On the Asymptotic Behaviour of the $M/G/1$ Retrial Queue With Priority Customers, Bernoulli Schedule and General Retrial Times

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Abstract—**In this work, we are interested in $M/G/1$ retrial queue with priority customers, Bernoulli schedule, FCFS orbit and general retrial times, which stochastic analysis have been performed in the literature. Our contribution consists in the study of the asymptotic behaviour of the number of customers in the retrial group under heavy traffic and also low retrial rate. We present some numerical illustrations to support the obtained theoretical results.**

Index Terms—**Retrial queue with priority, General retrial times, retrial group, waiting line, Asymptotic Behavior.**

I. Introduction

Retrial queues are characterized by the feature that arriving customers, who find the service area busy, join the retrial group (orbit) and reply for service at random intervals. The models in question arise in the analysis of different communication systems: cellular mobile networks, IP networks, LAN operating under transmission protocols like CSMA/CD, ... There is an extensive literature on retrial queues and their applications. For an accessible bibliography on this topic, we refer to [3], [4], [21].

Retrial queueing systems can be classified into two categories according to the number of different customers: models with single type of customers and models with several types of customers (these latter's can include the priority phenomenon). Several authors have studied retrial queues with priorities. A review of the main results exists in [2], [11]. In related bibliography, it is frequently assumed that the high priority customers are queued and served according to some discipline whereas low priority ones (in case of blocking) leave the service area and retry until they find the server idle. Moreover, the high priority customers have either preemptive or non-preemptive priority over low priority customers [5], [8], [9], [14]. In recent contributions, retrial queueing systems with priority mechanism are analyzed in various combinations [13], [18], [19], [23].

The greater part of the research on retrial queues is based on the fact that the retrials operate under classical retrial policy, which suppose that the intervals between successive repeated attempts are exponentially distributed with total rate $j\theta$ (j is the number of customers in the retrial group). There is another discipline, called constant retrial policy, where the total retrial rate does not depend on the number of customers in the retrial group: the customers form a queue and only the customer at the head of this queue can request a service. The discipline in question was introduced by Fayolle [15] for exponentially distributed retrial times and studied by Gomez-Corral [16] for single server queue with general retrial and service times. The recent contributions on this topic include [6], [10], [12], [20], [22], [24]. The stability of single server retrial queues under general distribution for retrial times was discussed in [17].

In this work, we consider a single server queueing system at which primary customers arrive according to a Poisson process with rate $\lambda > 0$. An arriving customer receives an immediate service if the server is idle; otherwise he decides either to enter the orbit with probability p or to join the waiting space (of infinite capacity) with probability $(1-p)$. We assume that only the customer at the head of the orbit is allowed for access to the server. The orbiting customer in question will repeatedly retry until the time at which he finds the server idle and starts his service. There is a retrial queue with FCFS orbit. The retrial times follow a general distribution with distribution function $A(x)$, Laplace-Stieltjes transform $\tilde{A}(s) = \int_0^\infty e^{-sx}dA(x), Re(s) > 0$ and k^{th} moment $\alpha_k = (-1)^k\tilde{A}^{(k)}(0)$. It is clear that an orbiting customer can be admitted for service only if the waiting space is empty. Thus, the queued customers (which are served in FCFS discipline) have non preemptive priority over those in the orbit. At any service completion, the server becomes idle only when the priority queue is empty and a competition between an exponential distribution of rate λ and general retrial time distribution of rate $\theta = \frac{1}{\alpha_1}$ determines the next customer to be served. The service times follow a general distribution with distribution function $B(x)$ and Laplace-Stieltjes transform $\tilde{B}(s) = \int_0^\infty e^{-sx}dB(x), Re(s) > 0$. Let $\beta_k = (-1)^k\tilde{B}^{(k)}(0)$ be the k^{th} moment of the service time about the origin, $\rho = \lambda\beta_1$ be the load of the system, $k_{ij} = \int_0^\infty \frac{(\lambda(1-p)x)^i}{i!}\frac{(\lambda px)^j}{j!}e^{-\lambda x}dB(x)$ be the joint distribution of the number of customers arriving at the priority queue and the orbit during a service time with generating function

$$K(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{ij}z^iz^j = \tilde{B}(\lambda - \lambda(1-p)z_1 - \lambda pz_2).$$

Finally, we admit the hypothesis of mutual independence between all random variables defined above.

The state of the system at time t can be described by means of the process

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$$\{C(t), N_q(t), N_o(t), \xi_1(t), \xi_2(t), t \geq 0\}, \quad (1)$$

where $N_q(t)$ and $N_o(t)$ are the numbers of customers in the priority queue and in the orbit, respectively; $C(t)$ is the state of the server at time t. We have that $C(t)$ is 0 or 1 depending on whether the server is idle or busy. If $C(t)=0$, $N_q(t)=0$ and $N_o(t)>0$, then $\xi_1(t)$ represents the elapsed retrial time of the customer at the head of the orbit (at time t). If $C(t)=1, \xi_2(t)$ is the elapsed service time of the customer in service at time t.

Stochastic analysis of this queueing model was performed in [7]. The authors studied the ergodicity of the associated embedded Markov chain and obtained its stationary distribution. By using the method of supplementary variables, they found the partial generating functions of the steady state distribution. These generating functions permitted to have the generating functions of the steady state system state distribution of the number of customers in the priority queue, in the orbit and in the system. The analysis in question included also the investigations on the stochastic decomposition property, convergence to a two-level priority queue without retrials as well as on the optimal control of the priority policy. Our contribution consists in the study of the asymptotic behavior of the number of customers in the retrial group under heavy traffic ($\rho \to \frac{\tilde{A}(\lambda)}{p+(1-p)\tilde{A}(\lambda)}$) and also under low retrial rate $\left(\tilde{A}(\lambda) \to 0\right)$. We present some numerical illustrations to support the obtained theoretical results.

The paper is organized as follows. In the next section, we review some existing results (ergodicity condition, steady state system state distribution, generating function of the steady state distribution of the number of customers in the orbit), which are related to the considered model. Section 3 deals with asymptotic analysis of the number of customers in the orbit in the case of low retrial rate and high traffic intensity. In section 4, we present numerical illustrations and their explanation.

