On Local Circulant and Residue Splitting Iterative Method for Toeplitz-structured Saddle Point Problems

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***Abstract*—By exploiting the special structure of the $(1,1)$-block in the coefficient matrix of saddle point problems, a local circulant and residue splitting (LCRS) iterative method is proposed to solve Toeplitz-structured saddle point problems, and the splitting matrix serve as a preconditioner to accelerate the convergence rate of Krylov subspace method such as GMRES. The advantage of these methods are that the elapsed CPU time of each iteration is reduced considerably by using of the fast Fourier transform (FFT). The convergence theorem is established under suitable conditions. Numerical experiments of a model Stokes problem are presented to show that the LCRS is used as either solver or preconditioner to GMRES method often outperform other tested methods in the elapsed CPU time.**

***Index Terms*—saddle point problems, local circulant and residue splitting (LCRS), iterative method, preconditioning, fast Fourier transform (FFT).**

I. Introduction

THE solution of large sparse saddle point problem with Toeplitz structure is considered:

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{or } \mathscr{A}u = b, \tag{1}$$

where, $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite (SPD) Toeplitz matrix or block-Toeplitz and Toeplitz-block (BTTB) matrix, $B \in \mathbb{R}^{m\times n}$ is a matrix of full rank, $x, f \in \mathbb{R}^n$, $y, g \in \mathbb{R}^m$, and $m \le n$. These assumptions guarantee the existence and uniqueness of the solution of linear systems (1); see [4], [8], [11], [12]. A is here called a BTTB matrix if $A = A_1 \otimes A_2 \otimes \cdots \otimes A_p$, where $A_i \in \mathbb{R}^{q\times q} (i = 1, 2, \cdots, p)$, $p, q \in \mathbb{Z}^+$, are all Toeplitz square matrices and $\otimes$ is a Kronecker tensor product.

The linear systems (1) arises in a variety of scientific computing and engineering applications, including computational fluid dynamics, constrained optimization, image reconstruction, mixed finite element approximations of elliptic PDEs and Stokes problems, numerical solutions of FDEs via finite difference method and so forth; in detail, one can see [10], [12], [21], [24] and references therein. These applications have motivated both mathematicians and engineers to develop specific algorithms for solving Toeplitz structure linear systems.

As the coefficient matrix $\mathscr{A}$ in (1) is usually large and sparse, iterative methods are recommended against direct methods. During the last decade, a large number of iterative methods for solving saddle point problems have been proposed. For example, Uzawa-type methods [4], [8], [15], [25], preconditioned Krylov subspace methods [11], Hermitian and skew-Hermitian splitting (HSS) iterative method and its accelerated variants [1], [7], [9], [16], restrictively preconditioned conjugate gradient methods [2], [6], [23] and so on. We can refer to a comprehensive survey [11] for algebraic properties and iterative methods for saddle point problems.

Due to the Toeplitz-like structure of the coefficient matrix, a lot of circulant preconditioners have been proposed for solving Toeplitz-like linear systems, included Strang's preconditioner, T. Chan's preconditioner, R. Chan's preconditioner, ω-circulant preconditioner etc; see [17] for more details. This paper focuses on the solving the saddle point problem with Toeplitz-structured and presents a new circulant splitting iterative method.

The paper is organized as follows. In Section II, a new circulant and residue splitting (CRS) iterative method is presented to solve the general Toeplitz linear system, and its convergence property is studied. In Section III, a local circulant and residue splitting (LCRS) iterative method for the saddle-point problem (1) is proposed, and the conditions for guaranteeing its convergence are studied. In Section IV, the splitting matrix of the LCRS iterative method serve as a preconditioner to accelerate the convergence rate of Krylov subspace methods and the procedure for computing the generalized residual equations is described. In Section V, numerical experiments of a model Stokes problem are given to show that the new splitting is used as either solver or preconditioner to GMRES method often do the best. Finally, the paper closes with concluding remarks in Section VI.

II. CRS iterative method

IN this section, we firstly review the definition of Toeplitz and circulant matrices, then present the circulant and residue splitting (CRS) iterative method for the Toeplitz linear systems

$$Tx = \tilde{b}, \tag{2}$$

where $T \in \mathbb{R}^{n\times n}$ is a symmetric Toeplitz or BTTB matrix, $x, \tilde{b} \in \mathbb{R}^n$.

Manuscript received December 15, 2017; This work was supported by the National Natural Science Foundation of China(11771193,11661033) and the Scientific Research Foundation for Doctor of Hexi University.

