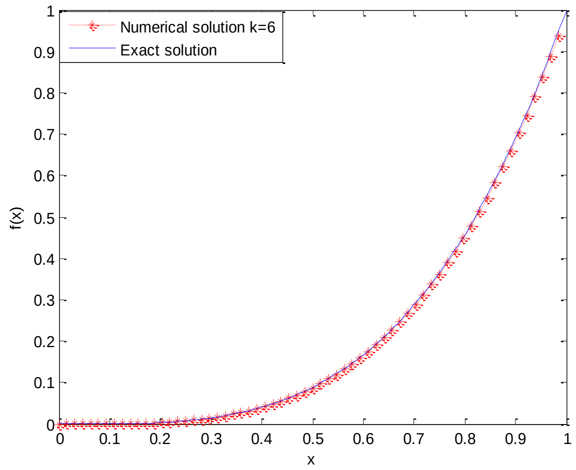

Fig. 4 Comparison of Num. sol. and Exa. Sol. of $k = 6, M = 2$.

TABLE II.
THE ABSOLUTE ERRORS FOR DIFFERENT VALUES OF $k, M = 2$.

x	$k=3$	$k=4$	$k=5$	$k=6$
0	0	0	0	0
1/8	2.24755e-004	4.86626e-005	6.63916e-006	7.06361e-006
2/8	5.68826e-004	8.92015e-005	4.52890e-005	8.47627e-006
3/8	8.06345e-004	7.09312e-005	3.13733e-005	4.04433e-006
4/8	2.37124e-003	2.36345e-004	7.36812e-005	9.10813e-006
5/8	2.80843e-003	7.10940e-004	2.44000e-004	3.74311e-005
6/8	3.16123e-003	2.50721e-003	3.80861e-004	4.30647e-005
7/8	3.35997e-003	3.03684e-003	6.01802e-004	6.80648e-005

VI. CONCLUSIONS

In the present manuscript, the application and scope of the Legendre wavelets have been extended to fractional order linear and nonlinear integro-differential equations successfully. We construct fractional orders generalized Legendre wavelets operational matrix of integration and use this to solve the fractional linear and nonlinear integro-differential equations numerically. By solving the linear and nonlinear system, numerical solutions are obtained. The convergence analysis of Legendre wavelets is proposed. The numerical results show that the approximation is in very good coincidence with the exact solution.

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