II. Stochastic analysis

In this section, we present some results, given in [7] which are necessary for our investigation on the asymptotic behaviour of the retrial queueing system under study. In the first time, consider the associated embedded Markov chain. Let ζ_n be the time of the n^{th} departure, $N_{q,n}=N_q(\zeta_n)$ and $N_{o,n}=N_o(\zeta_n)$ be the numbers of customers in the priority queue and in the orbit, respectively, just before the time ζ_n. The sequence of random vectors $\{N_{q,n}, N_{o,n}, n \geq 1\}$ forms a Markov chain which is the embedded Markov chain for the considered system. Its state space is $S = Z_+ \times Z_+$ and its fundamental equations are defined as

$$N_{q,n} = \begin{cases} N_{q,n-1} - 1 + v_{q,n} \text{ if } N_{q,n-1} > 0 \\ v_{q,n} \text{ if } N_{q,n-1} = 0 \end{cases}, \quad (2)$$

$$N_{o,n} = \begin{cases} N_{o,n-1} + v_{o,n} \text{ if } N_{q,n-1} \geq 1 \\ N_{o,n-1} - \delta_n + v_{o,n} \text{ if } N_{q,n-1} = 0, N_{o,n-1} \geq 1 \\ v_{o,n} \text{ if } N_{q,n-1} = 0 \text{ and } N_{o,n-1} = 0 \end{cases}, \quad (3)$$

where $v_{q,n}$ and $v_{o,n}$ are the numbers of customers arriving at the waiting space and the orbit during the n^{th} service time; $\delta_{q,n}$ is equal to 0 or 1 depending on whether the n^{th} served customer is the primary one or proceeds from the orbit. With the help of Foster's criterion and Kaplan's condition, it is established that the irreducible and aperiodic Markov chain $\{N_{q,n}, N_{o,n}, n \geq 1\}$ is ergodic if and only if $\rho < \frac{\tilde{A}(\lambda)}{p+(1-p)\tilde{A}(\lambda)}$.

Now consider the random process (1), assume that $\rho < \frac{\tilde{A}(\lambda)}{p+(1-p)\tilde{A}(\lambda)}$ and define the steady state distribution of the system state

$$P_0 = \lim_{t \to \infty} P(C(t), N_q(t)=0, N_o(t)=0);$$

$$P_{0,0,j} = \int_0^\infty \lim_{t \to \infty} \frac{d}{dx} P(C(t)=0, N_q(t)=0, N_o(t)=j, \xi_1(t) \leq x),$$

for $j \geq 1, x \geq 0$;

$$P_{1,i,j} = \int_0^\infty \lim_{t \to \infty} \frac{d}{dx} P(C(t)=1, N_q(t)=i, N_o(t)=j, \xi_2(t) \leq x),$$

for $i \geq 0, j \geq 0, x \geq 0$. The corresponding generating functions are given by

$$P_0(z_2) = \sum_{j=0}^\infty P_{0,0,j} z_2^j = P_0 \frac{z_2 (1-g(z_2)) \left(1-\tilde{A}(\lambda)\right)}{\tilde{A}(\lambda)(1-z_2) g(z_2) - z_2(1-g(z_2))}$$

and

$$\begin{aligned} P_1(z_1,z_2) &= \sum_{i=0}^\infty \sum_{j=0}^\infty P_{1,i,j} z_1^i z_2^j \\ &= p_0 \frac{\tilde{A}(\lambda)(1-z_2) K(z_1,z_2)}{\tilde{A}(\lambda)(1-z_2) g(z_2) - z_2(1-g(z_2))} \\ &\times \frac{z_1 - g(z_2)}{z_1 - K(z_1,z_2)} \frac{1-K(z_1,z_2)}{1-(1-p)z_1 - p z_2} \end{aligned}$$

with $p_0 = 1 - \frac{p+(1-p)\tilde{A}(\lambda)}{\tilde{A}(\lambda)} \rho$ and $g(z_2)$ as the unique root of z_1 of the equation $z_1 - K(z_1,z_2) = z_1 - \tilde{B}(\lambda - \lambda(1-p)z_1 - \lambda p z_2) = 0$ (see also [14]).

At present, the generating function of the steady state distribution of the number of customers in the orbit, $P(z)$, can be expressed in the following manner

$$P(z) = p_0 \times \frac{\tilde{A}(\lambda)}{p} \times \frac{1-(1-p)g(z) - pz}{\tilde{A}(\lambda)(1-z)g(z) - z(1-g(z))}. \quad (4)$$

From (2), we can see that $P(z)$ does not reveal the nature of the distribution in question. That is the reason to perform the asymptotic analysis of the orbit length under limit values of some parameters. We note that the asymptotical behaviour of the random variable representing the number of customers in the retrial group was early studied for $M/G/1$ retrial queue [14] and also $M^X/G/1$ retrial queue with impatient customers [1]. Here, the systems operated under classical retrial policy with exponential retrial times.

III. Asymptotic behaviour of the orbit length

A. Heavy traffic

First consider the case of heavy traffic when arrival rate increases in such a way that $\rho \to \frac{\tilde{A}(\lambda)}{p+(1-p)\tilde{A}(\lambda)}$.

Theorem 1. *If M/G/1 retrial queue, priority customers, FCFS orbit and general retrial times is in the steady state, and $\beta_2 < \infty$, then*

$$\lim_{z\to 1} P(z) = \frac{1}{1+s\frac{\beta_2}{2\beta_1^2}} \times \frac{1}{1+s\left[\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2} + \frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2}\right]}, \quad (5)$$

i.e. under heavy traffic orbit length $N_o(t)$ has asymptotically a two stage generalized Erlang distribution.

Proof. Let

$$\begin{aligned} P(z) &= \left[1-\frac{p+q\tilde{A}(\lambda)}{\tilde{A}(\lambda)}\rho\right] \times \frac{\tilde{A}(\lambda)}{p} \\ &\times \frac{1-qg(z)-pz}{\tilde{A}(\lambda)(1-z)g(z)-z(1-g(z))}. \end{aligned}$$

