The Quaternion Domain Fourier Transform and its Application in Mathematical Statistics

Mawardi Bahri, Amir Kamal Amir, Resnawati, and Chrisandi Lande

***Abstract*—Recently a generalization of the quaternion Fourier transform over quaternion domains so-called the quaternion domain Fourier transform (QDFT) has been introduced, including its properties such as shift, modulation, convolution theorem and uncertainty principle. In the present paper we explore more properties of the QDFT such as the correlation and product theorems and propose its application in probability theory and mathematical statistics.**

***Index Terms*—quaternion domain Fourier transform, quaternion random variable**

I. Introduction

It is well known that in signal and image processing, the classical Fourier transform is a very important tool (see, e.g., [6], [16]). The quaternion Fourier transform (QFT) (for e.g., [1], [2], [3], [4], [5], [7]) is also very useful tool for signal processing for quaternion signals with domain $\mathbb{R}^2$. The quaternion domain Fourier transform (QDFT) is a generalization of the QFT over the quaternion domain. The first work concerning the definition of the QDFT and its relation to the definition of the QFT was done by Hitzer [15]. The QDFT also can regarded as an extension of the classical Fourier transform (FT) using quaternion algebra. It transforms quaternion valued signals defined over a quaternion domain from a quaternion position space to a quaternion frequency space. A number of useful properties of the QDFT have been found including shift, modulation, convolution, correlation, differentiation, energy conservation, uncertainty principle and so on. It is well known that the classical Fourier transform plays crucial roles in probability theory and mathematical statistics. It is related to the characteristic function of any real-valued random variable to compute the distribution function. Therefore, in the present paper, we first investigate some important properties of the QDFT such as derivative, convolution, correlation and product theorems. We then establish the relationship between the quaternion characteristic function and the QDFT. We finally apply this relation to derive the properties of quaternion probability function and quaternion moments in the framework of the quaternion algebra of mathematical statistics.

The remainder of this paper is organized as follows. In Section II we briefly review the basic knowledge of quaternion and derivative operators used in the next section. In Section III we derive some useful properties of the QDFT such as the convolution, correlation and product theorems. In Section IV we discuss the application of the QDFT in probability theory and mathematical statistics. Finally, in Section V we give conclusion.

Manuscript received January 11, 2018; revised April 17, 2018. This work was supported by Research Collaboration Between Indonesian Universities (PEKERTI) with contract number: 622.48/UN28.2/PL/2017.

Mawardi Bahri is with the Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia (corresponding author: e-mail: mawardibahri@gmail.com)

Amir Kamal Amir is with the Department of Mathematics, Hasanuddin University, Makassar, Indonesia. e- mail: amirkamalamir@yahoo.com

Resnawati is with the Department of Mathematics, Tadulako University, Palu 94148, Indonesia. e- mail: r35n4w4t1@yahoo.com

Chrisandi Lande is with Politeknik Ilmu Pelayaran Makassar, Makassar 90165, Indonesia. e-mail: sandhylande@gmail.com

II. Quaternions

We first review the basic concepts and definition of quaternions. The quaternions, a generalization of complex numbers, are an associative but noncommutative over $\mathbb{R}$. The set of quaternions is denoted by $\mathbb{H}$. Every element of $\mathbb{H}$ can be written in the following form

$$\mathbb{H} = \{q = q_a + \mathbf{i}q_b + \mathbf{j}q_c + \mathbf{k}q_d \ ; \ q_a, q_b, q_c, q_d \in \mathbb{R}\}, \quad (1)$$

which obeys the following multiplication rules:

$$\begin{aligned} &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \quad (2)$$

For a quaternion $q = q_a + \mathbf{i}q_b + \mathbf{j}q_c + \mathbf{k}q_d \in \mathbb{H}$, q_a is called the *scalar* part of q denoted by $Sc(q)$ and $\mathbf{i}q_b + \mathbf{j}q_c + \mathbf{k}q_d$ is called the *vector* (or *pure*) part of q. The vector part of q is conventionally denoted by $\mathbf{q}$. Let p, $q \in \mathbb{H}$ and $\mathbf{p}$, $\mathbf{q}$ be their vector parts, respectively. From (2) we obtain the quaternionic multiplication qp as

$$qp = q_a p_a - \mathbf{q} \cdot \mathbf{p} + q_a \mathbf{p} + p_a \mathbf{q} + \mathbf{q} \times \mathbf{p}, \quad (3)$$

where

$$\mathbf{q} \cdot \mathbf{p} = q_b p_b + q_c p_c + q_d p_d \quad (4)$$

$$\mathbf{q} \times \mathbf{p} = \mathbf{i}\,(q_c p_d - q_d p_c) + \mathbf{j}\,(q_d p_b - q_b p_d) + \mathbf{k}\,(q_b p_c - q_c p_d). \quad (5)$$

Analogously as in the complex case, the quaternion conjugate of q is defined by

$$\bar{q} = q_a - \mathbf{i}q_b - \mathbf{j}q_c - \mathbf{k}q_3, \qquad q_a, q_b, q_c, q_d \in \mathbb{R}. \quad (6)$$

It is an anti-involution, i.e.

$$\overline{qp} = \bar{p}\bar{q}. \quad (7)$$

Notice that conjugate switches the order of multiplication. From (6) we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_a^2 + q_b^2 + q_c^2 + q_d^2}. \quad (8)$$

It is routine to check that

$$|qp| = |q||p| \quad \text{and} \quad |q+p| \leq |q| + |p|, \quad \forall p, q \in \mathbb{H}. \quad (9)$$

Using the conjugate (6) and the modulus of q, we get the inverse of a non-zero quaternion $q \in \mathbb{H}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2}, \quad (10)$$

(Advance online publication: 28 May 2018)

which shows that $\mathbb{H}$ is a normed division algebra. When $|q| = 1, q$ is a unit quaternion. A quaternion q with $q_a = 0$ is called a pure quaternion and its square is negative sum of three squares:

$$\mathbf{q}^2 = -(q_a^2 + q_b^2 + q_c^2) = -1. \tag{11}$$

According to (4) we can get a scalar part of two quaternions p, q as

$$p \cdot q = Sc(\bar{p}q) = \frac{1}{2}(p\bar{q} + q\bar{p}) = p_a q_a + p_b q_b + p_c q_c + p_d q_d. \tag{12}$$

A quaternion number q may be defined as a complex number with complex and imaginary parts.

$$q = z_1 + \mathbf{j} z_2, \quad z_1 = q_a + \mathbf{i} q_b, \quad z_2 = q_c + \mathbf{i} q_d. \tag{13}$$

Equation (13) is known as the Cayley-Dickson form.

