Positive Periodic Solutions of Second Order Functional Differential Equation with Impulses

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***Abstract*—This paper deals with a second order functional differential equation with periodic coefficients and impulses of the following form**

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)=f(t,x_t), t\neq t_j,\\ \Delta x|_{t=t_j}=I_j(x(t_j)),\\ -\Delta x'|_{t=t_j}=J_j(x(t_j)), t=t_j, j\in Z^+. \end{cases}$$

By using the fixed point theorem of cone expansion and cone compression of norm type, sufficient conditions for the existence of at least two periodic solutions of the equation are obtained. The results in the present paper generalize and improve many known conclusions.

***Index Terms*—Periodic solutions; Functional differential equations; Impulses.**

I. Introduction

It is well known that the theory of impulsive differential equations has become an important aspect of differential equations. Differential equations with impulses provide an adequate mathematical model of many evolutionary processes that suddenly change their states at certain moments.

Recently, impulsive differential equations have been studied both in theory and applications; see, for example, [1-6]. In [6], Tian et al. considered the following impulsive differential equations

$$\begin{cases} -x''+Mx=f(t,x), t\neq t_j,\\ \Delta x|_{t=t_j}=I_j(x(t_j)),\\ -\Delta x'|_{t=t_j}=J_j(x(t_j)), t=t_j, j\in Z^+. \end{cases}$$

By using the theory of fixed point index in cones, sufficient conditions are presented for the existence of one or two periodic solutions to the impulsive differential equations. In [8], Wang et al. considered the following second order nonlinear delay differential equation with periodic coefficients

$$\begin{aligned} &x''(t)+p(t)x'(t)+q(t)x(t)\\ &=r(t)x'(t-\tau(t))+f(t,x(t),x(t-\tau(t))). \end{aligned}$$

By using Krasnoselskii's fixed point theorem and the contraction mapping principle, established some criteria for the existence and uniqueness of periodic solution to the delay differential equation.

Motivated by the above statements, in this paper, by using a fixed point theorem on a cone to study a second order impulsive functional differential equations with periodic coefficients of the following form

$$\begin{cases} x''(t)+a(t)x'(t)+b(t)x(t)=f(t,x_t), t\neq t_j,\\ \Delta x|_{t=t_j}=I_j(x(t_j)),\\ -\Delta x'|_{t=t_j}=J_j(x(t_j)),\ t=t_j, j\in Z^+, \end{cases} \tag{1}$$

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where $I_j\in(R^+,R), J_j\in(R^+,R^+), \Delta x|_{t=t_j}=x(t_j^+)-x(t_j^-), \Delta x'|_{t=t_j}=x'(t_j^+)-x'(t_j^-)$, where $x^i(t_j^+)$ (respectively $x^i(t_j^-)$) denote the right limit (respectively left limit) of $x^i(t)$ at $t=t_j$, $i=0,1$. There exist a positive constant k such that $t_{j+k}=t_j+T, I_{j+k}(x(t_{j+k}))=I_j(x(t_j)), J_{j+k}(x(t_{j+k}))=J_j(x(t_j)), j\in Z^+$. Without loss of generality, we assume that $[0,T]\cap\{t_j, j\in Z^+\}=\{t_1,t_2,\cdots,t_k\}$.

$f(t,x_t)$ is a nonnegative function defined on $R\times BC$, where BC denotes the Banach space of bounded continuous functions $\phi:R\to R^+$ with the norm $\|\phi\|=\sup_{\theta\in R}|\phi(\theta)|$. If $x\in BC$, then $x_t\in BC$ for any $t\in R$ is defined by $x_t(\theta)=x(t+\theta)$ for $\theta\in R$. $f(t,x_t)$ is continuous in t, T-periodic whenever x is T-periodic.

In this paper, we shall use the following assumptions:

$(A1)$ $a,b:R\to R^+$ are all continuous T-periodic functions, $\int_0^T a(s)ds>0$, $\int_0^T b(s)ds>0$.

$(A2)$ $f(t,\xi)\geq 0$ for all $(t,\xi)\in R\times BC(R,R_+)$.