By using the stochastic decomposition property of the model [7], we can rewrite $P(z)$ in the following manner:

$$P(z) = X(z) \times Y(z), \quad (6)$$

where $X(z) = \frac{(1-\lambda\beta_1)(1-qg(z)-pz)}{(g(z)-z)p}$ is the generating function for the number of customers in the waiting space of the two-level priority queue without retrials and

$$\begin{aligned} Y(z) &= \frac{1}{(1-\lambda\beta_1)} \times \left[1-\frac{p+q\tilde{A}(\lambda)}{\tilde{A}(\lambda)}\rho\right] \times \tilde{A}(\lambda) \\ &\times \frac{g(z)-z}{\tilde{A}(\lambda)(1-z)g(z)-z(1-g(z))} \end{aligned}$$

is the generating function of the number of customers in the orbit given that the server is idle. Then

$$\lim_{z\to 1} P(z) = \lim_{z\to 1} X(z) \times \lim_{z\to 1} Y(z).$$

In the first time, we calculate

$$\lim_{z\to 1} X(z) = \frac{(1-\lambda\beta_1)(1-qg(z)-pz)}{(g(z)-z)p}.$$

Assume that $\varepsilon = (1-\rho) \to 0$ and $z = e^{-\varepsilon s}$. Thus $\varepsilon = (1-\rho) = 1-\lambda\beta_1$ and $\lambda = \frac{1}{\beta_1}(1-\varepsilon)$. Under this assumption, we have

$$\lim_{\varepsilon\to 0} X(e^{-\varepsilon s}) = \lim_{\varepsilon\to 0} \frac{\varepsilon(1-qg(e^{-\varepsilon s})-pe^{-\varepsilon s})}{(g(e^{-\varepsilon s})-e^{-\varepsilon s})p}.$$

Let $g(e^{-\varepsilon s}) = g(z)$, where $z = e^{-\varepsilon s}$ and $e^{-\varepsilon s} = 1-\varepsilon s + \frac{\varepsilon^2 s^2}{2} + o(\varepsilon^2)$. One can develop $g(z)$ in the following manner:

$$g(u) = g(1+(u-1)) = g(1)+(u-1)g^{'}(1)+\frac{1}{2}(u-1)^2 g^{''}(1),$$

this yields

$$g(e^{-\varepsilon s}) = 1-\varepsilon s g^{'}(1)+\frac{\varepsilon^2 s^2}{2}g^{'}(1)+\frac{\varepsilon^2 s^2}{2}g^{''}(1)+o\left(\varepsilon^2\right).$$

We have also that $g(1) = 1; g^{'}(1) = \frac{\rho p}{1-\rho q}; g^{''}(1) = \frac{\lambda^2 p^2 \beta_2}{(1-\rho q)^3}$. Therefore,

$$\begin{aligned} g(e^{-\varepsilon s}) &= 1-\frac{\varepsilon s p}{(1-(1-\varepsilon)q)}+\frac{\varepsilon^2 s p}{(1-(1-\varepsilon)q)} \\ &+ \frac{\varepsilon^2 s^2}{2}\frac{p}{(1-(1-\varepsilon)q)} \\ &+ \frac{\varepsilon^2 s^2}{2}\frac{(1-\varepsilon)^2\frac{\beta_2}{\beta_1^2}p^2}{(1-(1-\varepsilon)q)^3}+o\left(\varepsilon^2\right) \\ &= 1+\frac{1}{(1-(1-\varepsilon)q)^3} \\ &\times \left[-\varepsilon s p^3+\varepsilon^2 s p^2(p-2q)+\frac{\varepsilon^2 s^2}{2}p^2(p+\frac{\beta_2}{\beta_1^2})\right] \\ &+ o\left(\varepsilon^2\right). \end{aligned}$$

Taking into account that $\beta_2 < \infty$, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} X(e^{-\varepsilon s}) &= \lim_{\varepsilon\to 0} \frac{\varepsilon(1-qg(e^{-\varepsilon s})-pe^{-\varepsilon s})}{(g(e^{-\varepsilon s})-e^{-\varepsilon s})p} \\ &= \lim_{\varepsilon\to 0} \frac{\varepsilon^2 s q p^3+\varepsilon^2 s p^4}{\varepsilon^2 s p^2(p-2q+3q)+\frac{\varepsilon^2 s^2}{2}p^3(p+\frac{\beta_2}{\beta_1^2}-p)} \\ &= \lim_{\varepsilon\to 0} \frac{\varepsilon^2 s p^3}{\varepsilon^2 s p^3+\frac{\varepsilon^2 s^2}{2}p^3\frac{\beta_2}{\beta_1^2}} = \frac{1}{1+\frac{s}{2}\frac{\beta_2}{\beta_1^2}}. \end{aligned}$$

Now we calculate $\lim_{z\to 1} Y(z)$. In this case $\rho = \lambda\beta_1 < \frac{\tilde{A}(\lambda)}{p+q\tilde{A}(\lambda)}$. Assuming that $\varepsilon = \frac{\tilde{A}(\lambda)}{p+q\tilde{A}(\lambda)} - \rho \to 0$ and $z = e^{-\varepsilon s}$, thus $\varepsilon = \frac{\tilde{A}(\lambda)}{p+q\tilde{A}(\lambda)} - \lambda\beta_1$ and $\lambda = \frac{1}{\beta_1}\left(\frac{\tilde{A}(\lambda)}{p+q\tilde{A}(\lambda)}-\varepsilon\right)$. Define $\eta_1 = \frac{\tilde{A}(\lambda)}{p+q\tilde{A}(\lambda)}$. So $\lambda = \frac{1}{\beta_1}(\eta_1-\varepsilon)$ and $g(e^{-\varepsilon s})$ becomes

$$\begin{aligned} g(e^{-\varepsilon s}) &= 1-\frac{\varepsilon s(\eta_1-\varepsilon)p}{(1-(\eta_1-\varepsilon)q)}+\frac{\varepsilon^2 s^2}{2}\frac{(\eta_1-\varepsilon)p}{(1-(\eta_1-\varepsilon)q)} \\ &+ \frac{\varepsilon^2 s^2}{2}\frac{(\eta_1-\varepsilon)^2\frac{\beta_2}{\beta_1^2}p^2}{(1-(\eta_1-\varepsilon)q)^3}+o\left(\varepsilon^2\right) \\ &= 1+\frac{1}{(1-(\eta_1-\varepsilon)q)^3} \\ &\times \left(-\varepsilon s\eta_1 p(1-q\eta_1)^2+\varepsilon^2 s p(1-q\eta_1)(1-3q\eta_1)\right. \\ &+ \left.\frac{\varepsilon^2 s^2}{2}\eta_1 p((1-q\eta_1)^2+\eta_1\frac{\beta_2}{\beta_1^2}p)\right)+o\left(\varepsilon^2\right). \end{aligned}$$