We define derivative operators as

$$\begin{aligned}\tilde{\partial} &= \partial_{x_a} + \partial_{x_b}\mathbf{i} + \partial_{x_c}\mathbf{j} + \partial_{x_d}\mathbf{k} \\ \partial &= \partial_{x_a} - \partial_{x_b}\mathbf{i} - \partial_{x_c}\mathbf{j} - \partial_{x_d}\mathbf{k},\end{aligned} \tag{14}$$

where $\partial_{x_a} = \partial/\partial_{x_a}$ and so on.

Using the orthogonal planes split of $q \in \mathbb{H}$ with respect to the pure quaternion $\mu \in \mathbb{H}, \mu^2 = -1$ we define

$$\begin{aligned} q_{\pm} &= \frac{1}{2}(q \pm \mu q \mu), \quad q_- = q_a + q_\mu \mu, \\ q_+ &= q_\nu \nu + q_\eta \eta = (q_\nu + q_\eta \mu)\nu \end{aligned} \tag{15}$$

where rotation operator $R = (\mathbf{i} + \mu)\mathbf{i}, \nu = R\mathbf{j}R^{-1}$ and $\eta = R\mathbf{k}R^{-1}, \nu^2 = \eta^2 = -1, q_a, q_\mu, q_\nu, q_\eta \in \mathbb{R}$.

Similar to the complex case, we may define an inner product for two functions $f, g : \mathbb{H} \longrightarrow \mathbb{H}$ as

$$(f, g)_{L^2(\mathbb{H};\mathbb{H})} = \int_{\mathbb{H}} f(x)\overline{g(x)}\, d^4x, \tag{16}$$

where $x \in \mathbb{H}, d^4x = dx_a dx_b dx_c dx_d \in \mathbb{R}$. Notice that every quaternion domain function f maps $\mathbb{H} \longrightarrow \mathbb{H}$, its decomposition will take the form

$$f(x) = f_a(x) + f_b(x)\mathbf{i} + f_c(x)\mathbf{j} + f_d(x)\mathbf{j} \tag{17}$$

$$\begin{aligned} &= f_a(x_a, x_b, x_c, x_d) + f_b(x_a, x_b, x_c, x_d)\mathbf{i} \\ &\quad + f_c(x_a, x_b, x_c, x_d)\mathbf{j} + f_d(x_a, x_b, x_c, x_d)\mathbf{k}, x, y \in \mathbb{H}, \end{aligned} \tag{18}$$

where four coefficient functions f_a, f_b, f_c and f_d are in turn real valued quaternion domain function.

In particular, for $f = g$, we also may define the $L^2(\mathbb{H};\mathbb{H})$-norm as

$$\|f\| = \left(\int_{\mathbb{H}} |f(x)|^2\, d^4x \right)^{1/2}. \tag{19}$$

III. The Quaternion Domain Fourier Transform and its Useful Properties

In this section we introduce the quaternion domain Fourier transform (QDFT) and its basic properties, which is taken from [15]. We then make some observations about some further properties related the QDFT such as the correlation and product theorems, which will be very useful later.

Definition 1. *The quaternion domain Fourier transform of the quaternion function $f \in L^1(\mathbb{H};\mathbb{H})$ is given by the integral*

$$\mathcal{F}_\mu\{f)\}(\omega) = \int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x)\, d^4x, \tag{20}$$

where $x, \omega \in \mathbb{H}$ and some constant $\mu \in \mathbb{H}$.

Notice that constant pure quaternion μ can be chosen for each problem. For example, if we take $\mu = \mathbf{j}$ and the quaternion function f is decomposed as in (13), then the QDFT takes the form

$$\mathcal{F}_{\mathbf{j}}\{f\}(\omega) = \int_{\mathbb{H}} e^{\mathbf{j}\omega\cdot x} f(x)\, d^4x. \tag{21}$$

Expanding f in (21) into real and imaginary parts with respect to $\mathbf{i}$ and using the Euler formula for the quaternion Fourier kernel we obtain

$$\begin{aligned} &\mathcal{F}_{\mathbf{j}}\{f\}(\omega) \\ &= \int_{\mathbb{H}} (\cos(\omega\cdot x) + \mathbf{j}\sin(\omega\cdot x)(f_0(x) + f_1(x)\mathbf{i})\, d^4x \\ &= \int_{\mathbb{H}} f_0(x)\cos(\omega\cdot x)\, d^4x + \mathbf{j}\int_{\mathbb{H}} f_0(x)\sin(\omega\cdot x)\, d^4x \\ &+ \int_{\mathbb{H}} f_1(x)\cos(\omega\cdot x)\mathbf{i}\, d^4x + \mathbf{j}\int_{\mathbb{H}} f_1(x)\sin(\omega\cdot x)\mathbf{i}\, d^4x. \end{aligned} \tag{22}$$

Equation (22) clearly shows how the QDFT separates quaternion signal into the odd and even parts of real and imaginary parts in four different components in the QDFT domain.