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(Advance online publication: 28 May 2018)

Because any symmetric Toeplitz and BTTB matrix $T = (t_{i,j})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ possesses the splitting of the form

$$T = C + S,$$

where $S = T - C$ is the symmetric residue Toeplitz matrix, and

$$C = \mathrm{Circ}(c_0,\ c_1,\ c_2,\ \cdots, c_m,\ \cdots,\ c_2,\ c_1)$$
$$= \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_1 & \cdots & c_{m-1} & \cdots & c_3 & c_2 \\ & \vdots & & \vdots & & \vdots & & \\ c_2 & c_3 & c_4 & \cdots & \cdots & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & \cdots & \cdots & c_1 & c_0 \end{bmatrix} \tag{3}$$

is the symmetric circulant matrix [19], [20], [26], whose elements are decided as follows:

$$c_j = \Big\{ d_k:\ d_k, d_s \in \mathrm{mode}(t_{i,i+j},\ i = 1, 2, \cdots, n-j), \quad |d_k| \ge |d_s| \Big\}, \quad j = 0, 1, \cdots, m\,(m = \mathrm{fix}(n/2)). \tag{4}$$

Here, the function $\mathrm{mode}(t_{i,i+j},\ i = 1, 2, \cdots, n-j)$ finds the elements that appears most often in the (upper) j-th diagonals of the Toeplitz matrix T, and the function $\mathrm{fix}(n/2)$ rounds $n/2$ to the nearest integers towards zero.

Based on the above splitting of Toeplitz or BTTB matrix, we firstly present a new approach to solve the Toeplitz linear system (2), called the CRS iterative method, and it is described as follows.

Algorithm 1. *(CRS iterative method). Given an initial guess x_0, for $k = 0,1,2,\cdots$, until the iteration sequence $\{x_k\}$ converges to the exact solution, solve*

$$(\alpha I + C)x_{k+1} = (\alpha I - S)x_k + \tilde{b}, \tag{5}$$

where α is a given positive constant.

In the following, we deduce the convergence property for the CRS iterative method. Note that the iteration matrix of CRS iteration is

$$G_\alpha = (\alpha I + C)^{-1}(\alpha I - S). \tag{6}$$

Let $\rho(G_\alpha)$ denote the spectral radius of the iteration matrix G_α. Then the CRS iteration (5) is convergent if and only if $\rho(G_\alpha) < 1$ [3].

Theorem 1. *Suppose $T \in \mathbb{R}^{n\times n}$ is a SPD Toeplitz matrix, C defined in (3) and (4) is its circulant part, and α is a positive constant such that $(\alpha I_n + C)$ is a SPD circulant matrix. Assume x is an eigenvector of the iteration matrix G_α corresponding to its eigenvalue λ. Denote*

$$\gamma := \frac{x^*Cx}{x^*x} \quad \text{and} \quad \beta := \frac{x^*Tx}{x^*x}.$$

Then the spectral radius $\rho(G_\alpha) < 1$ and the CRS iterative method is convergent if and only if α, β, γ satisfy the conditions $0 < \beta < 2(\alpha + \gamma)$.

Proof. Let λ be an eigenvalue of G_α and x be its corresponding eigenvector. Then we have

$$(\alpha I - S)x = \lambda(\alpha I + C)x.$$

As $(\alpha I - S) = (\alpha I + C) - T$, by multiplying both sides of the this equality from the left with $\frac{x^*}{x^*x}$, we obtain

$$\lambda(\alpha + \gamma) = (\alpha + \gamma) - \beta,$$

where

$$\gamma = \frac{x^*Cx}{x^*x} \quad \text{and} \quad \beta = \frac{x^*Tx}{x^*x}.$$

By the assumption that $\alpha > 0,\ \beta > 0,\ \alpha + \gamma > 0$, it then follows that $|\lambda| = \frac{|\alpha + \gamma - \beta|}{\alpha + \gamma} < 1$ provided

$$0 < \beta < 2(\alpha + \gamma).$$

Thus we complete the proof. □

At the end of this section, the CRS iterative method is reformulated into the residual-updating form as follows.

Algorithm 1' Given an initial guess x_0 and positive parameter α.

(a) Set $r := \tilde{b} - Tx_0$;
(b) Solve the linear systems $(\alpha + C)z = r$ to obtain z by using the fast Fourier transform (FFT);
(c) Set $x_{k+1} := x_k + z$.

III. LCRS ITERATIVE METHOD

In this section, based on the CRS iterative method for the linear system $Tx = \tilde{b}$ in Section II, we present a local circulant and residue splitting (LCRS) iterative method for solving the Toeplitz-structured saddle point problem (1).

