Crank-Nicolson Difference Scheme for a Class of Space Fractional Differential Equations with High Order Spatial Fractional Derivative

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Abstract—**In this paper, we investigate the construction of unconditionally stable numerical methods for a class of space fractional differential equations with the order of the spatial fractional derivative belongs to (2,3), where the fractional derivative is defined in the sense of the Riemann-Liouville derivative. A Crank-Nicolson finite difference scheme is developed by use of the order reduction method and the weighted shifted Grünwald-Letnikov derivative approximation formula. Theoretical analysis of unique solvability, stability and convergence for the Crank-Nicolson difference scheme are fulfilled. In order to test the validity of the present algorithm, numerical experiments are carried out.**

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Index Terms—**Space fractional differential equation; High order fractional derivative; Finite difference scheme; Unconditionally stable**

I. Introduction

Fractional calculus has attracted much interest in recent years, and fractional differential equations (FDEs) are widely used as models to describe complex nonlinear phenomena in physics, biology, economics, engineering and other areas of science. Many real-world systems are translated into mathematical models via FDEs [1-8]. In [9], EDEKI etc researched one time-fractional order Black-Scholes model, which can be used for option pricing and assessment, and obtained analytical solutions of the fractional equation for European call option via a proposed relatively new semi-analytic technique hereby referred to as Projected Differential Transform Method. In [10], Zhong etc considered a time fractional convection diffusion equation with time-space variable coefficients, and proposed an effective numerical method for solving this equation. Many aspects of the subject of fractional differential equations have been studied so far, such as the existence and qualitative behaviors of their solutions. For example, we refer the readers to [11-14], and the references therein.

In order to better explain and understand the physical phenomena modeled by FDEs, it is necessary to obtain the solutions of FDEs. However, unlike differential equations of integer order, it is difficult to obtain exact solutions for most FDEs. Thus, obtaining numerical solutions have attracted much attention of many authors, and great efforts have been made to develop valid semi-analytical methods or numerical methods for solving FDEs. So far many valid semi-analytical methods have been developed, such as the reproducing kernel space method [15], the Adams-Bashforth-Moulton method [16], homotopy perturbation method [17], Adomian decomposition method [18,19], the variational iterative method [20,21], homotopy analysis method [22], and generalized differential transform method [23]. Also some valid numerical methods have been applied, such as the meshless method [24], the finite element method [25,26], the coupled fractional reduced differential transform method [27], the Bernstein polynomials method [28], the residual power series method [29], the Jacobi elliptic function method [30], the finite difference method and so on. For most of the semi-analytical methods, the common approach is to approximate the exact solutions in series forms, and usually it is difficult to obtain the closed forms of the approximating solutions. So it is not so easy to fulfil the convergence analysis of the series as well as the error estimate in the case that the closed forms of the approximating solutions can not be obtained. Among the numerical methods mentioned above, the finite difference method is the most widely used method to solve FDEs so far. Based on the finite difference method, many kinds of FDEs including the fractional subdiffusion equation, the fractional diffusion-wave diffusion equation, the fractional advection-diffusion equation, the fractional Schrödinger equation and the fractional Bloch-Torrey equaion etc. are numerically solved by the finite difference method (see [31-38] and the references therein for example).

We notice that most of the FDEs solved by the finite difference method involve fractional derivatives with the orders no more than two, and little attention has been paid so far on developing finite difference schemes for FDEs with the orders of fractional derivatives more than two. Motivated by the current research, in this paper, we consider the following space fractional differential equation involving nonhomogeneous source term:

$$\begin{aligned} u_t(x,t) = a(x)({}_0D_x^{\alpha}u(x,t) -{}_x D_L^{\alpha}u(x,t)) + f(x,t), \\ 2<\alpha<3,\ x\in[0,L],\ t\in[0,T], \end{aligned} \tag{1}$$

with the following initial boundary value condition

$$\begin{cases} u(x,0)=\varphi(x),\ x\in[0,L], \\ u(0,t)=u(L,t)=0,\ t\in[0,T], \end{cases} \tag{2}$$

where $a(x)\neq 0$, ${}_0D_x^{\alpha}u(x,t))$ and ${}_xD_L^{\alpha}u(x,t))$ denote the left-side Riemann-Liouville fractional derivative and the right-side Riemann-Liouville fractional derivative respectively.

The aim of this paper is to develop one unconditionally

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stable Crank-Nicolson finite difference scheme for the problems (1)-(2) by use of a combination of the order reduction method and the weighted shifted Grünwald-Letnikov derivative approximation method applied to the spatial fractional derivative.

The rest of this paper is organized as follows. In Section 2, we give some notations and preliminaries. Then in Section 3, the Crank-Nicolson finite difference scheme for the problems (1)-(2) is established. In Section 4, we fulfil analysis of unique solvability, stability and convergence for the present difference scheme. In Section 5, numerical example for testifying the present difference scheme is given. Finally, in Section 6, some conclusions are proposed.