Definition 2. *If $f \in L^1(\mathbb{H};\mathbb{H})$ and its QDFT $\mathcal{F}_\mu\{f\} \in L^1(\mathbb{H};\mathbb{H})$, then the inverse transform of the QDFT is given by the integral*

$$\mathcal{F}_\mu^{-1}[\mathcal{F}_\mu\{f\}](x) = f(x) = \frac{1}{2\pi}\int_{\mathbb{H}} e^{-\mu\omega\cdot x}\mathcal{F}_\mu\{f\}(\omega)\, d^4\omega. \tag{23}$$

Like the polynomial Fourier transform [19], the convolution of two quaternion functions $f, g \in L^1(\mathbb{H};\mathbb{H})$ is defined by

$$(f * g)(x) = \int_{\mathbb{H}} f(x-y)g(y)\, d^4y. \tag{24}$$

The following theorem provide the convolution theorem which describes how the QDFT behaves under the quaternion convolution.

Theorem 1. *Suppose $f \in L^1(\mathbb{H};\mathbb{H})$ and $g \in L^1(\mathbb{H};\mathbb{H})$ are integral functions. Then we have*

$$\begin{aligned} &\mathcal{F}_\mu\{f * g\}(\omega) \\ &= \mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{g\}(\omega) + \mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{g\}(-\omega). \end{aligned} \tag{25}$$

Moreover,

$$\begin{aligned} (f * g)(x) = \mathcal{F}_\mu^{-1}\big[&\mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{g\}(\omega) + \\ &+ \mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{g\}(-\omega)\big](x). \end{aligned} \tag{26}$$

Proof: Let $\mathcal{F}_\mu\{f\}$ and $\mathcal{F}_\mu\{g\}$ denote the QDFT of f and g, respectively. It follows from (20) *and* (24) *that*

$$\begin{aligned} \mathcal{F}_\mu\{f \star g\}(\omega) &= \int_{\mathbb{H}} e^{\mu\omega\cdot x}(f \star g)(x)\, d^4x \\ &= \int_{\mathbb{H}} \left(\int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x-y)g(y)\, d^4y \right) d^4x \\ &= \int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x-y) \left(\int_{\mathbb{H}} g(y)\, d^4y \right) d^4x. \end{aligned} \tag{27}$$

After the change of variables $z = x-y$, the above expression becomes

$$\begin{aligned}
&\mathcal{F}_\mu\{f\star g\}(\omega)\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot(y+z)} f(z)g(y)\, d^4y\, d^4z\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot z}\, e^{\mu\omega\cdot y} f(z)g(y)\, d^4z\, d^4y\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot z}\, e^{\mu\omega\cdot y} (f_-(z)+f_+(z))g(y)\, d^4z\, d^4y\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot z} (f_-(z)e^{\mu\omega\cdot y}+f_+(z)e^{-\mu\omega\cdot y})g(y)\, d^4z\, d^4y\\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot z} f_-(z)e^{\mu\omega\cdot y}g(y)\, d^4z\, d^4y\\
&\quad + \int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu\omega\cdot z} f_+(z)e^{-\mu\omega\cdot y}g(y)\, d^4z\, d^4y\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot z} f_-(z)\, d^4z \int_{\mathbb{H}} e^{\mu\omega\cdot y} g(y)\, d^4y\\
&\quad + \int_{\mathbb{H}} e^{\mu\omega\cdot z} f_+(z)\, d^4z \int_{\mathbb{H}} e^{-\mu\omega\cdot y} g(y)\, d^4y\\
&= \mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{g\}(\omega)+\mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{g\}(-\omega). \qquad (28)
\end{aligned}$$

In view of (28) *and inversion formula for the QDFT* (23)*, the relation* (26) *holds. This completes the proof of the theorem.* ■

Definition 3. *The correlation for the QDFT of two quaternion functions $f, g \in L^1(\mathbb{H};\mathbb{H})$ is given by*

$$(f\circ g)(x) = \int_{\mathbb{H}} f(x+y)\overline{g(y)}\, d^4y. \qquad (29)$$

We derive the following correlation theorem for the QDFT using the relationship between the quaternion convolution and quaternion correlation (compare to [17]).

Theorem 2. *Suppose $f \in L^1(\mathbb{H};\mathbb{H})$ and $g \in L^1(\mathbb{H};\mathbb{H})$ are integral functions. Thus*

$$\begin{aligned}
&\mathcal{F}_\mu\{f\circ g\}(\omega)\\
&= \mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{\bar g\}(-\omega)+\mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{\bar g\}(\omega). \qquad (30)
\end{aligned}$$

Proof: A simple computation yields

$$\begin{aligned}
&(f\circ g)(x)\\
&= \int_{\mathbb{H}} f(x+y)\overline{g(y)}\, d^4y\\
&= \int_{\mathbb{H}} f(x-u)\overline{g(-u)}\, d^4u\\
&= \int_{\mathbb{H}} f(x-u)h(u)\, d^4u\\
&= (f*h)(x)\\
&\overset{(26)}{=} \mathcal{F}_\mu^{-1}\Big[\mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{h\}(\omega)\\
&\qquad + \mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{h\}(-\omega)\Big](x). \qquad (31)
\end{aligned}$$

It is easily seen that

$$\mathcal{F}_\mu\{h\}(\omega) = \int_{\mathbb{H}} e^{\mu\omega\cdot u}\overline{g(-u)}\, d^4u = \mathcal{F}_\mu\{\bar g\}(-\omega), \qquad (32)$$

and

$$\mathcal{F}_\mu\{h\}(-\omega) = \int_{\mathbb{H}} e^{-\mu\omega\cdot u}\overline{g(-u)}\, d^4u = \mathcal{F}_\mu\{\bar g\}(\omega). \qquad (33)$$

Due to (32) and (33), equation (31) can be expressed as

$$\begin{aligned}
&(f\circ g)(x)\\
&= \mathcal{F}_\mu^{-1}\left[\mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{\bar g\}(-\omega)+\mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{\bar g\}(\omega)\right](x).
\end{aligned}$$