For the coefficient matrix of the Toeplitz-structured saddle point problem (1), we make the following splitting

$$\mathscr{A} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} = \begin{bmatrix} Q_1 + C & 0 \\ -B & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 - S & -B^T \\ 0 & Q_2 \end{bmatrix},$$

here, C and S are the symmetric circulant and residue parts of the SPD Toeplitz or BTTB matrix A, respectively, $Q_2 \in \mathbb{R}^{m\times m}$ is an SPD matrix and $Q_1 \in \mathbb{R}^{n\times n}$ is such that $Q_1 + C$ is an SPD circulant matrix. It is here noted that such matrix Q_1 does exist and the simplest case is $Q_1 = \alpha I$ with a suitable parameter α.

Then the local CRS iterative scheme for solving the saddle point problem (1) is given as follows:

$$\begin{bmatrix} Q_1 + C & 0 \\ -B & Q_2 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} Q_1 - S & -B^T \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}.$$

And it is described as below.

Algorithm 2. *(LCRS iterative method). Given an initial guess $u_0 = (x_0^*,\ y_0^*)^*$, for $k = 0,1,2,\cdots$, until the iteration sequence $u_k = (x_k^*,\ y_k^*)^*$ converges to the exact solution, solve*

$$\begin{cases} x_{n+1} = x_n + (Q_1 + C)^{-1}(f - Ax_n - B^T y_n), \\ y_{n+1} = y_n + Q_2^{-1}(Bx_{n+1} + g). \end{cases} \tag{7}$$

We see that the iterative method (7) is a special case of the parameterized inexact Uzawa (PIU) method studied in [8], and it is also similar to the local HSS iterative method presented in [16], [27], [28].

We emphasize that the coefficient matrix of the first linear sub-systems in LCRS iterative method is a circulant matrix, which can be efficiently solved by using FFT. Since FFT is highly parallizable and has been implemented on multiprocessors efficiently [18], the LCRS method is well-adapted for parallel computing and its computational workloads may be further saved.

In the following, the convergence of the LCRS iterative method will be studied. Note that the iteration can be written in a fixed-point form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \Gamma \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} Q_1+C & 0 \\ -B & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} f \\ g \end{bmatrix}, \tag{8}$$

where

$$\Gamma = \begin{bmatrix} Q_1+C & 0 \\ -B & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} Q_1-S & -B^T \\ 0 & Q_2 \end{bmatrix}$$

is called the iteration matrix of the LCRS method. The fixed-point iteration (8) converges for arbitrary initial guesses $u_0 = (x_0^*,\ y_0^*)^*$ and right-hand sides b to the solution $u^* = \mathscr{A}^{-1}b$ if and only if $\rho(\Gamma) < 1$, where $\rho(\Gamma)$ denotes the spectral radius of the iterative matrix Γ.

To determine the convergence conditions, some lemmas were developed for later use.

Lemma 2. *([8]) Both roots of the real quadratic equation* $\lambda^2 + \phi\lambda + \varphi = 0$ *have modulus less than one if and only if* $|\varphi| < 1$ *and* $|\phi| < 1 + \varphi$.

Lemma 3. *Assume that A is an SPD Toeplitz or BTTB matrix with the symmetric circulant part C, and B is a matrix of full rank. Let $Q_1 \in \mathbb{R}^{n\times n}$ be such that $Q_1 + C$ is an SPD circulant matrix, and $Q_2 \in \mathbb{R}^{m\times m}$ is an SPD matrix. If λ is an eigenvalue of the iteration matrix Γ, then $\lambda \neq 1$.*

Proof. Let λ be an eigenvalue of the iteration matrix Γ and $v = (v_1^*, v_2^*)^*$ be its corresponding eigenvector, where $v_1 \in \mathbb{C}^n$ and $v_2 \in \mathbb{C}^m$. Then it is true that

$$\begin{cases} \lambda(Q_1+C)v_1 - (Q_1-S)v_1 + B^T v_2 = 0, \\ \lambda B v_1 + (1-\lambda)Q_2 v_2 = 0. \end{cases} \tag{9}$$

If $\lambda = 1$ and $v = (v_1^*, v_2^*)^*$ is the corresponding eigenvector, then (9) produces

$$\begin{cases} Av_1 + B^T v_2 = 0, \\ Bv_1 = 0, \end{cases} \quad \text{or} \quad \mathscr{A} v = 0.$$

Since the coefficient matrix $\mathscr{A}$ is nonsingular, $v = (v_1^*, v_2^*)^* = 0$, which contradicts the assumption that $v = (v_1^*, v_2^*)^*$ is an eigenvector of the iteration matrix Γ. So $\lambda \neq 1$. □