$(A3)$ For any $L>0$ and $\varepsilon>0$, there exists $\delta>0$, such that

$$[\phi,\psi\in BC, \|\phi\|\leq L, \|\psi\|\leq L, \|\phi-\psi\|<\delta]$$

imply

$$|f(s,\phi_s))-f(s,\psi_t)|_0<\varepsilon.$$

For convenience, we first introduce the related definition and the fixed point theorem applied in the paper.

Definition 1. *Let X be a Banach space and K be a closed nonempty sunset of X, K is a cone if*

(1) $\alpha u+\beta v\in K$ *for all* $u,v\in K$ *and all* $\alpha,\beta\geq 0$ *;*
(2) $u,-u\in K$ *imply* $u=0$.

Theorem 1 ([9]). *(Fixed point theorem) Let K be a cone in a Banach space E, and Ω_1,Ω_2 be two bounded open sets in E such that $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $\mathbb{T}:K\cap(\overline{\Omega}_2\ \Omega_1)\to K$ be completely continuous operator. If*

(1) *There exists $u_0\in K\setminus\{0\}$ such that $u-\mathbb{T}u\neq\alpha u_0$, $u\in K\cap\partial\Omega_2$, $\alpha\geq 0$; $\mathbb{T}u\neq\mu u, u\in K\cap\partial\Omega_1, \mu\geq 1$, or*
(2) *There exists $u_0\in K\setminus\{0\}$ such that $u-\mathbb{T}u\neq\alpha u_0$, $u\in K\cap\partial\Omega_1$, $\alpha\geq 0$; $\mathbb{T}u\neq\mu u, u\in K\cap\partial\Omega_2, \mu\geq 1$,*

then $\mathbb{T}$ has at least one fixed point in $K\cap(\overline{\Omega}_2\ \Omega_1)$.

II. Preliminaries

In order to use Theorem 1 to prove the existence of periodic solutions of system (1), we shall consider the following spaces:

Let $J'=J\setminus\{t_1,t_2,\cdots,t_k\}$, then

$$\begin{aligned} PC(J,R)=\{x:J\to R: x|_{(t_j,t_{j+1})}\in C(t_j,t_{j+1}),\\ x(t_j^-)=x(t_j), \exists x(t_j^+), j=1,2,\cdots,k\} \end{aligned}$$

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is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in[0,T]}|x(t)|$. Let

$$PC^1(J,R) = \{x : J \to R : x|_{(t_j,t_{j+1})}, x'|_{(t_j,t_{j+1})} \in C(t_j,t_{j+1}), x(t_j^-) = x(t_j), x'(t_j^-) = x'(t_j), \exists x(t_j^+), x'(t_j^+), j = 1,2,\cdots,k\}$$

with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$, then $PC^1(J,R)$ is also a Banach space.

Lemma 1 ([7]). *Suppose that* $(A1)$ *hold and*

$$\frac{R_1[\exp(\int_0^T a(u)du) - 1]}{Q_1 T} \geq 1,$$

where

$$R_1 = \max_{t\in[0,T]}\left|\int_t^{t+T} \frac{\exp(\int_t^s a(u)du)}{\exp(\int_0^T a(u)du)-1} b(s)ds\right|,$$
$$Q_1 = \left(1 + \exp(\int_0^T a(u)du)\right)^2 R_1^2.$$

Then there exist continuous T*-periodic functions* p *and* q *such that* $q(t) > 0, \int_0^T p(u)du > 0$, *and*

$$p(t) + q(t) = a(t), q'(t) + p(t)q(t) = b(t),$$

for all $t \in R$. *Therefore*

$$p(t) + q(t) = a(t), q'(t) + p(t)q(t) = b(t), t \in R.$$

Lemma 2. *Suppose the conditions of Lemma 1 hold and* $\phi \in BC$. *Then the equation*

$$x''(t) + a(t)x'(t) + b(t)x(t) = \phi(t), \tag{2}$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds, \tag{3}$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u q(v)dv + \int_u^s p(v)dv]du}{[\exp(\int_0^T p(u)du) - 1][\exp(\int_0^T q(u)du) - 1]} + \frac{\int_s^{t+T} \exp[\int_t^u q(v)dv + \int_u^{s+T} p(v)dv]du}{[\exp(\int_0^T p(u)du) - 1][\exp(\int_0^T q(u)du) - 1]}.$$

Proof: Define $E_p = \exp(\int_0^T p(u)du) - 1, E_q = \exp(\int_0^T q(u)du) - 1$. By direct calculation, we can see that (3) is a T-periodic solution of (2).