Then

$$\begin{aligned} \lim_{z\to 1} Y(z) &= \lim_{\varepsilon\to 0} Y(e^{-\varepsilon s}) \\ &= \lim_{\varepsilon\to 0} \frac{\tilde{A}(\lambda)}{((1-\eta_1)+\varepsilon)}\left[1-\frac{p+q\tilde{A}(\lambda)}{\tilde{A}(\lambda)}(\eta_1-\varepsilon)\right] \\ &\times \frac{g(e^{-\varepsilon s})-e^{-\varepsilon s}}{\tilde{A}(\lambda)(1-e^{-\varepsilon s})g(e^{-\varepsilon s})-e^{-\varepsilon s}(1-g(e^{-\varepsilon s}))}. \end{aligned}$$

Taking into account that $\beta_2 < \infty$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} Y(e^{-\varepsilon s}) = \lim_{\varepsilon\to 0} \frac{1}{((1-\eta_1)+\varepsilon)} \\
\times \quad & \frac{\left(p+q\tilde{A}(\lambda)\right)\varepsilon\left(g(e^{-\varepsilon s})-e^{-\varepsilon s}\right)}{\tilde{A}(\lambda)\left(1-e^{-\varepsilon s}\right)g\left(e^{-\varepsilon s}\right)-e^{-\varepsilon s}\left(1-g(e^{-\varepsilon s})\right)} \\
= \quad & \lim_{\varepsilon\to 0} \frac{Y_1(e^{-\varepsilon s})}{Y_2(e^{-\varepsilon s})},
\end{aligned}$$

where

$$Y_1(e^{-\varepsilon s}) = \left(p+q\tilde{A}(\lambda)\right)\times\varepsilon\times\left(g(e^{-\varepsilon s})-e^{-\varepsilon s}\right)$$

and

$$\begin{aligned}
Y_2(e^{-\varepsilon s}) \quad = \quad & ((1-\eta_1)+\varepsilon)\tilde{A}(\lambda)\left(1-e^{-\varepsilon s}\right)g\left(e^{-\varepsilon s}\right) \\
- \quad & ((1-\eta_1)+\varepsilon)e^{-\varepsilon s}\left(1-g(e^{-\varepsilon s})\right).
\end{aligned}$$

The function $Y_1(e^{-\varepsilon s})$ can be expressed in the following manner

$$\begin{aligned}
Y_1(e^{-\varepsilon s}) \quad = \quad & \left(p+q\tilde{A}(\lambda)\right)\times\varepsilon\times\left(g(e^{-\varepsilon s})-e^{-\varepsilon s}\right) \\
= \quad & \left(p+q\tilde{A}(\lambda)\right)\frac{\varepsilon}{(1-(\eta_1-\varepsilon)q)^3} \\
\times \quad & (-\varepsilon s\eta_1 p(1-q\eta_1)^2 \\
+ \quad & \varepsilon^2 sp(1-q\eta_1)(1-3q\eta_1) \\
+ \quad & \frac{\varepsilon^2 s^2}{2}\eta_1 p((1-q\eta_1)^2+\eta_1\frac{\beta_2}{\beta_1^2}p) \\
+ \quad & \varepsilon s - \frac{\varepsilon^2 s^2}{2}) + o\left(\varepsilon^2\right) \\
= \quad & \left(p+q\tilde{A}(\lambda)\right) \\
\times \quad & \frac{-\varepsilon^2 s\eta_1 p(1-q\eta_1)^2+\varepsilon^2 s(1-q\eta_1)^3}{(1-(\eta_1-\varepsilon)q)^3}+o\left(\varepsilon^2\right) \\
= \quad & \frac{\left(p+q\tilde{A}(\lambda)\right)}{(1-(\eta_1-\varepsilon)q)^3} \\
\times \quad & \left[\varepsilon^2 s(1-q\eta_1)^2(1-q\eta_1-\eta_1 p)\right].
\end{aligned}$$

Then

$$Y_1(e^{-\varepsilon s}) = \frac{1}{(1-(\eta_1-\varepsilon)q)^3}\varepsilon^2 s\frac{p^3\left(1-\tilde{A}(\lambda)\right)}{\left(p+q\tilde{A}(\lambda)\right)^2}.$$

Now consider $Y_2(e^{-\varepsilon s})$ and assume that $Y_2(e^{-\varepsilon s})$=$l_1(e^{-\varepsilon s})$+$l_2(e^{-\varepsilon s})$, where

$$l_1(e^{-\varepsilon s}) = ((1-\eta_1)+\varepsilon)\tilde{A}(\lambda)\left(1-e^{-\varepsilon s}\right)g\left(e^{-\varepsilon s}\right)$$

and

$$l_2(e^{-\varepsilon s}) = -((1-\eta_1)+\varepsilon)e^{-\varepsilon s}\left(1-g(e^{-\varepsilon s})\right).$$

We transform the function $l_1(\varepsilon)$ as follows

$$\begin{aligned}
&l_1(e^{-\varepsilon s}) = ((1-\eta_1)+\varepsilon)\tilde{A}(\lambda)\left(1-e^{-\varepsilon s}\right)g\left(e^{-\varepsilon s}\right) \\
+ \quad & o(\varepsilon^2) \\
= \quad & \frac{1}{(1-(\eta_1-\varepsilon)q)^3}\Big(\varepsilon s\tilde{A}(\lambda)(1-\eta_1)(1-q\eta_1)^3 \\
+ \quad & \varepsilon^2 s\tilde{A}(\lambda)(1-q\eta_1)^2(3q(1-\eta_1)+(1-q\eta_1)) \\
- \quad & \frac{\varepsilon^2 s^2}{2}\tilde{A}(\lambda)(1-\eta_1)(1-q\eta_1)^2(1-q\eta_1+2\eta_1 p)\Big),
\end{aligned}$$

and the function $l_2(e^{-\varepsilon s})$ as follows

$$\begin{aligned}
&l_2(e^{-\varepsilon s}) = \frac{1}{(1-(\eta_1-\varepsilon)q)^3} \\
\times \quad & \Big(-\varepsilon sp\eta_1(1-\eta_1)(1-q\eta_1)^2 \\
+ \quad & \varepsilon^2 sp(1-q\eta_1)[(1-\eta_1)(1-3q\eta_1)-\eta_1(1-q\eta_1)] \\
+ \quad & \frac{\varepsilon^2 s^2}{2}\eta_1 p(1-\eta_1)\left[3(1-q\eta_1)^2+\eta_1 p\frac{\beta_2}{\beta_1^2}\right]\Big).
\end{aligned}$$