Lemma 4. *Under the assumptions in Lemma 3, let $v = (v_1^*, v_2^*)^*$ be an eigenvector of the iteration matrix Γ corresponding to the eigenvalue λ. Then $v_1 \neq 0$. If $v_2 = 0$, then $|\lambda| < 1$ provided $0 < \beta < 2(\eta+\gamma)$, where*

$$\eta = \frac{v_1^* Q_1 v_1}{v_1^* v_1}, \quad \gamma = \frac{v_1^* C v_1}{v_1^* v_1} \quad \textit{and} \quad \beta = \frac{v_1^* A v_1}{v_1^* v_1}.$$

Proof. If $v_1 = 0$, then the second equation in (9) produces

$$(1-\lambda)\ Q_2 v_2 = 0.$$

According to that Q_2 is SPD, we have $v_2 = 0$, which contradicts the assumption that $(v_1^*, v_2^*)^*$ is an eigenvector of the iteration matrix Γ. Therefore, $v_1 \neq 0$.

If $v_2 = 0$, then the first equation in (9) produces

$$\lambda(Q_1+C)v_1 - (Q_1-S)v_1 = 0.$$

Similar to the proof of Theorem 1, considering $\beta > 0, \eta + \gamma > 0$, it follows that $|\lambda| = \dfrac{|\eta+\gamma-\beta|}{\eta+\gamma} < 1$ provided $0 < \beta < 2(\eta+\gamma)$. □

Theorem 5. *Under the assumptions in Lemma 3, if $(v_1^*, v_2^*)^*$ is an eigenvector corresponding to an eigenvalue λ of the iteration matrix Γ, then the LCRS iterative method is convergent if the inequality*

$$0 \le \tau \le 4(\eta+\gamma) - 2\beta \tag{10}$$

holds, where

$$\gamma = \frac{v_1^* C v_1}{v_1^* v_1}, \ \beta = \frac{v_1^* A v_1}{v_1^* v_1}, \ \tau = \frac{v_1^* B^T Q_2^{-1} B v_1}{v_1^* v_1}, \ \eta = \frac{v_1^* Q_1 v_1}{v_1^* v_1}.$$

Proof. Let λ be an eigenvalue of the iteration matrix Γ and $(v_1^*, v_2^*)^*$ be the corresponding eigenvector. From Lemma 3 and Lemma 4, we have $\lambda \neq 1$ and $v_1 \neq 0$.

As $(1-\lambda)Q_2$ is nonsingular, Eliminating v_2 in (9) and using $Q_1 + C - A$ in place of $Q_1 - S$, we have

$$\lambda(Q_1+C)v_1 - (Q_1+C-A)v_1 + \frac{\lambda}{\lambda-1} B^T Q_2^{-1} B v_1 = 0.$$

Multiplying both sides of the above equation from left with $\frac{v_1^*}{v_1^* v_1}$, we obtain

$$\lambda(\eta+\gamma) - (\eta+\gamma-\beta) + \frac{\lambda}{\lambda-1}\tau = 0. \tag{11}$$

where

$$\eta := \frac{v_1^* Q_1 v_1}{v_1^* v_1} > 0, \quad \gamma := \frac{v_1^* C v_1}{v_1^* v_1}, \quad \beta := \frac{v_1^* A v_1}{v_1^* v_1} > 0$$

and

$$\tau := \frac{v_1^* B^T Q_2^{-1} B v_1}{v_1^* v_1} \ge 0.$$

When $Bv_1 = 0$, then $\tau = 0$. From the equation (11) we have

$$\lambda = \frac{\eta+\gamma-\beta}{\eta+\gamma},$$

which is similar to the Theorem 1, then $|\lambda| < 1$ provided $0 < \beta < 2(\eta+\gamma)$.

If $Bv_1 \neq 0$, then $\tau > 0$. Considering the assumption $\eta + \gamma > 0$, from (11) we obtain

$$\lambda^2 - \frac{2(\eta+\gamma)-\beta-\tau}{\eta+\gamma}\lambda + \frac{\eta+\gamma-\beta}{\eta+\gamma} = 0. \tag{12}$$

From Lemma 2, we known that both roots of the equation (12) satisfy $|\lambda| < 1$ if and only if

$$\left|\frac{\eta+\gamma-\beta}{\eta+\gamma}\right| < 1 \quad \text{and} \quad \left|\frac{2(\eta+\gamma)-\beta-\tau}{\eta+\gamma}\right| < 1 + \frac{\eta+\gamma-\beta}{\eta+\gamma}.$$

By straightforwardly solving the above two inequalities, the condition (10) is obtained immediately. □

Based on the above discussions, we known that $\rho(\Gamma) < 1$ hold true provided (10), i.e., the LCRS iterative method is convergent, and it convergence to the unique solution of the saddle point problem (1).