Suppose $x(t)$ is a T-periodic solution of (2), from Lemma 1, we have

$$x''(t)+p(t)x'(t)+q'(t)x(t)+q(t)x'(t)+p(t)q(t)x(t) = \phi(t),$$

which is equivalent to

$$(x'(t)e^{\int_0^t p(u)du})' + (q(t)x(t)e^{\int_0^t p(u)du})' = \phi(t)e^{\int_0^t p(u)du},$$

integrating it from t to $t+T$, we obtain

$$x'(t) + q(t)x(t) = \int_t^{t+T} \frac{\exp(\int_t^s p(u)du)}{\exp(\int_0^T p(u)du) - 1}\phi(s)ds.$$

Therefore,

$$\begin{aligned} x(t) &= \int_t^{t+T} \frac{\exp(\int_t^s q(u)du)}{\exp(\int_0^T q(u)du) - 1} \\ &\quad \times \left[\int_s^{s+T} \frac{\exp(\int_s^v p(u)du)}{\exp(\int_0^T p(u)du) - 1}\phi(v)dv\right] ds \\ &= \frac{1}{E_pE_q}\int_t^{t+T} \exp(\int_t^s q(u)du) \\ &\quad \times \left[\int_s^{s+T} \exp(\int_s^v p(u)du)\phi(v)dv\right] ds \\ &= \frac{1}{E_pE_q}\int_t^{t+T} \phi(s)ds \int_t^s \exp(\int_t^u q(v)dv \\ &\quad + \int_u^s p(v)dv)du \\ &\quad + \frac{1}{E_pE_q}\int_t^{t+T} \phi(s)ds \int_s^{t+T} \exp(\int_t^u q(v)dv \\ &\quad + \int_u^{s+T} p(v)dv)du \\ &= \int_t^{t+T} G(t,s)\phi(s)ds. \end{aligned}$$

This completes the proof.

So the equation

$$x''(t) + a(t)x'(t) + b(t)x(t) = f(t,x_t),$$

has a T-periodic solution, it can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)f(s,x_t)ds.$$

By (A2), we have

$$G(t,s)f(s,x_s) \geq 0, (t,s) \in R^2.$$

Corollary 1. *Green function* $G(t,s)$ *satisfies the following properties:*

$$G(t,t+T) = G(t,t), G(t+T,s+T) = G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s) = p(s)G(t,s) - \frac{\exp\int_t^s q(v)dv}{\exp\int_0^T q(v)dv - 1},$$

$$\frac{\partial}{\partial t}Gt,s) = -q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}.$$

Lemma 3 ([8]). *Let*

$$A = \int_0^T a(u)du, B = T^2\exp(\frac{1}{T}\int_0^T \ln b(u)du).$$

If

$$A^2 \geq 4B, \tag{4}$$

then

$$\min\left\{\int_0^T p(u)du, \int_0^T q(u)du\right\} \geq \tfrac{1}{2}\left(A - \sqrt{A^2-4B}\right),$$
$$\max\left\{\int_0^T p(u)du, \int_0^T q(u)du\right\} \leq \tfrac{1}{2}\left(A + \sqrt{A^2-4B}\right).$$

Let $\frac{1}{2}(A-\sqrt{A^2-4B}) := l, \frac{1}{2}(A+\sqrt{A^2-4B}) := m$, from Lemma 3, the function $G(t,s)$ satisfis

$$0 < N_1 =: \frac{T}{(e^m-1)^2} \le G(t,s) \le \frac{T\exp(\int_0^T a(u)du)}{(e^l-1)^2} := M_1, s\in[t,t+T],$$

$$\frac{\partial}{\partial s}G(t,s)|_{s=t_j} = p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}, \quad t_j\in[t,t+T],$$

here we assume that

$$\frac{\partial}{\partial s}G(t,s)|_{s=t_j} > 0, t_j\in[t,t+T].$$

Define

$$N_2 \le \frac{\partial}{\partial s}G(t,s)|_{s=t_j} \le M_2,$$

$$M = \max\{M_1, M_2\}, N = \min\{N_1, N_2\},$$

then

$$1 \ge \frac{G(t,s)}{M} \ge \frac{N}{M} = \sigma.$$

The following lemma is fundamental to our discussion. Since the method is similar to that in the literature [10], we omit the proof.