So,

$$\begin{aligned}
Y_2(e^{-\varepsilon s}) \quad = \quad & \frac{1}{(1-(\eta_1-\varepsilon)q)^3}\times\Bigg\{\varepsilon^2 s\frac{p^3\left(1-\tilde{A}(\lambda)\right)}{\left(p+q\tilde{A}(\lambda)\right)} \\
+ \quad & \frac{\varepsilon^2 s^2}{2}\Bigg(\frac{2p^4\tilde{A}(\lambda)\left(1-\tilde{A}(\lambda)\right)^2}{\left(p+q\tilde{A}(\lambda)\right)^4} \\
+ \quad & \frac{p^3\left(\tilde{A}(\lambda)\right)^2\left(1-\tilde{A}(\lambda)\right)\frac{\beta_2}{\beta_1^2}}{\left(p+q\tilde{A}(\lambda)\right)^3}\Bigg)\Bigg\}.
\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0} Y(e^{-\varepsilon s}) = \frac{1}{1+s\left[\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2}+\frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2}\right]}.$$

Finally, we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0} P(e^{-\varepsilon s}) = \lim_{\varepsilon\to 0} X(e^{-\varepsilon s})\times\lim_{\varepsilon\to 0} Y(e^{-\varepsilon s}) \\
= \quad & \frac{1}{1+\frac{s}{2}\frac{\beta_2}{\beta_1^2}}\times\frac{1}{1+s\left[\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2}+\frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2}\right]},
\end{aligned}$$

it is the product of two Laplace-Stieljes transforms of two exponential variables with different rates $\lambda_1 = \frac{\beta_2}{2\beta_1^2}$ and $\lambda_2 = \left[\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2}+\frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2}\right]$ which corresponds to a two-stage generalized Erlang's distribution (or two stage Hypo-exponential distribution) with mean $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$. □

B. Low rate of retrials

Now consider the case of low rate retrials, that is, as $\tilde{A}(\lambda)\to 0$. The following theorem describes the orbit length distribution.

Theorem 2. *If $\beta_2 < \infty$, then as $\tilde{A}(\lambda) \to 0$ the number of customers in the orbit is asymptotically Gaussian with mean* $\frac{p(1-\tilde{A}(\lambda))\left(\rho+\frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right)}{\tilde{A}(\lambda)-[p+q\tilde{A}(\lambda)]\rho}$ *and variance* $\frac{p\lambda}{(1-\rho^*)}\left(\beta_1+\frac{\lambda p\beta_2}{2(1-\rho)(1-pq)}\right)$, *where* $\rho^* = \frac{p+q\tilde{A}(\lambda)}{\tilde{A}(\lambda)}\rho$ *and* $\rho = \lambda\beta_1$.

Proof. Consider $\bar{n}_o = P'(1)$, that is the mean number of customers in the orbit. According to the stochastic decomposition property of the considered model [6], $\bar{n}_o = \bar{n}_\infty + \bar{n}_o^*$, where $\bar{n}_\infty = \frac{\lambda^2\beta_2}{2(1-\rho)}$ is the mean number of waiting customers in the ordinary two-level priority queue without retrials and

$$\bar{n}_o^* = \frac{P_0'(1)}{1-\rho} = \frac{p\left(1-\tilde{A}(\lambda)\right)}{\tilde{A}(\lambda)-\left[p+q\tilde{A}(\lambda)\right]\rho}\left(\rho+\frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right)$$

represents the mean number of customers in the orbit given that the server is idle.

Assume that $\nu = p\rho + \frac{\lambda^2 p^2\beta_2}{2(1-\rho)(1-\rho q)}$ and $\rho^* = \frac{p+q\tilde{A}(\lambda)}{\tilde{A}(\lambda)}\rho$. At present consider

$$\begin{aligned}\bar{n}^* &= \frac{\bar{n}_o-\bar{n}_o^*}{\frac{1}{\sqrt{\tilde{A}(\lambda)}}} = \frac{\bar{n}_o-\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)-[p+q\tilde{A}(\lambda)]\rho}\nu}{\frac{1}{\sqrt{\tilde{A}(\lambda)}}}\\ &= \sqrt{\tilde{A}(\lambda)}\times\bar{n}_o-\sqrt{\tilde{A}(\lambda)}\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)-\left[p+q\tilde{A}(\lambda)\right]\rho}\nu.\end{aligned}$$

The characteristic function $\mathbb{E}[\exp(it\bar{n}^*)]$ can be given in terms of the generating function $P(z)$ as follows:

$$\begin{aligned}\mathbb{E}[\exp(it\bar{n}^*)] &= P\left(e^{it\sqrt{\tilde{A}(\lambda)}}\right)\\ &\quad \exp\left\{-it\sqrt{\tilde{A}(\lambda)}\times\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\},\end{aligned}$$

where

$$P\left(e^{it\sqrt{\tilde{A}(\lambda)}}\right) = X(e^{it\sqrt{\tilde{A}(\lambda)}})\times Y(e^{it\sqrt{\tilde{A}(\lambda)}}),$$

with

$$X(e^{it\sqrt{\theta}}) = \frac{(1-\rho)\left(1-qg(e^{it\sqrt{\tilde{A}(\lambda)}})-pe^{it\sqrt{\tilde{A}(\lambda)}}\right)}{\left(g(e^{it\sqrt{\tilde{A}(\lambda)}})-e^{it\sqrt{\tilde{A}(\lambda)}}\right)p}$$

and

$$\begin{aligned}Y(e^{it\sqrt{\tilde{A}(\lambda)}}) &= \frac{1}{(1-\rho)}\\ \times\ & \frac{(1-\rho^*)\tilde{A}(\lambda)\times\left(g(e^{it\sqrt{\tilde{A}(\lambda)}})-e^{it\sqrt{\tilde{A}(\lambda)}}\right)}{\tilde{A}(\lambda)\left(1-e^{it\sqrt{\tilde{A}(\lambda)}}\right)g(e^{it})-e^{it\sqrt{\tilde{A}(\lambda)}}\left(1-g(e^{it\sqrt{\tilde{A}(\lambda)}})\right)}.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[\exp(it\bar{n}^*)] &= \lim_{\theta\to 0} P\left(e^{it\sqrt{\tilde{A}(\lambda)}}\right)\\ &\quad \exp\left\{-it\sqrt{\tilde{A}(\lambda)}\times\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}\\ &= \lim_{\theta\to 0} X(e^{it\sqrt{\tilde{A}(\lambda)}})\\ &\times \lim_{\theta\to 0} Y(e^{it\sqrt{\tilde{A}(\lambda)}})\\ &\quad \exp\left\{-it\sqrt{\tilde{A}(\lambda)}\times\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}.\end{aligned}$$