IV. Krylov subspace acceleration

THE LCRS iterative method (7) for solving the saddle point problem (1) belong to the class of stationary iterative methods, and it convergence to the unique solution of the saddle point problem (1). Although the LCRS iteration is very simple and very easy to implement, but its convergence may be typically too slow for the method to be competitive even with the optimal choice of the parameter matrices Q_1 and Q_2. Since the splitting matrix of the LCRS iteration can serve as a preconditioner for Krylov subspace methods, called the LCRS preconditioner, the convergence rate of preconditioned Krylov subspace methods such as GMRES can be greatly improved for solving the saddle point linear system (1) in this section.

It follows from (8) that the linear system $\mathscr{A}u = b$ is equivalent to the linear system

$$(I-\Gamma)u = M^{-1}\mathscr{A}u = M^{-1}b.$$

This equivalent linear system is considered as a left-preconditioned system and it can be solved using non-symmetric Krylov subspace methods such as GMRES [22]. Hence, the matrix M, which is induced by the LCRS iterative method, can be utilized as a preconditioner for GMRES. In other words, we can say that GMRES is used to accelerate the convergence of the splitting iteration applied to $\mathscr{A}u = b$ [22]. In general, a clustered spectrum of the preconditioned matrix $M^{-1}\mathscr{A}$ often translates in rapid convergence of GMRES.

Since $(I-\Gamma) = M^{-1}\mathscr{A}$ and $\rho(\Gamma) < 1$ in Theorem 5, we know that the spectra of the preconditioned matrix $M^{-1}\mathscr{A}$ are located inside a circle centred at (1, 0) with radius 1 on the complex plane, which is a desirable property for Krylov subspace acceleration.

Another aspect of preconditioned krylov subspace method need consider is how to solve the general residual equations. When the LCRS preconditioner is used to accelerate the convergence rate of Krylov subspace methods, it is necessary to solve sequences of generalized residual equations

$$\begin{bmatrix} Q_1+C & 0 \\ -B & Q_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{13}$$

where $r = (r_1^*, r_2^*)^*$ and $z = (z_1^*, z_2^*)^*$ are the current and the generalized residual vectors, respectively.

The residual equation (13) can be dealt with by first solving $(Q_1+C)z_1 = r_1$ and then solving $Q_2z_2 = r_2 + Bz_1$. By taking advantage of the structures of the matrices Q_1, C and Q_2, we can compute the generalized residual vector $z = (z_1^*, z_2^*)^*$ in (13) by the following procedure.

(a) solve z_1 from $(Q_1+C)z_1 = r_1$ by using FFT;
(b) compute $\tilde{r}_2 = r_2 + Bz_1$;
(c) solve z_2 from $Q_2z_2 = \tilde{r}_2$ by using CG or direct method.

V. Numerical results

In this section, the effectiveness and advantages of the proposed splitting iterative method are illustrated by using numerical example, which coming from the finite difference discretization of the two-dimensional Stokes problem. This problem is chosen for numerical experiments because it is widely known and well-understood test problems. We note that the (1, 1)-block in saddle point problems (1) is an SPD BTTB matrix. Our aim here is to show that our circulant splitting method serve as a solver may be competitive with the well-known Uzawa, GSOR (generalized successive over relaxation) [4], GMRES methods, while it serve as a preconditioner may be superior to the local shift splitting (LSS) [13] preconditioner, classic Uzawa (Uzawa) preconditioner [11] and generalized parameterized inexact Uzawa (GPIU) [15], [25] preconditioner. In order to express these preconditioners clearly, the following matrix splitting is given

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} = \begin{bmatrix} A+W_1 & W_4 \\ -B+W_3 & W_2 \end{bmatrix} - \begin{bmatrix} W_1 & -B^T+W_4 \\ W_3 & W_2 \end{bmatrix} \tag{14}$$

And the parameter matrices W_1, W_2, W_3 and W_4 are list in Table I.

TABLE I: The selection of parameter matrices W_1, W_2, W_3 and W_4.