II. Notations and preliminaries

Let $M,\ N$ be two positive integers, and $h=\dfrac{L}{M},\ \tau=\dfrac{T}{N}$ denote the spatial and temporal step size respectively. Define $x_i=i*h(0\le i\le M),\ t_n=n\tau(0\le n\le N),\ \Omega_h=\{x_i|0\le i\le M\},\ \Omega_\tau=\{t_n|0\le n\le N\},\ (i,n)=(x_i,t^n)$, and then the domain $[0,L]\times[0,T]$ is covered by $\Omega_h\times\Omega_\tau$. Let $V_h=\{u_i^n|0\le i\le M,\ 0\le n\le N\}$ be the grid function on the mesh $\Omega_h\times\Omega_\tau$. $U_i^n=u(x_i,t^n)$ and u_i^n denote the exact solution and numerical solution at the point (i,n) respectively. $U^n=(U_1^n,\ U_2^n,...,\ U_M^n)^T,\ u^n=(u_1^n,\ u_2^n,...,\ u_M^n)^T$.

For any mesh function u_i^n, we use the following notations:

$$\delta_t u_i^n=\frac{u_i^n-u_i^{n-1}}{\tau},\ \delta_{\hat{x}}u_i^n=\frac{u_{i+1}^n-u_{i-1}^n}{2h}, u_i^{n-\frac{1}{2}}=\frac{u_i^n+u_i^{n-1}}{2}.$$

Definition 1. For $n-1\le\beta<n,\ n\in\mathbf{N}$, the left-side Riemann-Liouville fractional derivative and the right-side Riemann-Liouville fractional derivative of order β for the function $u(x)$ are defined by

$${}_{-\infty}D_x^{\beta}u(x)=\frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x}(x-\sigma)^{n-1-\beta}u(\sigma)d\sigma)$$

and

$${}_xD_{\infty}^{\beta}u(x)=(-1)^n\frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)}\int_{x}^{\infty}(\sigma-x)^{n-1-\beta}u(\sigma)d\sigma)$$

respectively.

Similarly, we have

$${}_0D_x^{\beta}u(x)=\frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)}\int_{0}^{x}(x-\sigma)^{n-1-\beta}u(\sigma)d\sigma)$$

and

$${}_xD_L^{\beta}u(x)=(-1)^n\frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)}\int_{x}^{L}(\sigma-x)^{n-1-\beta}u(\sigma)d\sigma).$$

From the definition of the Riemann-Liouville derivatives one can see that for some $k\in\mathbf{N}$,

$${}_0D_x^{\beta+k}u(x)=\frac{d^k}{dx^k}({}_0D_x^{\beta}u(x)),$$

$${}_xD_L^{\beta+k}u(x)=(-1)^k\frac{d^k}{dx^k}({}_xD_L^{\beta}u(x)).$$

Definition 2. The left-side shifted Grünwald difference operator is defined as follows [39]:

$$A_{h,p}^{\alpha}u(x)=\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}g_k^{(\alpha)}u(x-(k-p)h),$$

where p is an integer, and

$g_0^{(\alpha)}=1,\ g_k^{(\alpha)}=(1-\dfrac{\alpha+1}{k})g_{k-1}^{(\alpha)},\ k=1,2,....$

Property 1. The left-side Riemann-Liouville fractional derivative can be approximated by the left-side shifted Grünwald difference operator uniformly with first order accuracy, that is,

$$A_{h,p}^{\alpha}u(x)={}_{-\infty}D_x^{\alpha}u(x)+O(h).$$

Similarly, the right-side shifted Grünwald difference operator is defined by

$$B_{h,p}^{\alpha}u(x)=\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}g_k^{(\alpha)}u(x+(k-p)h),$$

and right-side Riemann-Liouville fractional derivative can be approximated by the right-side shifted Grünwald difference operator uniformly with first order accuracy, that is,

$$B_{h,p}^{\alpha}u(x)={}_xD_{\infty}^{\alpha}u(x)+O(h).$$

Lemma 1 [40]. Let $u\in L_1(\mathbf{R})$, ${}_{-\infty}D_x^{\alpha+2}u(x)$, ${}_xD_{\infty}^{\alpha+2}u(x)$ and their Fourier transforms belong to $L_1(\mathbf{R})$, and define the weighted and shifted Grünwald difference operators by

$${}_LD_{h,p,q}^{\alpha}u(x)=\frac{\alpha-2q}{2(p-q)}A_{h,p}^{\alpha}u(x)+\frac{2p-\alpha}{2(p-q)}A_{h,q}^{\alpha}u(x),$$

and

$${}_RD_{h,p,q}^{\alpha}u(x)=\frac{\alpha-2q}{2(p-q)}B_{h,p}^{\alpha}u(x)+\frac{2p-\alpha}{2(p-q)}B_{h,q}^{\alpha}u(x).$$

Then we have

$${}_LD_{h,p,q}^{\alpha}u(x)={}_{-\infty}D_x^{\alpha}u(x)+O(h^2)$$

and

$${}_RD_{h,p,q}^{\alpha}u(x)={}_xD_{\infty}^{\alpha}u(x)+O(h^2)$$

uniformly for $x\in\mathbf{R}$, where $p,\ q$ are integers and $p\neq q$.