Lemma 4. *$x\in PC^1(J)\cap C^2(J')$ is a solution of problem (1) if and only if $x\in PC(J)$ is a solution of the equation*

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)f(s,x_s)ds \\ &+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \frac{\partial G(t,s)}{\partial s}|_{s=t_j} I_j(x(t_j)). \end{aligned} \quad (5)$$

Let K be a cone in $PC(JR)$, which is defined as

$$K = \{x\in PC(JR) : x(t) \ge \sigma\|x\|_{PC}, t\in J\}.$$

Define an operator

$$\begin{aligned} (\mathbb{T}x)(t) = &\int_t^{t+T} G(t,s)f(s,x_s)ds \\ &+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \frac{\partial G(t,s)}{\partial s}|_{s=t_j} I_j(x(t_j)), \end{aligned}$$

that is

$$\begin{aligned} (\mathbb{T}x)(t) = &\int_t^{t+T} G(t,s)f(s,x_s)ds \\ &+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \Big(p(t_j)G(t,t_j) \\ &- \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\Big) I_j(x(t_j)). \end{aligned}$$

Then we have the following lemma.

Lemma 5. *$T : K \to K$ is well defined.*

Proof: For each $x\in K$, by $(A3)$, we have $(Tx)(t)$ is continuous in t and

$$\begin{aligned} &(\mathbb{T}x)(t+T) \\ =& \int_{t+T}^{t+2T} G(t,s)f(s,x_s)ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \Big(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\Big) \\ &\times I_j(x(t_j)) \\ =& \int_t^{t+T} G(t+T,v+T)f(v+T,x_{v+T})dv \\ &+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \Big(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\Big) \\ &\times I_j(x(t_j)) \\ =& \int_t^{t+T} G(t,v)f(v,x_v)dv + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} \Big(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\Big) \\ &\times I_j(x(t_j)) \\ =& (\mathbb{T}x)(t). \end{aligned}$$

Hence, for $x\in K$, we have

$$\begin{aligned} \|(\mathbb{T}x)\| \le\ & M\Big(\int_t^{t+T} f(s,x_s)ds + \sum_{j:t_j\in[t,t+T]} J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} I_j(x(t_j))\Big), \\ \text{and} & \\ (\mathbb{T}x)(t) \ge\ & N\Big(\int_t^{t+T} f(s,x_s)ds + \sum_{j:t_j\in[t,t+T]} J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} I_j(x(t_j))\Big) \\ =\ & \frac{N}{M}M\Big(\int_t^{t+T} f(s,x_s)ds + \sum_{j:t_j\in[t,t+T]} J_j(x(t_j)) \\ &+ \sum_{j:t_j\in[t,t+T]} I_j(x(t_j))\Big) \\ \ge\ & \sigma\|\mathbb{T}x\|. \end{aligned} \quad (6)$$

Therefore, $Tx\in K$, this complete the proof.

Lemma 6. *$T : K \to K$ is completely continuous.*

Proof: We first show that T is continuous. By (A3), for any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\{\phi, \psi\in BC, \|\phi\| \le L, \|\psi\| \le L, \|\phi-\psi\| < \delta\},$$

imply

$$\sup_{0\le s\le T}|f(s,\phi_s)) - f(s,\psi_s)|_0 < \frac{\varepsilon}{3MT},$$

and since $I_j \in C(R^+, R), J_j \in (R^+, R^+)$, we have

$$\|I_j(\phi) - I_j(\psi)\| < \frac{\varepsilon}{3Mk},\ \|J_j(\phi) - J_j(\psi)\| < \frac{\varepsilon}{3Mk}.$$