	W_1	W_2	W_3	W_4
LSS	$-\frac{1}{2}A$	$\frac{\alpha}{2}I$	$\frac{1}{2}B$	$\frac{1}{2}B^T$
Uzawa	0	δI	0	0
GPIU	δA	BW_1B^T	$tB(t>0)$	0
LCRS	$A-Q1-C^1$	δI	0	0

In actual numerical experiments, $Q_1 = \alpha\mu I$ with $\mu = \text{mode}\big(A(i,i), i = 1, 2, \cdots, n\big)$ is chosen for use in the LCRS iterative method. All the tests are performed in MATLAB R2013a with machine precision 10^{-16}, and terminated when the current residual satisfies $\|r_k\|/\|r_0\| < 10^{-6}$ or the number of the prescribed iteration $k_{\max} = 1{,}000$ is exceeded, where r_k is the residual at the k-th iteration. The zero vector serve as the initial guess, and the right-hand side vector b is selected such that the exact solution of the saddle point problem is $u^* = (x^*, y^*)^* = (1, 1, \cdots, 1)^*$.

The problem under consideration is the Stokes problem, which is firstly constructed and used in [3] and latter in other papers [1], [4], [13], i.e.,

$$\begin{cases} -\upsilon\Delta\mathbf{u} + \nabla p & = \tilde{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} & = \tilde{g}, \quad \text{in } \Omega, \\ \mathbf{u} & = 0, \quad \text{on } \partial\Omega, \\ \int_\Omega p(x)dx & = 0, \end{cases} \tag{15}$$

where $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω, υ stands for the viscosity scalar, Δ is the componentwise Laplace operator, $\mathbf{u}=(u^T, v^T)^T$ is a vector-valued function representing the velocity, and p is a scalar function representing the pressure. By discretizing (15) with the upwind scheme and taking $\upsilon = 1$, we obtain the following linear system with saddle point form (1),

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix},$$

where

$$A = \begin{bmatrix} I\otimes T + T\otimes I & 0 \\ 0 & I\otimes T + T\otimes I \end{bmatrix} \in \mathbb{R}^{2p^2\times 2p^2},$$

$$B^T = \begin{bmatrix} I\otimes F \\ F\otimes I \end{bmatrix} \in \mathbb{R}^{2p^2\times p^2},$$

$$T = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p\times p},$$

$$F = \frac{1}{h}\text{tridiag}(-1, -1, 0) \in \mathbb{R}^{p\times p}$$

with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{p+1}$ the discretization meshsize, p is a positive integer, the total number of varablesis $m + n = 2p^2 + p^2 = 3p^2$; see [13].

The LCRS iterative method is compare with the classic Uzawa, GSOR, GMRES when they are used as solvers for the tested example, from the point of view of the number of total iteration steps (denoted by "IT"), the elapsed CPU time in seconds (denoted by "CPU") and the absolute error norm $\|u^k - u^*\|_\infty$ (denoted by "ERR"), where u^k is the approximate solution satisfied the terminated condition of iterative methods, and u^* is the exact solution of the saddle point problem. The concrete numerical results and the corresponding parameters used in iterative methods are given in Table II.

Moreover, The LCRS iteration is compare with the classic Uzawa, generalized parameterized inexact Uzawa (GPIU) and local shift-splitting (LSS) when they are used as preconditioners to accelerate GMRES for the tested problem, from the point of view of the number of total iteration steps and the elapsed CPU time in seconds, denoted by "IT" and "CPU" respectively. The empirical optimal parameters used in preconditioners and concrete numerical results are given in Table III. Furthermore, the convergence history is shown in Figure 1.

As shown in Table II and Figure 1, all iterative methods can successfully produce approximate solution of the tested saddle point problem. The elapsed CPU time and number of iterations increased as the scale of the problem increased. In terms of the number of iterations, the Uzawa iteration is the fewest, and the LCRS iterative method are fewer than GMRES but more than GSOR. In terms of the elapsed CPU time, the LCRS iteration is the least among all iterative methods.

When the LCRS is used as a preconditioner to accelerate GMRES method, from Table III, we can see that the iteration counts of GPIU preconditioner is the fewest, and those of the LCRS preconditioner is the most. But the elapsed CPU time of LCRS preconditioner is the lest. Moreover, as FFT could be paralleled and implemented on multiprocessors efficiently, the elapsed CPU time of the LCRS iterative method and the LCRS preconditioned GMRES method may be further saved.

Clearly, with the optimal parameters, the LCRS is used as either solver or preconditioner to GMRES method often do best in the elapsed CPU time, but whose iteration counts are more than Uzawa and GSOR iterative methods, also more than Uzawa, GPIU and LSS used as preconditioners to GMRES. This may be attributable to the following two reasons. The first is that the convergence speed of the LCRS iterative method may be slow, which directly increases the number of iterations. The other is that the FFT is used in LCRS iterative method or preconditioner, which can save a lot of time in each iteration steps. In conclusion, the LCRS is a good iterative method for solving saddle point problem (1), which is much more effective than the other tested method, and it can be used as a good preconditioner to accelerate the convergence of GMRES for the test problems.