Remark 1. From Lemma 1 one can see that various approximations for the left-side and right-side Riemann-Liouville fractional derivatives can be obtained by different choices of $p,\ q$, and in some cases these approximations can be used to establish unconditionally stable difference schemes for time or space fractional differential equations with the order of the spatial fractional derivative α belongs to $(0,2)$. For example, when $\alpha\in(0,1),\ (p,q)=(0,-1)$, Lemma 1 was used to construct unconditionally stable difference scheme for a class of time fractional differential equations [41], while when $\alpha\in(1,2),\ (p,q)=(1,0)$, Lemma 1 was used to construct unconditionally stable difference scheme for one dimensional space fractional diffusion equation [40]. In [40], the authors also proved that when $\alpha\in(1,2),\ (p,q)=(1,-1)$, unconditionally stable difference scheme can also be obtained. However, for those difference schemes constructed in other choices of $p,\ q$, and even for $\alpha>2$, the stability analysis is difficult to fulfil.

Lemma 2. Under the conditions of Lemma 1, if $\alpha \in (1,2),\ (p,q)=(1,0)$, then we have

$$\begin{cases} \frac{1}{h^\alpha}\sum_{k=0}^{\infty}\omega_k^{(\alpha)}u(x-(k-1)h) = {}_{-\infty}D_x^\alpha u(x)+O(h^2),\\ \frac{1}{h^\alpha}\sum_{k=-\infty}^{0}\omega_k^{(\alpha)}u(x+(k-1)h) = {}_x D_\infty^\alpha u(x)+O(h^2). \end{cases} \tag{3}$$

where

$$\begin{cases} \omega_0^{(\alpha)} = \frac{\alpha}{2}g_0^{(\alpha)} = \frac{\alpha}{2},\\ \omega_k^{(\alpha)} = \frac{\alpha}{2}g_k^{(\alpha)} + \frac{2-\alpha}{2}g_{k-1}^{(\alpha)},\ k=1,2,.... \end{cases}$$

Similarly, for $u \in C(\mathbf{R})$, and $u(0)=u(L)=0$, we have the following approximation formulas

$$\begin{cases} \frac{1}{h^\alpha}\sum_{k=0}^{[x/h]+1}\omega_k^{(\alpha)}u(x-(k-1)h) = {}_0D_x^\alpha u(x)+O(h^2),\\ \frac{1}{h^\alpha}\sum_{k=0}^{[(L-x)/h]+1}\omega_k^{(\alpha)}u(x+(k-1)h) = {}_xD_L^\alpha u(x)+O(h^2). \end{cases} \tag{4}$$

III. The Crank-Nicolson finite difference scheme

In this section, we derive the Crank-Nicolson finite difference scheme for solving the problems (1)-(2) by use of a combination of the order reduction method and the weighted shifted Grünwald-Letnikov derivative approximation method. To use the order reduction method, set $\beta=\alpha-1$. Then $\beta \in (1,2)$ for $\alpha \in (2,3)$.

Based on Lemma 2 one has the following approximation at the grid point (i,n)

$$\begin{aligned} [{}_0D_x^\beta u(x,t) + {}_xD_L^\beta u(x,t)]_{(i,n)} &= \frac{1}{h^\beta}\sum_{k=0}^{i+1}\omega_k^{(\beta)}U_{i-k+1}^n\\ &+\frac{1}{h^\beta}\sum_{k=0}^{M-i+1}\omega_k^{(\beta)}U_{i+k-1}^n + O(h^2)\\ &= \frac{1}{h^\beta}\sum_{k=-M+i}^{i}\lambda_k^{(\beta)}U_{i-k}^n + O(h^2),\ 1\le i\le M-1, \end{aligned} \tag{5}$$

where

$$\begin{cases} \lambda_0^{(\beta)} = 2\omega_1^{(\beta)},\\ \lambda_1^{(\beta)} = \omega_2^{(\beta)}+\omega_0^{(\beta)},\\ \lambda_k^{(\beta)} = \omega_{k+1}^{(\beta)},\ k=2,3,...,\\ \lambda_{-k}^{(\beta)} = \lambda_k^{(\beta)},\ k=1,2,.... \end{cases}$$

If we set $\kappa(x,t) = {}_0D_x^\beta u(x,t) + {}_xD_L^\beta u(x,t)$. Then the following center difference formula holds provided that $\kappa(x,t)\in C^{(3,1)}([0,L]\times[0,T])$:

$$\frac{d}{dx}\kappa(x,t) = \frac{\kappa(x+h,t)-\kappa(x-h,t)}{2h} + O(h^2). \tag{6}$$

On the other hand, according to the property of the Riemann-Liouville fractional derivative one can obtain that $\frac{d}{dx}\kappa(x,t) = {}_0D_x^\alpha u(x,t) - {}_xD_L^\alpha u(x,t)$.