Or, equivalently,

$$\begin{aligned}
&\mathcal{F}_\mu\{f\circ g\}(\omega)\\
&= \mathcal{F}_\mu\{f_-\}(\omega)\mathcal{F}_\mu\{\bar g\}(-\omega)+\mathcal{F}_\mu\{f_+\}(\omega)\mathcal{F}_\mu\{\bar g\}(\omega),
\end{aligned}$$

which was to be proved. ■

Theorem 3. *Suppose $f \in L^1(\mathbb{R};\mathbb{H})$ such that $f(x)$ is continuous n-times differentiable, then for $\lim_{x\to\pm\infty} f(x) = 0$ the following holds*

$$\mathcal{F}_\mu\{\tilde\partial^n f\}(\omega) = \omega^n\,(-\mu)^n\mathcal{F}_\mu\{f\}(\omega), n\in\mathbb{N}. \qquad (34)$$

Proof: Consider first the case $n = 1$. Indeed, we have

$$\begin{aligned}
&\mathcal{F}_\mu\{\tilde\partial f\}(\omega)\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot x}(\partial_{x_a}+\partial_{x_b}\mathbf{i}+\partial_{x_c}\mathbf{j}+\partial_{x_d}\mathbf{k})f(x)\, d^4x\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot x}\partial_{x_a}f(x)\, d^4x + \int_{\mathbb{H}} e^{\mu\omega\cdot x}\mathbf{i}\partial_{x_a}f(x)\, d^4x\\
&\quad + \int_{\mathbb{H}} e^{\mu\omega\cdot x}\mathbf{j}\partial_{x_a}f(x)\, d^4x + \int_{\mathbb{H}} e^{\mu\omega\cdot x}\mathbf{k}\partial_{x_a}f(x)\, d^4x\\
&= -\int_{\mathbb{H}} (\partial_{x_a}e^{\mu\omega\cdot x})f(x)\, d^4x - \int_{\mathbb{H}} (\partial_{x_b}e^{\mu\omega\cdot x})f(x)\, d^4x\mathbf{i}\\
&\quad - \int_{\mathbb{H}} (\partial_{x_b}e^{\mu\omega\cdot x})f(x)\, d^4x\mathbf{j} - \int_{\mathbb{H}} (\partial_{x_b}e^{\mu\omega\cdot x})f(x)\, d^4x\mathbf{k}\\
&= \int_{\mathbb{H}} (\omega_a+\omega_b\mathbf{i}+\omega_c\mathbf{j}+\omega_d\mathbf{k})(-\mu)\, e^{\mu\omega\cdot x}f(x)\, d^4x\\
&= \omega\,(-\mu)\int_{\mathbb{H}} e^{\mu t\cdot x}f(x)\, d^4x.
\end{aligned}$$

Using mathematical induction we can finish the proof of the theorem. ■

The following theorem describes the relationship between the product of two quaternion functions and its QDFT.

Theorem 4. *Let $f, g \in L^2(\mathbb{H};\mathbb{H})$. Then the QDFT of product of two quaternion functions f and g is given by*

$$\begin{aligned}
&\mathcal{F}_\mu\{fg\}(\omega)\\
&= \frac{1}{(2\pi)^2}\Big[(\mathcal{F}_q\{f_-\}*\mathcal{F}_\mu\{g\})(\omega)+(\mathcal{F}_\mu\{\bar f_+\}\circ\mathcal{F}_q\{g\})(\omega)\Big]. \qquad (35)
\end{aligned}$$

Proof: Applying the QDFT definition we obtain

$$
\begin{aligned}
&\mathcal{F}_\mu\{fg\}(\omega)\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x)g(x)\, d^4x\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x)\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{H}} e^{-\mu u\cdot x}\mathcal{F}_\mu\{g\}(u)\, d^4u\Big)\, d^4x\\
&= \int_{\mathbb{H}} e^{\mu\omega\cdot x} f(x)\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{H}} e^{-\mu u\cdot x}\mathcal{F}_\mu\{g\}(u)\, d^4u\Big)\, d^4x\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{H}} e^{\mu\omega\cdot x}(f_-(x)+f_+(x))\\
&\quad\times\Big(\int_{\mathbb{H}} e^{-\mu u\cdot x}\mathcal{F}_\mu\{g\}(u)\, d^4u\Big)\, d^4x\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\int_{\mathbb{H}} e^{-\mu u\cdot x}e^{\mu\omega\cdot x} f_-(x)\mathcal{F}_\mu\{g\}(u)\, d^4u\, d^4x\\
&+ \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\int_{\mathbb{H}} e^{\mu u\cdot x}e^{\mu\omega\cdot x} f_+(x)\mathcal{F}_\mu\{g\}(u)\, d^4u\, d^4x\\
&= \frac{1}{(2\pi)^4}\int_{\mathbb{H}}\int_{\mathbb{H}}\int_{\mathbb{H}}\Big(e^{-\mu u\cdot x}e^{\mu\omega\cdot x}e^{-\mu v\cdot x}\\
&\quad\times\mathcal{F}_\mu\{f_-\}(v)\mathcal{F}_\mu\{g\}(u)\Big)\, d^4u\, d^4v\, d^4x\\
&+ \frac{1}{(2\pi)^4}\int_{\mathbb{H}}\int_{\mathbb{H}}\int_{\mathbb{H}}\Big(e^{\mu u\cdot x}e^{\mu\omega\cdot x}e^{-\mu v\cdot x}\\
&\quad\times\mathcal{F}_\mu\{f_+\}(v)\mathcal{F}_\mu\{g\}(u)\Big)\, d^4u\, d^4v\, d^4x\\
&= \frac{1}{(2\pi)^4}\int_{\mathbb{H}}\int_{\mathbb{H}}\Bigg[\Big(\int_{\mathbb{H}} e^{\mu(\omega-u-v)\cdot x}\, d^4x\Big)\\
&\quad\times\mathcal{F}_\mu\{f_-\}(v)\mathcal{F}_\mu\{g\}(u)\Bigg]\, d^4u\, d^4v\\
&+ \frac{1}{(2\pi)^4}\int_{\mathbb{H}}\int_{\mathbb{H}}\Bigg]\Big(\int_{\mathbb{H}} e^{\mu(\omega+u-v)\cdot x}\, d^4x\Big)\\
&\quad\times\mathcal{F}_\mu\{f_+\}(v)\mathcal{F}_\mu\{g\}(u)|bigg]\, d^4u\, d^4v\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\int_{\mathbb{H}}\delta(w-u-v)\mathcal{F}_\mu\{f_-\}(v)\mathcal{F}_\mu\{g\}(u)\, d^4u\, d^4v\\
&+ \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\int_{\mathbb{H}}\delta(\omega+u-v)\mathcal{F}_\mu\{f_+\}(v)\mathcal{F}_\mu\{g\}(u)\, d^4u\, d^4v\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\mathcal{F}_\mu\{f_-\}(\omega-u)\mathcal{F}_\mu\{g\}(u)\, d^4u\\
&+ \frac{1}{(2\pi)^2}\int_{\mathbb{H}}\mathcal{F}_\mu\{f_+\}(\omega+u)\mathcal{F}_\mu\{g\}(u)\, d^4u. \qquad (36)
\end{aligned}
$$