We have that

$$\lim_{\theta\to 0} X(e^{it\sqrt{\tilde{A}(\lambda)}}) = \frac{(1-\rho)\left(1-qg(e^{it\sqrt{\tilde{A}(\lambda)}})-pe^{it\sqrt{\tilde{A}(\lambda)}}\right)}{\left(g(e^{it\sqrt{\tilde{A}(\lambda)}})-e^{it\sqrt{\tilde{A}(\lambda)}}\right)p} \to 1$$

and

$$\begin{aligned}&\lim_{\theta\to 0} Y(e^{it\sqrt{\tilde{A}(\lambda)}})\exp\left\{-it\sqrt{\tilde{A}(\lambda)}\times\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}\\ =\ & \lim_{\theta\to 0} Y(e^{it\sqrt{\tilde{A}(\lambda)}})\exp\left\{-\left(e^{it\sqrt{\tilde{A}(\lambda)}}-1\right)\frac{(1-\tilde{A}(\lambda))\nu}{\tilde{A}(\lambda)(1-\rho^*)}\right\}\\ \times\ & \exp\left\{\left(\left(e^{it\sqrt{\tilde{A}(\lambda)}}-1\right)-it\sqrt{\tilde{A}(\lambda)}\right)\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}.\end{aligned}$$

Let

$$\begin{aligned}f_1\left(\tilde{A}(\lambda)\right) &= \lim_{\theta\to 0}\exp\left\{\left(\left(e^{it\sqrt{\tilde{A}(\lambda)}}-1\right)-it\sqrt{\tilde{A}(\lambda)}\right)\right.\\ &\quad\times \left.\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}\end{aligned}$$

and

$$\begin{aligned}f_2\left(\tilde{A}(\lambda)\right) &= \lim_{\theta\to 0} Y(e^{it\sqrt{\tilde{A}(\lambda)}})\times\exp\left\{-\left(e^{it\sqrt{\tilde{A}(\lambda)}}-1\right)\right.\\ &\quad\times\left.\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right\}.\end{aligned}$$

We have $e^{it\sqrt{\tilde{A}(\lambda)}} = 1+it\sqrt{\tilde{A}(\lambda)}-\frac{t^2}{2}\tilde{A}(\lambda)+o(t^2)$. So

$$\begin{aligned}f_1\left(\tilde{A}(\lambda)\right) &= \lim_{\theta\to 0}\exp\left\{\frac{1-\tilde{A}(\lambda)}{\tilde{A}(\lambda)(1-\rho^*)}\nu\right.\\ \times\ & \left.\left(1+it\sqrt{\tilde{A}(\lambda)}-\frac{t^2}{2}\tilde{A}(\lambda)-1-it\sqrt{\tilde{A}(\lambda)}\right)\right\}\\ =\ & \exp\left\{-\frac{t^2}{2}\frac{\nu}{(1-\rho^*)}\right\}\\ =\ & \exp\left\{-\frac{t^2}{2}\frac{1}{(1-\rho^*)}\left(p\rho+\frac{\lambda^2p^2\beta_2}{2(1-\rho)(1-\rho q)}\right)\right\}\end{aligned}$$

and with the help of the Hospital rule, we find that

$$f_2\left(\tilde{A}(\lambda)\right) = \lim_{\theta\to 0}\frac{1}{(1-\rho)}\times\frac{-p\rho\times(-g'(1)+1)}{-g'(1)},$$

where $g'(1) = \frac{p\rho}{1-q\rho}$. So

$$\begin{aligned}f_2\left(\tilde{A}(\lambda)\right) &= \frac{1}{(1-\rho)}\frac{-p\rho\times(-g'(1)+1)}{-g'(1)}\\ &= \frac{1}{(1-\rho)}\frac{p\rho(-p\rho+1-\rho+p\rho)}{1-q\rho}\frac{1-q\rho}{p\rho} = 1.\end{aligned}$$

Finally,

$$\mathbb{E}\left[e^{it\bar{n}^*}\right] = \exp\left\{-\frac{t^2}{2}\frac{1}{(1-\rho^*)}\left(p\rho+\frac{\lambda^2p^2\beta_2}{2(1-\rho)(1-\rho q)}\right)\right\}.$$

This is the characteristic function of Gaussian random variable with mean equal to 0 and variance $\frac{1}{(1-\rho^*)}\left(p\rho+\frac{\lambda^2p^2\beta_2}{2(1-\rho)(1-\rho q)}\right)$. □

IV. Numerical illustrations

This section deals with numerical illustrations to discuss the theoretical results obtained in theorem 1 and 2.

In the first time, consider (5). It is easy to see that $\lim_{z\to 1} P(z)$ is the product of two Laplace-Stieltjes transforms corresponding to the sum of two independent exponential random variables:

$$\tilde{F}(s)=\frac{1}{1+\frac{s}{2}\frac{\beta_2}{\beta_1^2}}\times\frac{1}{1+s\left[\frac{p(1-\tilde{A}(\lambda))\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2}+\frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2}\right]}.$$

The mean of random variable having the Laplace-Stieltjes transforms $\tilde{F}(s)$ is $m_1=(-1)\tilde{F}'(s=0)$. In fact,

$$m_1=\frac{\beta_2}{2\beta_1^2}\left[1+\frac{\left(\tilde{A}(\lambda)\right)^2}{\left(p+q\tilde{A}(\lambda)\right)}\right]+\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2}.$$

To illustrate the behaviour of the mean number of the customers in the orbit, under heavy traffic, we assume that the service rate $\gamma=\frac{1}{\beta_1}=1$ and consider the following service time distributions:

- Exponential $(E): B_E(x)=1-e^{-\gamma x}, x\geq 0$, with $\beta_1=1$ and $\beta_2=2$;
- Two stage Erlang $(E_2): B_{E_2}(x)=1-e^{-2\gamma x}-2\gamma e^{-2\gamma x}, x\geq 0$, with $\beta_1=1$ and $\beta_2=1.5$.
  Thus, we have $M/M/1$ and $M/E_2/1$ retrial models, respectively.