VI. Concluding remarks

The sparse symmetric Toeplitz or BTTB structured saddle point problems are common in many field. The local circulant and residue splitting (LCRS) iterative method and its preconditioned form are proposed to solve the saddle point problems with the SPD Toeplitz or BTTB $(1,1)$-block. In fact, the new splitting iterative method belongs to a class of the parameterized inexact Uzawa (PIU) method studied in [8], and it is also similar to the local HSS iterative method presented in [16], [27], [28]. The splitting matrix of the LCRS iterative method can serve as a preconditioner, called the LCRS preconditioner, to accelerate GMRES method for solving the saddle point problems. The main idea is the Toeplitz matrix could be split to a sum of circulant and residue matrix, and the linear sub-systems with circulant coefficient matrix could be solved by using FFT. Theoretical analysis have shown that the LCRS iterative method is feasible and effective. Numerical experiments shown that LCRS iterative method and the GMRES method incorporated with the LCRS preconditioner outperform other test solvers or preconditioners in elapsed CPU time.

Based on the following splitting of the saddle point matrix $\mathscr{A}$:

$$\mathscr{A} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} = \begin{bmatrix} G & B^T \\ -B & 0 \end{bmatrix} - \begin{bmatrix} A-G & 0 \\ 0 & 0 \end{bmatrix} := M - N,$$

where M is called the constraint preconditioner, the main ideas in this paper could be generalized to the constraint preconditioner by further exploiting the Toeplitz structure. In [14]the convergence of the constraint preconditioned Krylov subspace iterative method was study, and it may be interesting to study the convergence of a class of splitting iterative methods based on the local circulant and residual splitting. In addition, future work should also focus on determining the choice of the optimal parameter matrices Q_1 and Q_2.

Acknowledgements

The author would like to thank the anonymous referees for his/her careful reading of the manuscript.

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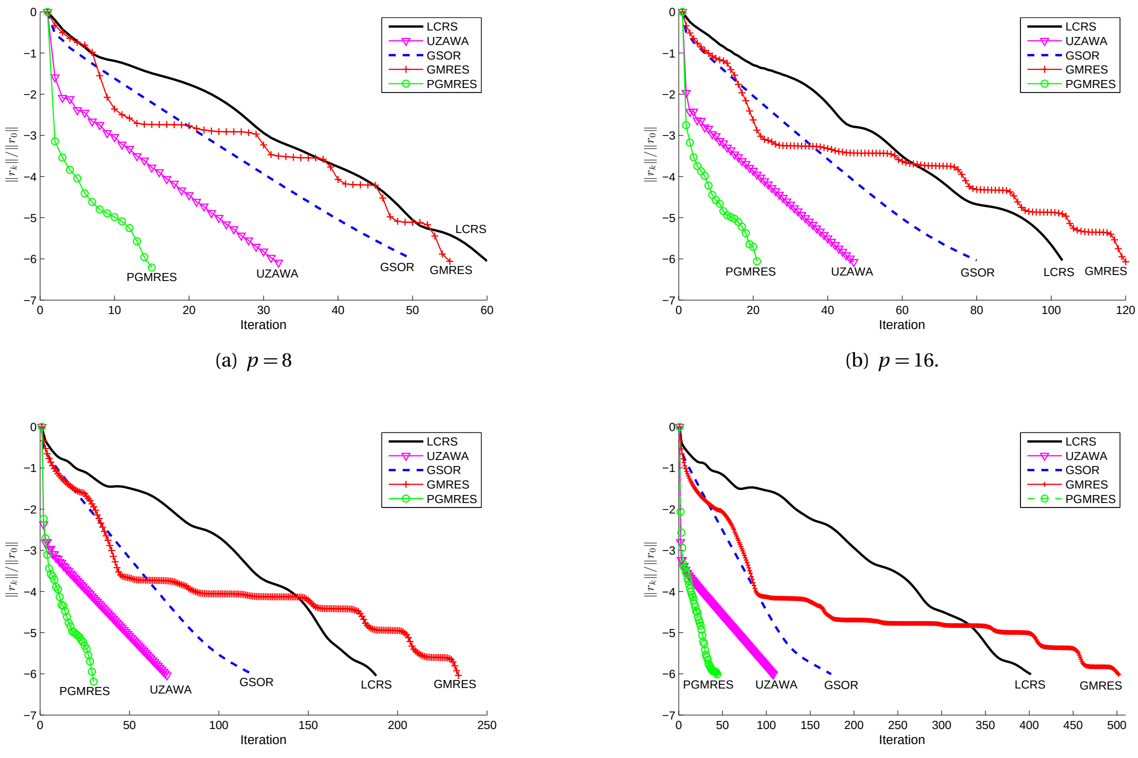
TABLE II: Numerical results of iterative methods.