This gives the required result. ■

As an immediate consequence of the above theorem, we get the following corollary.

Corollary 1. *Let $f, g \in L^2(\mathbb{H};\mathbb{H})$. Assume that the QDFT of g is a real-valued function, then Theorem 4 will reduce to*

$$\mathcal{F}_q\{fg\}(\omega) = \frac{1}{(2\pi)^2}(\mathcal{F}_q\{f\} * \mathcal{F}_q\{g\})(\omega). \qquad (37)$$

IV. Application of the Quaternion Domain Fourier Transform in Mathematical Statistics

Following [8], [9] we define the probability density function of a quaternion random variable, which is algebraically similar to real probability density function of the corresponding associated probability quantity [6].

Definition 4. *Let $X = X_a + X_b\mathbf{i} + X_c\mathbf{j} + X_d\mathbf{k}$ be a quaternion random variable. A quaternion function $f_X(x) = f_{X_a}(x) + f_{X_b}(x)\mathbf{i} + f_{X_c}(x)\mathbf{j} + f_{X_d}(x)\mathbf{k}$ of the quaternion variable $x = x_a + x_b\mathbf{i} + x_c\mathbf{j} + x_d\mathbf{k}$ is called the quaternion probability density function (qpdf) of X if*

$$\int_{\mathbb{H}} f_{X_i}(x)\, d^4x = 1, \quad \text{and} \quad f_{X_i}(x) \geq 0\, \forall x \in \mathbb{H}, i = a, b, c, d.$$

We also define the quaternion cumulative distribution function (compare to [18])

$$f_X(x) = \tilde{\partial} F_X(x),$$

where the probability Pr is related to F_X given by

$$F_X(x) = Pr(X_a \leq x_a, X_b \leq x_b, X_c \leq x_c, X_d \leq x_d).$$

Definition 5 (Expected value)**.** *Let X be a quaternion-valued random variable with quaternion density function $f(x)$. The expected value $m = E[X]$ is defined by*

$$
\begin{aligned}
m &= E[X]\\
&= \int_{\mathbb{H}} x f_X(x)\, d^4x\\
&= \int_{\mathbb{H}} x\big(f_{X_a}(x) + f_{X_b}(x)\mathbf{i} + f_{X_c}(x)\mathbf{j} + f_{X_d}(x)\mathbf{k}\big)\, d^4x\\
&= \int_{\mathbb{H}} x f_{X_a}(x)\, d^4x + \int_{\mathbb{H}} x f_{X_b}(x)\mathbf{i}\, d^4x\\
&\quad + \int_{\mathbb{H}} x f_{X_c}(x)\mathbf{j}\, d^4x + \int_{\mathbb{H}} x f_{X_d}(x)\mathbf{k}\, d^4x\\
&= E[X_a] + E[X_b]\mathbf{i} + E[X_c]\mathbf{j} + E[X_d]\mathbf{k}, \qquad (38)
\end{aligned}
$$

provided the integral exists. The expected value of $X \in \mathbb{H}$ is usually called the mean.

Two quaternion random variables $X = X_a + X_b\mathbf{i} + X_c\mathbf{j} + X_d\mathbf{k}$ and $Y = Y_a + Y_b\mathbf{i} + Y_c\mathbf{j} + Y_d\mathbf{k}$ are independent if $(X_a, Y_a), (X_b, Y_b), (X_c, Y_d)$ and (X_d, Y_d) are independent. Using (2) we can obtain the product of two quaternion random variables X and Y. If the quaternion variables X and Y are independent, then $E[XY] = E[X]E[Y]$.

Corollary 2. *If X and Y are quaternion random variables, then*

1) *$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$ for any constants $\alpha, \beta \in \mathbb{H}$;*
2) *$|E[X]| \leq E[|X|]$ for any the quaternion probability density function $f_X(x) \in \mathbb{R}, f_X(x) > 0 \forall x \in \mathbb{H}$.*

Definition 6. *If X is a quaternion random variable with the quaternion density function $f_X(x)$, then the characteristic function $\phi_X(t)$ of the random variable X or the distribution function $F(x)$ is defined by formula*

$$
\begin{aligned}
\phi_X(t) &= E[e^{\mu t\cdot X}]\\
&\overset{(12)}{=} E[e^{\frac{1}{2}\mu(t\bar{X} + X\bar{t})}]\\
&= \int_{\mathbb{H}} e^{\mu t\cdot x} f_X(x)\, d^4x. \qquad (39)
\end{aligned}
$$