So furthermore by a combination of (5) and (6) we have

$$\begin{aligned} &[{}_0D_x^\alpha u(x,t) - {}_xD_L^\alpha u(x,t)]_{(i,n)}\\ &= \delta_{\hat{x}}[\frac{1}{h^\beta}\sum_{k=-M+i}^{i}\lambda_k^{(\beta)}U_{i-k}^n] + O(h)\\ &= \frac{1}{2h}[\frac{1}{h^\beta}\sum_{k=-M+i+1}^{i+1}\lambda_k^{(\beta)}U_{i+1-k}^n\\ &\quad -\frac{1}{h^\beta}\sum_{k=-M+i-1}^{i-1}\lambda_k^{(\beta)}U_{i-1-k}^n] + O(h)\\ &= \frac{1}{2h^\alpha}\sum_{k=0}^{M}\mu_{i-k}^{(\alpha)}U_k^n + O(h),\ 1\le i\le M-1. \end{aligned} \tag{7}$$

where

$$\begin{cases} \mu_0^{(\alpha)} = \lambda_1^{(\beta)} - \lambda_{-1}^{(\beta)} = 0,\\ \mu_k^{(\alpha)} = \lambda_{k+1}^{(\beta)} - \lambda_{k-1}^{(\beta)},\ k=1,2,...,\\ \mu_{-k}^{(\alpha)} = -\mu_k^{(\alpha)},\ k=1,2,.... \end{cases}$$

For the approximation of Eq. (1), we have the following observation at the point $(i,n+\frac{1}{2})$:

$$\delta_t U_i^n = \frac{a_i}{2h^\alpha}\sum_{k=0}^{M}\mu_{i-k}^{(\alpha)}U_k^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}} + O(\tau^2+h). \tag{8}$$

So the Crank-Nicolson finite difference scheme approximating the problems (1)-(2) can be denoted as follows:

$$\begin{cases} \delta_t u_i^n = \frac{a_i}{2h^\alpha}\sum_{k=0}^{M}\mu_{i-k}^{(\alpha)}u_k^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}},\\ \quad 1\le n\le N,\ i=1,2,...,M-1,\\ u_i^0 = \varphi(x_i),\ i=1,2,...,M-1. \end{cases} \tag{9}$$

IV. Theoretical analysis of the difference scheme

In this section, we fulfil analysis of unique solvability, stability and convergence for the Crank-Nicolson difference scheme (9) presented above are fulfilled. To this end, the definition of the function $u(x,t)$ and $\varphi(x)$ are extended as following

$$v(x,t) = \begin{cases} u(x,t),\ x\in[0,L],\\ 0,\ x\in(-\infty,0)\bigcup(L,\infty), \end{cases}$$

$$\phi(x) = \begin{cases} \varphi(x),\ x\in[0,L],\\ 0,\ x\in(-\infty,0)\bigcup(L,\infty), \end{cases}$$

So $u(x,t)$ and $\varphi(x)$ can be seen the restriction of $v(x,t)$ and $\phi(x)$ on $[0,L]$ respectively, and $v(x,t)$ is continuous on $\mathbf{R}$. Thus the numerical solution v_i^n $(i=0,\pm1,\pm2,...,\ 1\le n\le N)$ and the exact solution $V_i^n = v(x_i,t^n)$ can be similarly defined, and $V^n = (...,V_{-2}^n,\ V_{-1}^n,\ V_0^n,\ V_1^n,\ V_2^n,...)^T$, $v^n = (...,v_{-2}^n,\ v_{-1}^n,\ v_0^n,\ v_1^n,v_2^n,...)^T$.

Setting $r=\frac{\tau a_i}{2h^\alpha}$, and (9) can be rewritten as follows:

$$\begin{cases} v_i^n - v_i^{n-1} = r\sum_{k=-\infty}^{\infty}\mu_{i-k}^{(\alpha)}v_k^{n-\frac{1}{2}} + \tau f_i^{n-\frac{1}{2}},\\ \quad 1\le n\le N,\ i=0,\pm1,\pm2,...,\\ v_i^0 = \phi(x_i),\ i=0,\pm1,\pm2,.... \end{cases} \tag{10}$$

A. Unique solvability

Define the grid functions spaces $V_h = \{v|v = (...,v_{-2},\ v_{-1},\ v_0,\ v_1,\ v_2,...)\}$ and $V_h^0 = \{v|v\in V_h,\ \lim_{|i|\to\infty} v_i = 0\}$. For $u,\ v\in V_h^0$, the discrete inner product is defined as $(u,v) = h\sum_{i=-\infty}^{\infty}u_iv_i$, while the discrete L_2 norm is defined by $\|v\| = \sqrt{(v,v)} = (\sum_{i=-\infty}^{\infty}h|v_i|^2)^{\frac{1}{2}}$.