For retrial times, we choose:

1) Exponential $(E): A_E(x)=1-e^{-\theta x}, x\geq 0$;
2) Two stage Erlang $(E_2): A_{E_2}(x)=1-e^{-2\theta x}-2\theta xe^{-2\theta x}, x\geq 0$;
3) Two stage Hyperexponentiel $(H_2): A_{H_2}(x)=1-\zeta_1e^{-\theta_1 x}-\zeta_2e^{-\theta_2 x}$, $x\geq 0, \frac{\zeta_1}{\theta_1}+\frac{\zeta_2}{\theta_2}=\frac{1}{\theta}, \zeta_1+\zeta_2=1, \theta_1=2\zeta_1\theta, \theta_2=2\zeta_2\theta$.

Table I presents the numerical values of the traffic intensity ρ^* calculated according to $\rho^*=\frac{\left(p+(1-p)\tilde{A}(\lambda)\right)}{\tilde{A}(\lambda)}\times\lambda\beta_1$ when the arrival rate λ, the probability p and the retrial time distribution are varied. We have assumed that the retrial intensity $\theta=2$. Since the service rate $\gamma=\frac{1}{\beta_1}=1$, we obtain the same values of ρ^* for the two considered models.

Fig.1-3 show the asymptotic behaviour of the mean number of customers in the orbit, m_1, with respect to the arrival rate λ for the $M/M/1$ retrial model; whereas Fig.4-6 are concerned by the $M/E_2/1$ retrial model. The numerical results are obtained for different values of the probability p ($p=0.25$, $p=0.5$ and $p=0.75$) as well as for above mentioned retrial time distributions : exponential (solid curve), two-stage Erlang (dash curve) and two-stage hyperexponential (dots curve). It is easy to see that the probability p has little effect on the numerical values of m_1.

TABLE I: Traffic intensity ρ^* for $M/M/1$ and $M/E_2/1$ retrial models.

$\theta=2$		*Retrial time E*		*Retrial time E_2*		*Retrial time H_2*	
p	λ	$\tilde{A}_E(\lambda)$	ρ^*	$\tilde{A}_{E_2}(\lambda)$	ρ^*	$\tilde{A}_{H_2}(\lambda)$	ρ^*
$\frac{1}{4}$	0.1	0.9524	0.1013	0.9518	0.1013	0.9531	0.1012
	0.2	0.9091	0.2050	0.9070	0.2051	0.9128	0.2048
	0.3	0.8696	0.3113	0.8653	0.3117	0.8771	0.3105
	0.4	0.8333	0.4200	0.8265	0.4210	0.8450	0.4183
	0.5	0.8000	0.5313	0.7901	0.5332	0.8159	0.5282
	0.6	0.7692	0.6450	0.7561	0.6484	0.7892	0.6401
	0.7	0.7407	0.7613	0.7243	0.7666	0.7646	0.7539
	0.8	0.7143	0.8800	0.6944	0.8880	0.7418	0.8696
$\frac{1}{2}$	0.1	0.9524	0.1025	0.9518	0.1025	0.9531	0.1025
	0.2	0.9091	0.2100	0.9070	0.2103	0.9128	0.2096
	0.3	0.8696	0.3225	0.8653	0.3233	0.8771	0.3210
	0.4	0.8333	0.4400	0.8265	0.4420	0.8450	0.4367
	0.5	0.8000	0.5625	0.7901	0.5664	0.8159	0.5564
	0.6	0.7692	0.6900	0.7561	0.6968	0.7892	0.6801
	0.7	0.7407	0.8225	0.7243	0.8332	0.7646	0.8077
	0.8	0.7143	0.9600	0.6944	0.9760	0.7418	0.9392
$\frac{3}{4}$	0.1	0.9524	0.1038	0.9518	0.1038	0.9531	0.1037
	0.2	0.9091	0.2150	0.9070	0.2154	0.9128	0.2143
	0.3	0.8696	0.3337	0.8653	0.3350	0.8771	0.3315
	0.4	0.8333	0.4600	0.8265	0.4630	0.8450	0.4550
	0.5	0.8000	0.5936	0.7901	0.5996	0.8159	0.5846
	0.6	0.7692	0.7350	0.7561	0.7451	0.7892	0.7202
	0.7	0.7407	0.8838	0.7243	0.8998	0.7646	0.8616

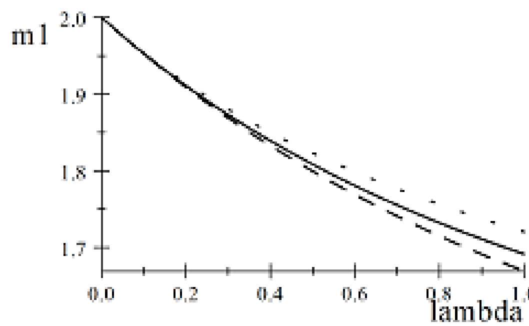

Fig. 1: M/M/1 retrial model. $\theta=2, p=0.25$.

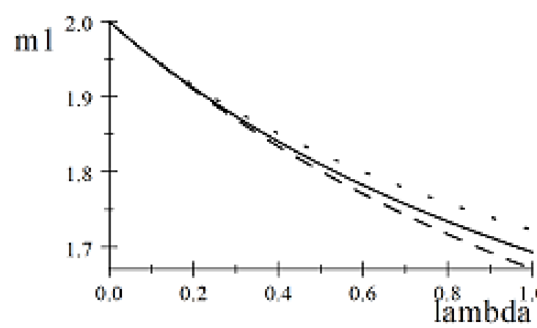

Fig. 2: M/M/1 retrial model. $\theta=2, p=0.5$.

We have also examined the effects of the retrial intensity θ on the asymptotic behaviour of the mean number of customers in the orbit when the traffic intensity ρ^* goes to 1.

From Fig.7, we can see that increasing the retrial intensity θ results in important increasing of the considered performance measure, and this for the three considered retrial time distribu-

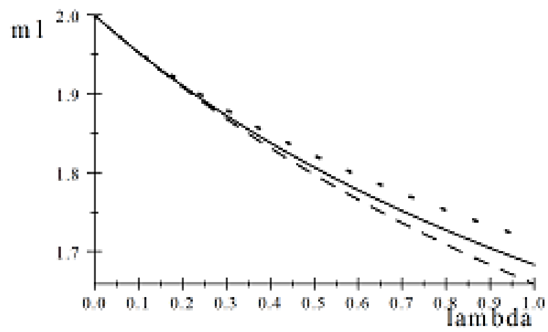

Fig. 3: M/M/1 retrial model. $\theta = 2, p = 0.75$.