This shows that the characteristic function $\phi_X(t)$ can be regarded as the quaternion domain Fourier transform of the density function $f_X(x)$. Applying (12) we can rewrite (39) mentioned above in the form

$$\begin{aligned}
&\phi_X(t)\\
&=\int_{\mathbb{H}} e^{\mu(t_a x_a+t_b x_b+t_c x_c+t_d x_d)}\\
&\quad\times\big(f_{X_a}(x)+f_{X_b}(x)\mathbf{i}+f_{X_c}(x)\mathbf{j}+f_{X_d}(x)\mathbf{k}\big)\,d^4x.\\
&=\int_{\mathbb{H}} e^{\mu(t_a x_a+t_b x_b+t_c x_c+t_d x_d)} f_{X_a}(x)\,d^4x\\
&\quad+\int_{\mathbb{H}} e^{\mu(t_a x_a+t_b x_b+t_c x_c+t_d x_d)} f_{X_b}(x)\,d^4x\mathbf{i}\\
&\quad+\int_{\mathbb{H}} e^{\mu(t_a x_a+t_b x_b+t_c x_c+t_d x_d)} f_{X_c}(x)\,d^4x\mathbf{j}\\
&\quad+\int_{\mathbb{H}} e^{\mu(t_a x_a+t_b x_b+t_c x_c+t_d x_d)} f_{X_d}(x)\,d^4x\mathbf{k}\\
&=\phi_{X_a}(t)+\phi_{X_b}\mathbf{i}+\phi_{X_c}\mathbf{j}+\phi_{X_d}\mathbf{k}. \qquad (40)
\end{aligned}$$

Some basic properties of the characteristic function are listed in the following corollary.

Corollary 3. *Let X be a quaternion random variable with quaternion density function $f_X(x)$.*

1) *If quaternion random variables X and Y are independent, then*

$$\phi_{X+Y}(t)=\phi_X(t)+\phi_Y(t); \qquad (41)$$

2) $\phi_{\alpha+\beta X}(t)=e^{\mu t\cdot\alpha}\phi_X(t\beta)$.

Proof: For the first assertion, simple computations gives

$$\begin{aligned}
&\phi_{X+Y}(t)\\
&=E[e^{\mu t\cdot(X+Y)}]\\
&=\phi_{X_a+Y_a}(t)+\phi_{X_b+Y_b}(t)\mathbf{i}+\phi_{X_c+Y_c}(t)\mathbf{j}+\phi_{X_d+Y_d}(t)\mathbf{k}\\
&=\big(\phi_{X_a}(t)+\phi_{X_b}(t)\mathbf{i}+\phi_{X_c}(t)\mathbf{j}+\phi_{X_d}(t)\mathbf{k}\big)\\
&\quad+\big(\phi_{Y_a}(t)+\phi_{Y_b}(t)\mathbf{i}+\phi_{Y_c}(t)\mathbf{j}+\phi_{Y_d}(t)\mathbf{k}\big)\\
&=\phi_X(t)+\phi_Y(t). \qquad (42)
\end{aligned}$$

For the second, In fact, we have

$$\begin{aligned}
&\phi_{\alpha+\beta X}(t)\\
&=E[e^{\mu t\cdot(\alpha+\beta X)}]\\
&=e^{\mu t\cdot\alpha}E[e^{\mu t\cdot\beta X}]\\
&=e^{\mu t\cdot\alpha}\big(\phi_{\beta_a X_a}(t)+\phi_{\beta_b X_b}(t)\mathbf{i}+\phi_{\beta_c X_c}(t)\mathbf{j}+\phi_{\beta_d X_d}(t)\mathbf{k}\big)\\
&=e^{\mu t\cdot\alpha}\phi_X(t\beta). \qquad (43)
\end{aligned}$$

This is the desired result. ■

The following theorem is an extension of the Riemann-Lebesgue lemma to the quaternion density function.

Theorem 5 (Riemann-Lebesgue lemma of density function)**.** *For a quaternion density function $f_X\in L^1(\mathbb{H};\mathbb{H})$ the quaternion characteristic function $\phi_X(t)$ satisfies*

$$\lim_{|t|\to\infty}|\phi_X(t)|=0, \qquad (44)$$

Proof: Because of

$$e^{\mu tx}=-e^{\mu t\cdot(x+\frac{\bar{t}\pi}{|t|^2})}, \qquad (45)$$

we have

$$\phi_X(t)=\int_{\mathbb{H}} e^{\mu t\cdot x}f_X(x)\,d^4x=-\int_{\mathbb{H}} e^{\mu t\cdot(x+\frac{\bar{t}\pi}{|t|^2})}f_X(x)\,d^4x.$$

By substitution, we have

$$\begin{aligned}
&\phi_X(t)\\
&=\frac{1}{2}\left[\int_{\mathbb{H}} e^{\mu t\cdot x}f_X(x)\,d^4x-\int_{\mathbb{H}} e^{\mu t\cdot(x+\frac{\bar{t}\pi}{|t|^2})}f_X(x)\,d^4x\right]\\
&=\frac{1}{2}\left[\int_{\mathbb{H}} e^{\mu t\cdot y}f(y)\,d^4y-\int_{\mathbb{H}} e^{\mu t\cdot y}f(y-\frac{\bar{t}\pi}{|t|^2})\,d^4y\right]\\
&=\frac{1}{2}\int_{\mathbb{H}} e^{\mu t\cdot y}\left[f(y)-f(y-\frac{\bar{t}\pi}{|t|^2})\right]\,d^4y.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{|t|\to\infty}|\phi_X(t)|&\le\frac{1}{2}\lim_{|t|\to\infty}\int_{\mathbb{H}}|e^{\mu t\cdot y}|\left|f(y)-f(y-\frac{\bar{t}\pi}{|t|^2})\right|\,d^4y\\
&=\frac{1}{2}\lim_{|t|\to\infty}\int_{\mathbb{H}}\left|f(y)-f(y-\frac{\bar{t}\pi}{|t|^2})\right|\,d^4y=0.
\end{aligned}$$

The proof is complete. ■

The following theorem describes an important property of the characteristic function.