Furthermore, define $W_h = \{u|u = (u_0,\ u_1,\ u_2, ..., u_M)\}$ and $W_h^0 = \{u|u \in W_h,\ u_0 = u_M = 0\}$. For $u,\ v \in W_h^0$, the discrete inner product is defined as $(u, v) = h \sum\limits_{i=1}^{M-1} u_i v_i$, while the discrete L_2 norm is defined by $\|u\|_W = \sqrt{(u, u)} = (\sum\limits_{i=1}^{M-1} h|u_i|^2)^{\frac{1}{2}}$.

Lemma 3. If $v \in V_h^0$, then for any integer k, it holds that

$$\sum_{i=-\infty}^{\infty} v_{i-k} v_i = \sum_{i=-\infty}^{\infty} v_{i+k} v_i.$$

In fact, if we set $j = i - k$, then $\sum\limits_{i=-\infty}^{\infty} v_{i-k} v_i = \sum\limits_{j=-\infty}^{\infty} v_j v_{j+k} = \sum\limits_{i=-\infty}^{\infty} v_{i+k} v_i$.

Remark 2. It can be seen from the difference schemes (9) and (10) that $\|u^n\|_W = \|v^n\|$.

Theorem 1. The difference scheme denoted by (9) is uniquely solvable.

proof. In order to analyze the unique solvability of the difference scheme denoted by (9), it is adequate to verify there is only zero solution for the corresponding homogeneous difference equation of (10), that is,

$$v_i^n = \frac{r}{2} \sum_{k=-\infty}^{\infty} \mu_{i-k}^{(\alpha)} v_k^n = \frac{r}{2} \sum_{k=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n. \tag{11}$$

Multiplying hv_i^n on both sides of Eq. (11) and a summation with respect to i from $-\infty$ to ∞, together with Lemma 3 one can deduce that

$$\begin{aligned}
\|v^n\|^2 &= \frac{rh}{2} \sum_{k=-\infty}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n v_i^n] \\
&= \frac{rh}{2} \sum_{k=-\infty}^{-1} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n v_i^n] \\
&\quad + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n v_i^n] + \frac{rh}{2} \sum_{i=-\infty}^{\infty} \mu_0^{(\alpha)} v_i^n v_i^n \\
&= \frac{rh}{2} \sum_{k=-\infty}^{-1} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i+k}^n v_i^n] \\
&\quad + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n v_i^n] + \frac{rh}{2} \mu_0^{(\alpha)} \|v^n\|^2 \\
&= \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_{-k}^{(\alpha)} v_{i-k}^n v_i^n] + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} v_{i-k}^n v_i^n] \\
&= \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} (\mu_{-k}^{(\alpha)} + \mu_k^{(\alpha)}) v_{i-k}^n v_i^n] \\
&= 0.
\end{aligned}$$

So $\|v^n\|^2 = 0$, which implies that there is only zero solution for the homogeneous difference equation (11). So the difference scheme (9) is uniquely solvable. The proof is complete.

B. Stability

For the sake of stability analysis, consider the following problem

$$\begin{cases} \widetilde{v}_i^n - \widetilde{v}_i^{n-1} = r \sum\limits_{k=-\infty}^{\infty} \mu_{i-k}^{(\alpha)} \widetilde{v}_k^{n-\frac{1}{2}} + \tau \widetilde{f}_i^{n-\frac{1}{2}}, \\ \qquad 1 \le n \le N,\ i = 0, \pm 1, \pm 2, ..., \\ \widetilde{v}_i^0 = \widetilde{\phi}(x_i),\ i = 0, \pm 1, \pm 2, ... \end{cases} \tag{12}$$

Set $\epsilon_i^n = v_i^n - \widetilde{v}_i^n,\ y_i^{n-\frac{1}{2}} = f_i^{n-\frac{1}{2}} - \widetilde{f}_i^{n-\frac{1}{2}}$. Then from (10) and (12) one can obtain that

$$\begin{cases} \epsilon_i^n - \epsilon_i^{n-1} = r \sum\limits_{k=-\infty}^{\infty} \mu_{i-k}^{(\alpha)} \epsilon_k^{n-\frac{1}{2}} + \tau y_i^{n-\frac{1}{2}}, \\ \qquad 1 \le n \le N,\ i = 0, \pm 1, \pm 2, ..., \\ \epsilon_i^0 = \phi(x_i) - \widetilde{\phi}(x_i),\ i = 0, \pm 1, \pm 2, ... \end{cases} \tag{13}$$