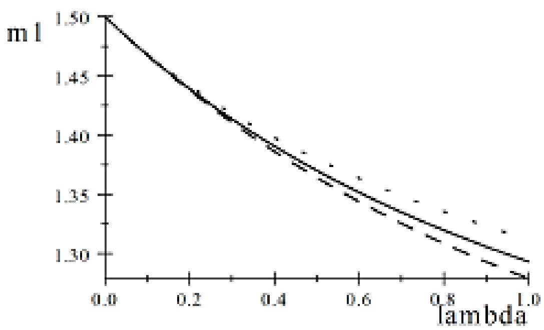

Fig. 4: $M/E_2/1$ retrial model. $\theta = 2, p = 0.25$.

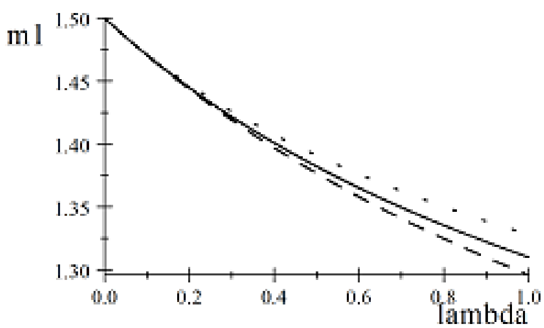

Fig. 5: $M/E_2/1$ retrial model. $\theta = 2, p = 0.5$.

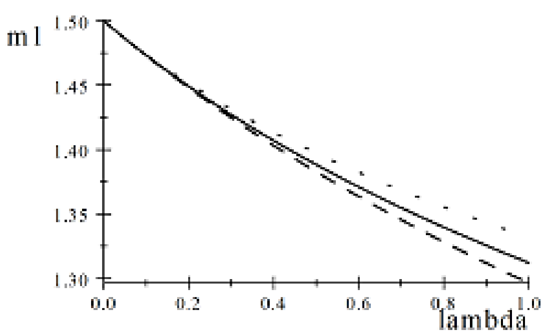

Fig. 6: $M/E_2/1$ retrial model. $\theta = 2, p = 0.75$.

tions: exponential (solid curve), two-stage Erlang (dash curve) and hyperexponential (dots curve).

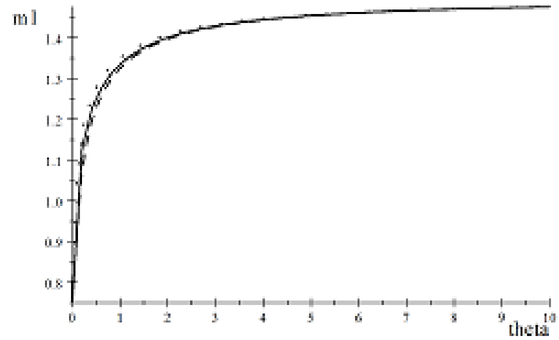

Fig. 7: $M/E_2/1$ retrial model. $p = 0.5, \lambda = 0.4$.

At present, our intention is made to the result established in theorem 2. To show the behaviour of the mean and variance of the number of customers in the orbit (when the retrial intensity $\theta \to 0$), the $M/E_2/1$ retrial queue is considered. Fig $8-9$ present the numerical values of the Gaussian parameters when the retrial time is distributed according to exponential law (solid curve), two-stage Erlang law (dash curve) and two-stage hyperexponential one (dots curve). It is easy to see that the convergence is faster in the case of two-stage hyperexponential retrial times.

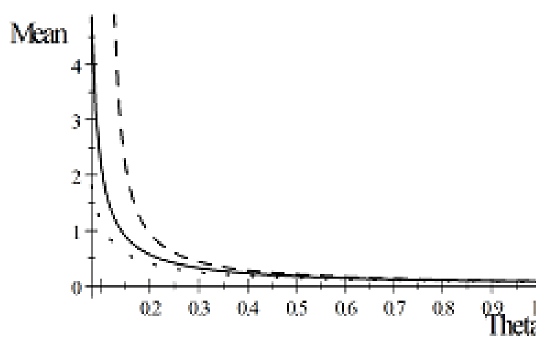

Fig. 8: $M/E_2/1$ retrial model. $p = 0.5, \lambda = 0.3, \beta_1 = 1$.

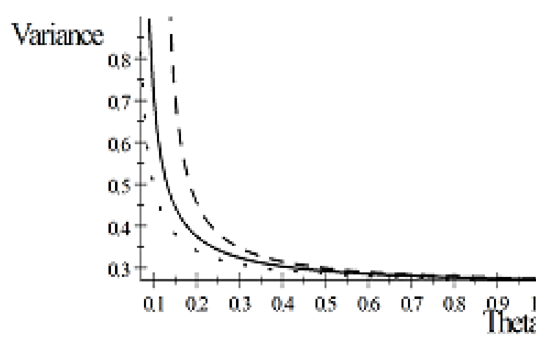

Fig. 9: $M/E_2/1$ retrial model. $p = 0.5, \lambda = 0.3, \beta_1 = 1$.

V. Concluding remarks

We therefore established that for model of type M/G/1 with retrials, priority, FCFS orbit and general retrial times, the bellow conclusions apply successfully. The asymptotic behaviour of the steady state distribution of the number of customers in orbit, which did not reveal the nature of the variable studied, allowed us to approximate the latter by distributions whose characteristics are known.

- In case of heavy traffic, the distribution of the number of customers in orbit goes to two-stage generalized Erlang distribution, then the mean number of customers in orbit is:

$$\overline{n_o} = \frac{2\beta_1^2}{\beta_2} + \left[\frac{p\left(1-\tilde{A}(\lambda)\right)\tilde{A}(\lambda)}{\left(p+q\tilde{A}(\lambda)\right)^2} + \frac{\left(\tilde{A}(\lambda)\right)^2\beta_2}{2\left(p+q\tilde{A}(\lambda)\right)\beta_1^2} \right]^{-1}.$$

- In case of low rate of retrials, the distribution of the number of customers in orbit goes to Gaussian distribution, then the mean number of customers in orbit is:

$$\overline{n_o} = \frac{p\left(1-\tilde{A}(\lambda)\right)\left(\rho + \frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right)}{\tilde{A}(\lambda) - \left[p+q\tilde{A}(\lambda)\right]\rho}.$$

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