Theorem 6 (Continuity)**.** *If the quaternion density function $f\in L^1(\mathbb{H};\mathbb{H})$, then the characteristic function $\phi_X(t)$ of a quaternion-valued random variable X is continuous function on $\mathbb{H}$.*

Proof: It follow directly from the QDFT definition (20) that

$$\begin{aligned}
&|\phi_X(t+h)-\phi_X(t)|\\
&=\left|E\left[e^{\mu(t+h)\cdot X}\right]-E\left[e^{\mu t\cdot X}\right]\right|\\
&=\left|\int_{\mathbb{H}} e^{\mu t\cdot x}e^{\mu h\cdot x}f_X(x)\,d^4x-\int_{\mathbb{H}} e^{\mu t\cdot x}f_X(x)\,d^4x\right|\\
&=\left|\int_{\mathbb{H}} e^{\mu t\cdot x}(e^{\mu h\cdot x}-1)f_X(x)\,d^4x\right|\\
&\le\int_{\mathbb{H}}|(e^{\mu h\cdot x}-1)||f_X(x)|\,d^4x. \qquad (46)
\end{aligned}$$

We know from the triangle inequality for quaternions that

$$|e^{\mu h\cdot x}-1|\le|e^{\mu h\cdot x}|+1=2.$$

Therefore,

$$|\phi_X(t+h)-\phi_X(t)|\le 2\int_{\mathbb{H}}|f_X(x)|\,d^4x. \qquad (47)$$

Applying the Lebesgue dominated convergence theorem to (46) gives

$$\lim_{h\to 0}|\phi_X(t+h)-\phi_X(t)|=0. \qquad (48)$$

This shows the characteristic function $\phi_X(t)$ is continuous on $\mathbb{H}$. ■

Theorem 7 (Parseval identity)**.** *If the quaternion characteristic functions $\phi_X(t)$ and $\psi_X(t)$ of the random variable X are defined by*

$$\phi_X(t)=\int_{\mathbb{H}} e^{\mu t\cdot x}f_X(x)\,d^4x,\quad \psi_X(t)=\int_{\mathbb{H}} e^{\mu t\cdot x}g_X(x)\,d^4x, \qquad (49)$$

where $f_X(x)$ and $g_X(x)$ are quaternion density functions with respect to $\phi_X(t)$ and $\psi_X(t)$, respectively. Then we have

$$\int_{\mathbb{H}} g_X(t)e^{-\mu t\cdot y}\phi(t)\,d^4t = \int_{\mathbb{H}} \psi_{X-}(x-y)f_X(x)\,d^4x + \int_{\mathbb{H}} \psi_{X+}(y-x)f_X(x)\,d^4x. \quad (50)$$

Proof: Applying the characteristic function (39) we obtain

$$e^{-\mu t\cdot y}\phi_X(t) = \int_{\mathbb{H}} e^{\mu t\cdot(x-y)}f_X(x)\,d^4x. \quad (51)$$

Multiplying both sides of the above identity by $g_X(t)$ and then integrating with respect to d^4t we immediately get

$$\begin{aligned}
&\int_{\mathbb{H}} g_X(t)e^{-\mu t\cdot y}\phi_X(t)\,d^4t \\
&= \int_{\mathbb{H}}\int_{\mathbb{H}} g_X(t)e^{\mu t\cdot(x-y)}f_X(x)\,d^4x\,d^4t \\
&= \int_{\mathbb{H}}\left[\int_{\mathbb{H}} \left(g_{X-}(t)+g_{X+}(t)\right)e^{\mu\cdot t(x-y)}\,d^4t\right]f_X(x)\,d^4x \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{H}} e^{\mu t\cdot(x-y)}g_{X-}(t)\,d^4t\right)f_X(x)\,d^4x \\
&\quad + \int_{\mathbb{H}}\left(\int_{\mathbb{H}} e^{-\mu t\cdot(x-y)}g_{X+}(t)\,d^4t\right)f_X(x)\,d^4x \\
&= \int_{\mathbb{H}} \psi_{X-}(x-y)f_X(x)\,d^4x + \int_{\mathbb{H}} \psi_{X+}(y-x)f_X(x)\,d^4x.
\end{aligned} \quad (52)$$

This is the desired result. ■

The next, we first observe that

$$\mathcal{F}_\mu\{\tilde{\partial}F_X\}(t) = \mathcal{F}_\mu\{f\}(t) = \phi_X(t), \quad (53)$$

where $F_X(x)$ is quaternion distribution function of random variable X. Furthermore, application of (34) we easily get

$$\phi(t) = t\,(-\mu)\mathcal{F}_\mu\{F_X\}(t), \quad (54)$$

and thus

$$\mathcal{F}_\mu\{F_X\}(t) = t^{-1}\mu\,\phi_X(t). \quad (55)$$

As easy consequence of (55), we obtain the following corollary.

Corollary 4. *Let X be a quaternion random variable. If the composition of two quaternion distribution functions $F_X(x)$ and $G_X(x)$ is given by*

$$H(x) = F_X(x) * G_X(x) = \int_{\mathbb{H}} F(x-y)\tilde{\partial}G(y)\,d^4y \quad (56)$$

the following holds

$$\varphi(t) = \phi_{X-}(t)\psi_X(t)) + \phi_{X+}(t)\psi_X(-t)), \quad (57)$$

where $\phi_X(t)$ and $\psi_X(t)$ are the characteristic functions of the distributions functions $F_X(x)$ and $G_X(x)$, respectively.

Proof: Simple computation shows that

$$\begin{aligned}
&t^{-1}\,\mu\varphi(t) \\
&= \mathcal{F}_\mu\left\{\int_{\mathbb{H}} F(x-y)\tilde{\partial}G(y)\,d^4y\right\}(t) \\
&\overset{(55)}{=} \mathcal{F}_\mu\{F_{X-}\}(t)\mathcal{F}_\mu\{\tilde{\partial}G\}(t) + \mathcal{F}_\mu\{F_{X+}\}(t)\mathcal{F}_\mu\{\tilde{\partial}G\}(-t) \\
&= t^{-1}\mu\,\phi_{X-}(t)\psi_X(t)) + t^{-1}\mu\,\phi_{X+}(t)\psi_X(-t)).
\end{aligned}$$

The proof is complete. ■

From (38) we introduce the nth moment of a quaternion-valued random variable X defined by

$$m_n = E[X^n] = \int_{\mathbb{H}} x^n f_X(x)\,d^4x, \quad n = 1,2,3,\ldots, \quad (58)$$

provided the integral exists. It is obvious that for $n = 1$ in (58) we obtain the first moment m_1 (simply m), which is called the expectation of X. This gives the following result.