Multiplying $h\epsilon_i^{n-\frac{1}{2}}$ on both sides of the first equation of (13) and a summation with respect to i from $-\infty$ to ∞, together with Lemma 3 we have

$$\begin{aligned}
&\frac{\|\epsilon^n\|^2 - \|\epsilon^{n-1}\|^2}{2} \\
&= rh \sum_{k=-\infty}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] + \tau h \sum_{i=-\infty}^{\infty} y_i^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}} \\
&= \frac{rh}{2} \sum_{k=-\infty}^{-1} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \frac{rh}{2} \sum_{i=-\infty}^{\infty} \mu_0^{(\alpha)} u_i^n u_i^n + \tau(y^{n-\frac{1}{2}}, \epsilon^{n-\frac{1}{2}}) \\
&= \frac{rh}{2} \sum_{k=-\infty}^{-1} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i+k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \frac{rh}{2} \mu_0^{(\alpha)} \|\epsilon^{n-\frac{1}{2}}\|^2 + \tau(y^{n-\frac{1}{2}}, \epsilon^{n-\frac{1}{2}}) \\
&= \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_{-k}^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} \mu_k^{(\alpha)} \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] + \tau(y^{n-\frac{1}{2}}, \epsilon^{n-\frac{1}{2}}) \\
&= \frac{rh}{2} \sum_{k=1}^{\infty} [\sum_{i=-\infty}^{\infty} (\mu_{-k}^{(\alpha)} + \mu_k^{(\alpha)}) \epsilon_{i-k}^{n-\frac{1}{2}} \epsilon_i^{n-\frac{1}{2}}] \\
&\quad + \tau(y^{n-\frac{1}{2}}, \epsilon^{n-\frac{1}{2}}) \\
&= \tau(y^{n-\frac{1}{2}}, \epsilon^{n-\frac{1}{2}}) \\
&\le \tau[\frac{1}{2+\tau} \|\epsilon^{n-\frac{1}{2}}\|^2 + \frac{2+\tau}{4} \|y^{n-\frac{1}{2}}\|^2] \\
&= \tau[\frac{1}{4(2+\tau)} (\|\epsilon^n + \epsilon^{n-1}\|^2) + \frac{2+\tau}{4} \|y^{n-\frac{1}{2}}\|^2] \\
&\le \tau[\frac{1}{2(2+\tau)} (\|\epsilon^n\|^2 + \|\epsilon^{n-1}\|^2) + \frac{2+\tau}{4} \|y^{n-\frac{1}{2}}\|^2],
\end{aligned}$$

that is,

$$\frac{2}{2+\tau} \|\epsilon^n\|^2 \le \frac{2+2\tau}{2+\tau} \|\epsilon^{n-1}\|^2 + \frac{\tau(2+\tau)}{2} \|y^{n-\frac{1}{2}}\|^2,$$

which is followed by

$$\|\epsilon^n\|^2 \le (1+\tau)\|\epsilon^{n-1}\|^2 + \tau(1+\frac{\tau}{2})^2\|y^{n-\frac{1}{2}}\|^2.$$

Furthermore, we have

$$\begin{aligned}
&\|\epsilon^n\|^2 \le (1+\tau)^n\|\epsilon^0\|^2 \\
&\quad + \sum_{m=0}^{n-1}(1+\tau)^m\tau(1+\frac{\tau}{2})^2\|y^{n-m-\frac{1}{2}}\|^2 \\
&\le (1+\tau)^n\|\epsilon^0\|^2 + [\sum_{m=0}^{n-1}(1+\tau)^m]\tau(1+\frac{\tau}{2})^2 \max_{1\le k\le n}\|y^{k-\frac{1}{2}}\|^2 \\
&\le (1+\tau)^n[\|\epsilon^0\|^2 + (1+\frac{\tau}{2})^2 \max_{1\le k\le n}\|y^{k-\frac{1}{2}}\|^2] \\
&\le \exp^{n\tau}\|\epsilon^0\|^2 + \exp^{(n+1)\tau}\max_{1\le k\le n}\|y^{k-\frac{1}{2}}\|^2 \\
&\le \exp^{T}\|\epsilon^0\|^2 + \exp^{2T}\max_{1\le k\le n}\|y^{k-\frac{1}{2}}\|^2 \\
&\le C(\|\epsilon^0\|^2 + \max_{1\le k\le n}\|y^{k-\frac{1}{2}}\|^2),
\end{aligned}$$

where $C = \exp^{2T}$ is a constant. From the inequality above one can see that small changes occurring on the initial value and the right-side hand source term in the difference scheme (10) also lead to small perturbation for the solution, and furthermore it is the case for the difference scheme (9). So we have the following theorem.

Theorem 2. The Crank-Nicolson difference scheme denoted by (9) is unconditionally stable on the initial value and the right source term.

C. Convergence

Let $\varepsilon_i^n = V_i^n - v_i^n,\ i = 1,2,...,M,\ n = 0,1,...,N$ denote the errors between the exact solutions and the numerical solutions, and $\varepsilon^n = (...,\varepsilon_{-2}^n,\ \varepsilon_{-1}^n,\ \varepsilon_0^n,\ \varepsilon_1^n,\ \varepsilon_2^n,...)^T$. Then from (8)-(10) we have

$$\begin{cases} \varepsilon_i^n - \varepsilon_i^{n-1} = r\sum\limits_{k=-\infty}^{\infty}\mu_{i-k}^{(\alpha)}\varepsilon_k^{n-\frac{1}{2}} + \tau R(\tau,h), \\ \qquad 1\le n\le N,\ i = 0,\pm1,\pm2,..., \\ \varepsilon_i^0 = 0,\ i = 0,\pm1,\pm2,..., \end{cases} \tag{14}$$

where $R(\tau,h) = O(\tau^2+h)$.