		p=8	p=16	p=24	p=32	p=40	p=48	p=56	p=64	p=72	p=80
LCRS	α	0.26	0.21	0.20	0.20	0.20	0.20	0.19	0.19	0.19	0.19
	δ	1.28	1.03	1.06	1.13	1.13	1.12	1.12	1.08	1.09	1.08
	IT	59	102	158	187	242	296	359	401	459	516
	CPU	**0.0029**	**0.0068**	**0.0195**	**0.0304**	**0.0525**	**0.1000**	**0.1743**	**0.2598**	**0.3334**	**0.5389**
	ERR	7.90E-5	1.26E-4	1.54E-4	2.87E-4	2.94E-4	7.07E-4	5.62E-4	8.25E-4	7.29E-4	7.39E-4
Uzawa	δ	0.55	0.53	0.52	0.51	0.51	0.5	0.5	0.5	0.5	0.5
	IT	31	46	59	70	80	91	99	107	115	122
	CPU	0.0204	0.0282	0.0667	0.1291	0.2629	0.5683	0.7189	1.286	2.3341	2.9978
	ERR	6.41E-5	2.53E-4	5.05E-4	8.25E-4	1.33E-3	1.49E-3	2.16E-3	2.85E-3	3.55E-3	4.42E-3
GSOR	ω	1.2	1.23	1.22	1.23	1.23	1.23	1.23	1.23	1.23	1.23
	τ	0.87	0.87	0.91	0.91	0.92	0.93	0.93	0.93	0.94	0.94
	IT	49	79	98	118	134	148	161	173	184	193
	CPU	0.0161	0.0558	0.1724	0.3148	0.7977	1.5938	2.4809	4.1128	10.7579	17.964
	ERR	1.81E-4	6.51E-4	1.38E-3	2.21E-3	3.24E-3	4.41E-3	5.89E-3	7.56E-3	8.93E-3	1.11E-2
GMRES	IT	54	119	176	233	312	376	439	501	563	625
	CPU	0.0377	0.1302	0.3909	1.2338	3.0164	5.7188	9.8161	14.9753	24.059	36.0489
	ERR	2.44E-4	9.99E-4	4.66E-3	1.42E-2	3.04E-2	4.16E-2	5.63E-2	7.50E-2	9.68E-2	1.21E-1

TABLE III: Numerical results of preconditioned GMRES with different preconditioner.

		p=8	p=16	p=24	p=32	p=40	p=48	p=56	p=64	p=72	p=80
Non	IT	54	119	176	233	312	376	439	501	563	625
	CPU	0.0377	0.1302	0.3909	1.2338	3.0164	5.7188	9.8161	14.9753	24.059	36.0489
LCRS	α	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	δ	0.75	0.48	0.56	0.53	0.41	0.30	0.31	0.35	0.36	0.44
	IT	14	19	24	27	29	31	33	35	37	39
	CPU	**0.0042**	**0.0112**	**0.0343**	**0.0633**	**0.0956**	**0.1361**	**0.1773**	**0.2482**	**0.3265**	**0.3917**
Uzawa	δ	0.74	0.63	0.54	0.53	0.51	0.59	0.58	0.57	0.56	0.55
	IT	8	9	10	10	11	11	11	11	11	11
	CPU	0.0092	0.0287	0.0790	0.1451	0.2249	0.3182	0.4288	0.6133	0.9722	1.1538
GPIU	α	0.03	0.4	0.05	0.06	0.07	0.08	0.08	0.09	0.08	0.09
	t	0.33	0.33	0.35	0.34	0.33	0.32	0.32	0.33	0.34	0.33
	IT	3	3	3	3	3	3	3	3	3	3
	CPU	0.0034	0.0176	0.0732	0.2159	0.5783	1.4742	3.7813	8.5324	18.1374	32.2067
LSS	α	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
	IT	7	7	8	9	9	9	9	9	9	9
	CPU	0.0047	0.0171	0.0438	0.0933	0.1575	0.1997	0.2808	0.4685	0.5099	0.6485



(a) $p = 8$ (b) $p = 16$.

(c) $p = 32$ (d) $p = 64$

Fig. 1: Convergence of LCRS, Uzawa, GSOR, GMRES and LCRS preconditioned GMRES methods.