If $x, y, \in K$ with $\|x\| \le L, \|y\| \le L, \|x - y\| \le \delta$, then

$$\begin{aligned}
&|(\mathbb{T}x)(t) - (\mathbb{T}y)(t)|_0 \\
\le& \int_t^{t+T} |G(t,s)||f(s,x_s) - f(s,y_s)|_0 ds \\
&+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)|J_j(x(t_j)) - J_j(y(t_j))| \\
&+ \sum_{j:t_j\in[t,t+T]} \frac{\partial G(t,s)}{\partial s}|_{s=t_j}||I_j(x(t_j)) - I_j(y(t_j)) \\
\le& M\int_0^T |f(s,x_s) - f(s,y_s)|_0 ds \\
&+M \sum_{j:t_j\in[t,t+T]} |J_j(x(t_j)) - J_j(y(t_j))| \\
&+M \sum_{j:t_j\in[t,t+T]} |I_j(x(t_j)) - I_j(y(t_j))| \\
<& \varepsilon,
\end{aligned}$$

for all $t \in [0, T]$, this yields $\|Tx - Ty\| < \varepsilon$, thus T is continuous.

Next we show that T maps any bounded sets in K into relatively compact sets. Now we first prove that f maps bounded sets into bounded sets.

Indeed, let $\varepsilon = 1$, by (H3), for any $\mu > 0$, there exists $\delta > 0$ such that $\{x, y \in BC, \|x\| \le \mu, \|y\| \le \mu, \|x - y\| \le \delta, 0 \le s \le T\}$ imply

$$|f(s,x_s) - f(s,y_s)|_0 < 1.$$

Choose a positive integer N such that $\frac{\mu}{N} < \delta$. Let $x \in BC$ and define $x^k(t) = \frac{x(t)k}{N}, k = 0, 1, 2, \cdots, N$. If $\|x\| < \mu$, then

$$\begin{aligned}
\|x^k - x^{k-1}\| &= \sup_{t\in R}\left|\frac{x(t)k}{N} - \frac{x(t)(k-1)}{N}\right| \\
&\le \|x\|\frac{1}{N} \le \frac{\mu}{N} < \delta.
\end{aligned}$$

Thus

$$|f(s,x_s^k) - f(s,x_s^{k-1})|_0 < 1$$

for all $s \in [0, T]$, this yields

$$\begin{aligned}
|f(s,x_s)|_0 &= |f(s,x_s^N)| \\
&\le \sum_{k=1}^N |f(s,x_s^k) - f(s,x_s^{k-1})|_0 + |f(s,0)|_0 \\
&< N + |f(s,0)|_0 =: W, \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
|I_j(x(t_j))|_0 &= |I_j(x^N(t_j))| \\
&\le \sum_{k=1}^N |I_j(x^N(t_j)) - I_j(x^{N-1}(t_j))|_0 \\
&\quad +|I_j(0)|_0 < N + |I_j(0)|_0 =: U, \qquad (8)\\
|J_j(x(t_j))|_0 &= |J_j(x^N(t_j))| \\
&\le \sum_{k=1}^N |J_j(x^N(t_j)) - J_j(x^{N-1}(t_j))|_0 \\
&\quad +|J_j(0)|_0 < N + |J_j(0)|_0 =: V. \qquad (9)
\end{aligned}$$

It follows from (6) that for $t \in [0, R]$

$$\begin{aligned}
\|\mathbb{T}x\| &= \sup_{t\in R}|(\mathbb{T}x)(t)| \\
&\le M\int_0^T |f(s,x_s)|ds + M \sum_{j:t_j\in[t,t+T]} J_j(x(t_j)) \\
&\quad +M \sum_{j:t_j\in[t,t+T]} I_j(x(t_j)) \\
&\le MTW + Mk(U + V).
\end{aligned}$$

Finally, for $t \in R$, we have

$$\begin{aligned}
&(\mathbb{T}x)'(T) \\
=& \int_t^{t+T} \big[- q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\big] f(s,x_s)ds \\
&+ \sum_{j:t_j\in[t,t+T]} \big[- q(s)G(t,s) \\
&+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\big] J_j(x(t_j)) \\
&+ \sum_{j:t_j\in[t,t+T]} \big[p(t_j)\big[- q(t_j)G(t,t_j) \\
&+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\big] \\
&-\frac{\exp\int_t^s q(v)dv}{\exp\int_0^T q(v)dv - 1}\big] I_j(x(t_j)), \qquad (10)
\end{aligned}$$

combine (7)-(10) and Corollary 2.1, we obtain

$$\begin{aligned}
\left\|\frac{d}{dt}(\mathbb{T}x)(t)\right\| &= \sup_{t\in R}|(\mathbb{T}x)'(t)| \\
&\le (TW + kV + kPU)(M\|Q\| \\
&\quad +\frac{e^m}{e^l - 1}) + \frac{e^m}{e^l - 1}k\|Q\|U,
\end{aligned}$$

where $\|Q\| = \max_{0\le t\le T}|q(t)|, \|P\| = \max_{0\le t\le T}|p(t)|$.