Following in a similar manner as in the analysis of stability one can deduce that

$$\begin{aligned}
\|\varepsilon^n\|^2 &\le \exp^T\|\varepsilon^0\|^2 + \exp^{2T}\|R(\tau,h)\|^2 \\
&= \exp^{2T}\|R(\tau,h)\|^2.
\end{aligned}$$

Furthermore, $\|\varepsilon^n\| \le \exp^T\|R(\tau,h)\| \le C_1\tau^2 + C_2h$, where $C_1,\ C_2$ are two positive constants. So we have the following theorem.

Theorem 3. The Crank-Nicolson difference scheme denoted by (9) is convergent.

V. Numerical Experiments

In this section, we present one numerical example for the Crank-Nicolson difference scheme (9). Consider the problems (1)-(2) with an exact analytical solution

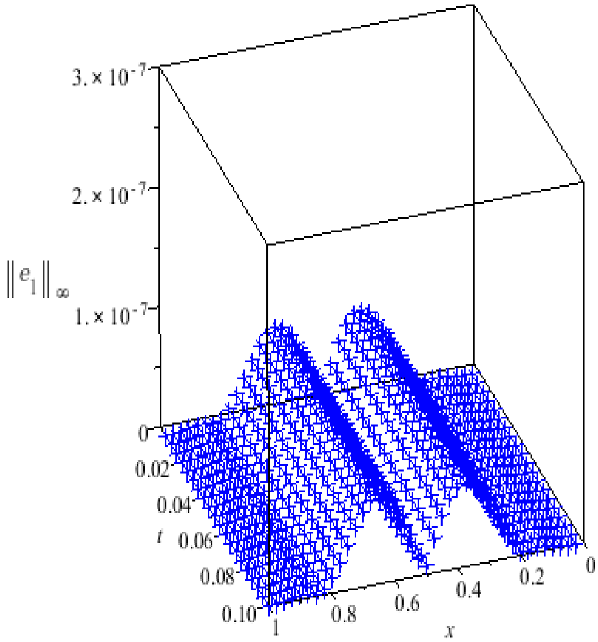

Fig 1. The maximum errors with h=1/41;τ=0.001;α=2.5

$$u(x,t) = \begin{cases} (t^6+1)x^4(1-x)^4,\ x\in(0,1), \\ 0,\ x\in(-\infty,0]\bigcup[1,\infty), \end{cases}$$

and satisfies

$$\begin{cases} a(x) = -\dfrac{1}{2\cos(\frac{1}{2}\alpha\pi)}x^4(1-x)^4, \\ f(x,t) = 6t^5x^4(1-x)^4 + \dfrac{1}{2\cos(\frac{1}{2}\alpha\pi)} \\ \qquad \sum\limits_{m=4}^{8}[\dfrac{c_m m!x^{-\alpha+m}}{\Gamma(1-\alpha+m)} - \dfrac{c_m m!(1-x)^{-\alpha+m}}{\Gamma(1-\alpha+m)}] \\ \qquad (t^6+1)x^4(1-x)^4, \\ u(x,0) = \varphi(x) = x^4(1-x)^4, \end{cases}$$

where $x^2(1-x)^2 = \sum\limits_{m=2}^{4}c_m x^m$.

For further use, let $\|e_1\|_\infty = \max\limits_i|U_i^n - u_i^n|$ and $\|e_2\|_\infty = \max\limits_i|(\dfrac{U_i^n - u_i^n}{U_i^n}| \times 100)$ denote the maximum absolute error and the maximum relative error respectively.

In Fig. 1, the maximum absolute errors are shown under some selected parameters, while in Fig. 2, comparison between the exact solutions and the numerical solutions is demonstrated.

In Tables 1-2, the maximum relative errors are listed under some certain conditions.

Table 1: The maximum relative errors at

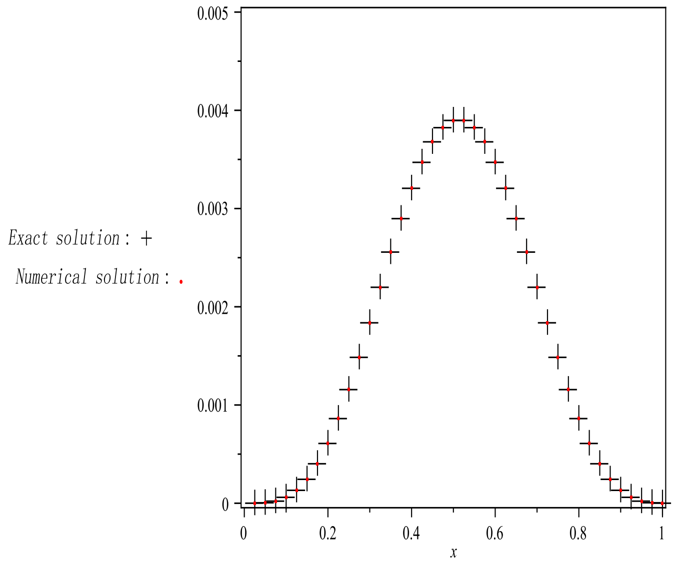

Fig. 2 Comparison between exact solutions and numerical solutions with α=2.5,h=1/41,τ=0.001,t=0.02