Theorem 8. *If X is quaternion random variable, then there exists n-th continuous derivatives for the quaternion characteristic function $\phi_X(t)$ which is given formula*

$$\tilde{\partial}^k\phi_X(t) = \mu^k\int_{\mathbb{H}} x^k e^{\mu t\cdot x}f_X(x)\,d^4x. \quad (59)$$

Moreover,

$$m_k = E[X^k] = (-\mu)^k\tilde{\partial}^k\phi_X(0), \quad k = 1,2,3,\ldots,n. \quad (60)$$

Proof: The proof of this theorem is quite similar to the proof of Theorem 3. ■

Definition 7. *Let X be a any quaternion random variable. The variance of X is defined by*

$$\begin{aligned}
\sigma^2 &= \mathrm{var}(X) \\
&= E\left[(X - E[X])\overline{(X-E[X])}\right] \\
&= \left(E[X_a^2] - E[X_a]\right)^2 + \left(E[X_b^2] - E[X_b]\right)^2 + \left(E[X_c^2]\right. \\
&\quad \left.- E[X_c]\right)^2 + \left(E[X_d^2] - E[X_d]\right)^2 \\
&= \mathrm{var}(X_a) + \mathrm{var}(X_b) + \mathrm{var}(X_c) + \mathrm{var}(X_d).
\end{aligned} \quad (61)$$

Next, we can obtain the variance σ^2 of a quaternion random variable in terms of the characteristic function as

$$\begin{aligned}
\sigma^2 &= \int_{\mathbb{H}} (x-m)\overline{(x-m)}f_X(x)\,d^4x \\
&= (-\mu)^2\tilde{\partial}^2\phi_X(0) - \{(-\mu)\tilde{\partial}\phi_X(0)\}^2 \\
&= \{\tilde{\partial}\phi_X(0)\}^2 - \tilde{\partial}^2\phi_X(0).
\end{aligned}$$

Example 1. *Find the moments of the normal distribution defined by the density function (compare to [6])*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)\overline{(x-m)}}{2\sigma^2}}. \quad (62)$$

It follows from (39) that

$$\phi(t) = \int_{\mathbb{H}} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{|x-m|^2}{2\sigma^2}}e^{\mu t\cdot x}\,d^4x.$$

Making the change of variable $x - m = y$ we obtain

$$\begin{aligned}
\phi(t) &= \int_{\mathbb{H}} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{|y|^2}{2\sigma^2}}e^{\mu t\cdot(m+y)}\,d^4y \\
&= \frac{e^{\mu t\cdot m}}{\sigma\sqrt{2\pi}}\int_{\mathbb{H}} e^{-\frac{|y|^2}{2\sigma^2}}e^{\mu t\cdot y}\,d^4y \\
&= \frac{e^{\mu t\cdot m}}{\sigma\sqrt{2\pi}}\sqrt{2\sigma^2\pi}\,e^{-\frac{\sigma^2|t|^2}{2}} \\
&= e^{\mu t\cdot m - \frac{\sigma^2|t|^2}{2}}.
\end{aligned}$$

Therefore,

$$\tilde{\partial}\phi(t) = (\mu m - t\sigma^2)e^{\mu tm - \frac{\sigma^2|t|^2}{2}}. \quad (63)$$

Combining (60) and (63) yields

$$\begin{aligned} m_1 &= (-\mu)\tilde{\partial}\phi(0) \\ &= (-\mu)(\mu m) = m \\ m_2 &= m^2 + \sigma^2 \\ m_3 &= m(m^2 + 3\sigma^2). \end{aligned} \tag{64}$$

This means that the variance of the normal distribution is

$$\sigma^2 = m_2 - m_1^2. \tag{65}$$

V. Conclusion

In this paper, we derived more properties of the QDFT such as the convolution, correlation and product theorems. We presented the probability density function of a quaternion random variable in the framework of quaternion algebra. We studied the application of the QDFT in probability theory and mathematical statistics.

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Mawardi Bahri received his B.S. degree in mathematics from Hasanuddin University, Makassar, Indonesia, in 1997 and his M.S. degree in mathematics from the Bandung Institute of Technology, Bandung, Indonesia, in 2001. He received Ph.D. degree in 2007 from the University of Fukui, Japan. Presently, he is a professor with the Department of Mathematics, Hasanuddin University. His research interests are Clifford algebra and wavelet and Fourier transforms in quaternion algebra.

Amir Kamal Amir received his B.S degree in mathematics from Hasanuddin University, Makassar, Indonesia, in 1990 and his M.Sc. degree in industrial mathematics from Kaiserslautern University, Germany, in 1998. He received Ph.D. degree from the Bandung Institute of Technology, Bandung, Indonesia, in 2011. He is a professor with the Department of Mathematics, Hasanuddin University. His research interests are ring theory specially skew polynomial ring. In recent time, he is also working in skew polynomial ring over quaternion.

Resnawati received her B.S. degree in mathematics from from Hasanuddin University, Makassar, Indonesia, in 2005 and her M.S. degree in mathematics from Hasanuddin University, Makassar, Indonesia, in 2013. Her research interests are in the areas of the quaternion Fourier transform, quaternion linear canonical transform, and time-frequency signal processing.

Chrisandi Lande received his B.S. degree in mathematics from Hasanuddin University, Makassar, Indonesia, in 2008 and his M.S. degree in mathematics from Hasanuddin University, Makassar, Indonesia, in 2017. His research interests are in the areas of the linear canonical transform, Stockwell transform, and time-frequency signal processing.