Hence $\{\mathbb{T}x : x \in K, \|x\| \le \mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0, T]$. By a theorem of Arzela-Ascoli, we know that the function $\mathbb{T}$ is completely continuous.

For convenience in the following discussion, we introduce the following notations:

$$I^0 = \max_{0\le u\le T} \sum_{j:t_j\in[t,t+T]} I_j(u),$$

$$J^0 = \max_{0\le u\le T} \sum_{j:t_j\in[t,t+T]} J_j(u),$$

$$I_0 = \min_{0\le u\le T} \sum_{j:t_j\in[t,t+T]} I_j(u),$$

$$J_0 = \min_{0\le u\le T} \sum_{j:t_j\in[t,t+T]} J_j(u).$$

III. Main results

Theorem 2. *Suppose that $(A_1) - (A_3)$ hold, and there are positive constants r_1, r_2 and r_3 with $r_1 < r_3 < r_2$ such that*

(A_4) $\quad I_0 + J_0 < \frac{r_1}{2N}; I^0 + J^0 > \frac{r_3}{2M};$

(A_5) $\quad \inf_{\|\phi\|=r_1,\phi\in K} \int_0^T |f(s,\phi_s)|_0\, ds > \frac{r_1}{2N},$

$\quad \inf_{\|\phi\|=r_2,\phi\in K} \int_0^T |f(s,\phi_s)|_0\, ds > \frac{r_2}{2N};$

$$(A_6) \qquad \sup_{\|\phi\|=r_3,\phi\in K}\int_0^T |f(s,\phi_s)|_0\,ds < \frac{r_3}{2M}.$$

Then system (1) has at least two positive T-periodic solutions.

Proof: Let $\Omega_1 = \{u \in X : \|u\| < r_1\}$. Then for any $u \in K\cap\ \partial\Omega_1$, we have $u - \mathbb{T}u \neq \alpha u_0, u_0 \in K\setminus\{0\}, \alpha \geq 0$. For the sake of contradiction, we choose $u_0 = (1, 1, \cdots, 1)^T \in R^n$. Suppose that there exists $\bar{u} \in K \cap \partial\Omega_1$ such that $\bar{u} - \mathbb{T}\bar{u} = \alpha_0 u_0$ for some $\alpha_0 > 0$. Then, we have

$$\bar{u}(t) = (\mathbb{T}\bar{u})(t) + \alpha_0.$$

From this, the definition of $\mathbb{T}$, it follows that

$$\begin{aligned}
|\bar{u}x| &= \int_t^{t+T} G(t,s)f(s,x_s)ds \\
&\quad + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) \\
&\quad + \sum_{j:t_j\in[t,t+T]} \Big(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}\Big) \\
&\quad \times I_j(x(t_j)) + \alpha_0 \\
&> N\big(\int_t^{t+T} f(s,x_s)ds + \sum_{j:t_j\in[t,t+T]} J_j(x(t_j)) \\
&\quad + \sum_{j:t_j\in[t,t+T]} I_j(x(t_j))\big) + \alpha_0 \\
&> N\big(\int_t^{t+T} f(s,x_s)ds\big) + I_0 + J_0.
\end{aligned}$$

Hence, we have

$$r_1 = \|\bar{u}\| > N\big(\int_t^{t+T} |f(s,x_s)ds|_0\big) + I_0 + J_0 > r_1,$$

which is a contradiction. Therefore, we derive that

$$u - \mathbb{T}u \neq \alpha u_0, \forall u_0 \in K\setminus\{0\}, \alpha \geq 0. \qquad (11)$$

Let $\Omega_2 = \{u \in X : \|u\| < r_2\}$. Then for any $u \in K\cap\ \partial\Omega_2$, applying the second inequality in (A_5), similarly to the proof of (11), we have $u - \mathbb{T}u \neq \alpha u_0, u_0 \in K\setminus\{0\}, \alpha \geq 0$.