$\tau = 10^{-3}$ after 50 time steps

	$\alpha = 2.3$	$\alpha = 2.4$	$\alpha = 2.5$
h	$\|e_2\|_\infty$	$\|e_2\|_\infty$	$\|e_2\|_\infty$
$\frac{1}{10}$	0.0237543	0.0176157	0.0089474
$\frac{1}{15}$	0.0233580	0.0159321	0.0056928
$\frac{1}{20}$	0.0224223	0.0147233	0.0034654
$\frac{1}{25}$	0.0225080	0.0145044	0.0023444
$\frac{1}{30}$	0.0224372	0.0142964	0.0018136

Table 2: The maximum relative errors at $\tau = 10^{-2}$ after 50 time steps

	$\alpha = 2.3$	$\alpha = 2.4$	$\alpha = 2.5$
h	$\|e_2\|_\infty$	$\|e_2\|_\infty$	$\|e_2\|_\infty$
$\frac{1}{10}$	1.76682	1.70644	1.62076
$\frac{1}{15}$	1.76295	1.68989	1.58874
$\frac{1}{20}$	1.75372	1.67796	1.56682
$\frac{1}{25}$	1.75467	1.67593	1.55578
$\frac{1}{30}$	1.75395	1.67385	1.55056

From Fig. 1 one can see that the maximum absolute errors can be bounded to a low level, and the increase of the the time steps do not lead to sharp change of the the maximum absolute errors, which coincides with the previous stability analysis, while Fig 2. shows that the numerical solutions can approximate the exact solutions satisfactory. From the results of Tables 1-2 one can see that the relative errors are small with small time step size, and will increase with the increment of the time step size. Yet the relative errors are still acceptable.

VI. Conclusions

In this paper, by use of the order reduction method and the weighted shifted Grünwald-Letnikov derivative approximation formula, we have constructed a Crank-Nicolson finite difference scheme for a class of space fractional differential equations, where the order of the fractional derivative belongs to $(2, 3)$. This scheme is proved to be unconditionally stable, convergent, and with the local truncating error $O(\tau^2 + h)$. Numerical experiments are carried out to verify the theoretical analysis. It is worth to notice that the accuracy of the present difference scheme in this paper can be further improved by use of the compact method. Besides, we note that this handling process in this paper can be applied to construct unconditionally stable difference schemes for other types of space, time or space-time fractional differential equations. For example, one can further consider two types of fractional differential equations. One is the two-dimensional extension of Eq. (1), which is denoted as follows

$$\begin{cases} u_t(x,y,t) = k_1({}_0D_x^\alpha u(x,y,t) -{}_x D_{L_1}^\alpha u(x,y,t)) \\ \qquad +k_2({}_0D_y^\alpha u(x,y,t) -{}_y D_{L_2}^\alpha u(x,y,t)) \\ \qquad +f(x,y,t),\ 1<\alpha,\ \beta<2, \\ \qquad (x,y)\in\Omega,\ t\in[0,T], \\ u(x,y,0)=\varphi(x,y),\ (x,y)\in\Omega, \\ u(x,y,t)=0,\ (x,y)\in\partial\Omega, \end{cases} \tag{15}$$

where $\Omega = [0, L_1] \times [0, L_2],\ k_1,\ k_2 > 0$, the function u is smooth enough, and the fractional derivatives are defined in the sense of the Riemann-Liouville derivatives. The other is the following initial boundary value problem for space-time fractional differential equation:

$$\begin{cases} {}_0^CD_t^\gamma u(x,t) = a(x)({}_aD_x^\alpha u(x,t) -{}_x D_b^\alpha u(x,t)) \\ \qquad +f(x,t),\ 0<\gamma<1, \\ u(x,0)=h(x),\ x\in[a,b], \\ u(a,t)=u(b,t)=0,\ t\in[0,T], \end{cases} \tag{16}$$

where $a(x) > 0$, ${}_0^CD_t^\gamma u(x,t)$, ${}_aD_x^\alpha u(x,t)$ and ${}_xD_b^\alpha u(x,t)$ denote the Caputo fractional derivative, the left-side Riemann-Liouville fractional derivative and the right-side Riemann-Liouville fractional derivative respectively.

By the known $L1$ formula, one can approximate the time fractional derivative. Then together with the process demonstrated in Sections II and III, one can obtain corresponding finite difference schemes with unconditional stability for the problem above.

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