On the other hand, Let $\Omega_3 = \{u \in X : \|u\| < r_3\}$. Then for any $u \in K\cap\ \partial\Omega_3$, from the definition of $\mathbb{T}$, we have

$$|\mathbb{T}u| \leq M\big(\int_t^{t+T} f(s,x_s)ds\big) + I^0 + J^0.$$

Hence, in view of (A_5), one has

$$\|Tu\| \leq M\big(\int_t^{t+T} |f(s,x_s)ds|_0\big) + I^0 + J^0 < r_3,$$

that is ,

$$\|\mathbb{T}u\| < \|u\|\, \forall u \in K \cap \partial\Omega_3.$$

Therefore,

$$\mathbb{T}u \neq \mu u, \forall u \in K \cap \partial\Omega_3, \mu \geq 1.$$

It is clear that $\Omega_1 \subset \Omega_3 \subset \Omega_2$, by Theorem 1, we can conclude that $\mathbb{T}$ has two fixed points $u_1 \in K\cap(\overline{\Omega}_3\setminus\Omega_1)$ and $u_2 \in K\cap(\overline{\Omega}_2\setminus\Omega_3)$ with $r_1 < \|u_1\| < r_3, r_3 < \|u_2\| < r_2$. Therefore, $u_1(t)$ and $u_2(t)$ are positive solutions of system (1). This complete the proof.

Theorem 3. *Suppose that $(A_1) - (A_3)$ hold, and that there are positive constants R_1, R_2 and R_3 with $R_1 < R_3 < R_2$ such that*

$$(A_6) \qquad I_0 + J_0 < \frac{R_1}{2M}; I^0 + J^0 > \frac{R_3}{2N};$$

$$(A_7) \qquad \sup_{\|\phi\|=R_1,\phi\in K}\int_0^T |f(s,\phi_s)|_0\,ds < \frac{R_1}{M},$$

$$\sup_{\|\phi\|=R_2,\phi\in K}\int_0^T |f(s,\phi_s)|_0\,ds < \frac{R_2}{M};$$

$$(A_8) \qquad \inf_{\|\phi\|=R_3,\phi\in K}\int_0^T |f(s,\phi_s)|_0\,ds > \frac{R_3}{N},$$

Then the system (1) has at least two positive T-periodic solutions.

Proof: By condition (A_7), from the proof of Theorem 2, we know that

$$\mathbb{T}u \neq \mu u, \forall u \in \partial\Omega_4, \mu \geq 1,$$

$$\mathbb{T}u \neq \mu u, \forall u \in \partial\Omega_5, \mu \geq 1,$$

where $\Omega_4 = \{\mathbb{T} \in X : \|\mathbb{T}\| < R_1\}, \Omega_5 = \{\mathbb{T} \in X : \|T\| < R_2\}$.

From condition (A_8), Let $\Omega_6 = \{\mathbb{T} \in X : \|\mathbb{T}\| < R_3\}$, for any $u \in K \cap \Omega_6$, it is similar to the proof of (11), we have

$$u - \mathbb{T}u \neq \alpha u_0, u_0 \in K\setminus\{0\}, \alpha \geq 0.$$

It is clear that $\Omega_4 \subset \Omega_6 \subset \Omega_5$, by Theorem 1, we can conclude that $\mathbb{T}$ has two fixed points $u_3 \in K\cap(\overline{\Omega}_6\setminus\Omega_4)$ and $u_4 \in K\cap(\overline{\Omega}_5\setminus\Omega_6)$ with $R_1 < \|u_3\| < R_3, R_3 < \|u_4\| < R_2$. Therefore, $u_3(t)$ and $u_4(t)$ are positive solutions of system (1). This complete the proof.

IV. Conclusion

This paper studied the existence problem for a second order impulsive functional differential equations. Some existence results of multiplicity positive periodic solutions are obtained. The proof techniques used in this paper are new and can be used to many other functional differential equations, for example [11-16].

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