E-Bayesian Estimation for Burr-X Distribution Based on Type-I Hybrid Censoring Scheme

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***Abstract*—In this paper, Burr-X distribution with Type-I hybrid censored data is considered. E-Bayesian estimation (expectation of the Bayesian estimate) and the corresponding maximum likelihood and Bayesian estimation methods are discussed for the distribution parameter and the reliability function. Bayesian and E-Bayesian estimates are derived by using LINEX and squared error loss (SEL) functions. By applying Markov chain Monte Carlo (MCMC) techniques Bayesian and E-Bayesian estimates are obtained. An illustrative examples of Type-I hybrid censored samples and real data set are presented. Finally, a comparison among the proposed estimation methods is conducted.**

***Index Terms*—Bayesian estimation, Burr-X distribution, E-Bayesian estimation, Hybrid censoring scheme, Maximum likelihood estimation, MCMC method.**

I. Introduction

BURR-X distribution is a member of Burr distributions family which was suggested by [1]. This distribution is important in statistics and operations research. It is applied in many fields such as health, agriculture and biology. The probability density function (PDF) of Burr-X distribution is given as follows:

$$f(x;\alpha) = 2\alpha x\exp(-x^2)(1-\exp(-x^2))^{\alpha-1},\ \ x>0,\ \alpha>0, \tag{1}$$

hence the cumulative distribution function (CDF) is given by

$$F(x;\alpha) = (1-\exp(-x^2))^{\alpha},\ \ x>0,\ \alpha>0, \tag{2}$$

where α is the shape parameter.

The reliability function $R(t)$ and the hazard rate function $h(t)$ for Burr-X distribution are, respectively, given by

$$R(t) = 1-(1-\exp(-t^2))^{\alpha},\ \ t>0, \tag{3}$$

and

$$h(t) = \frac{2\alpha t\exp(-t^2)(1-\exp(-t^2))^{\alpha-1}}{1-(1-\exp(-t^2))^{\alpha}},\ \ t>0. \tag{4}$$

Recently, there have been many publications on Burr-X distribution, for example, ([2], [3], [4], [5] and [6]).

Hybrid censoring scheme (HCS) is a mixture of Type-I and Type-II censoring schemes. In this type, the experiment is terminated when a pre-fixed number r, out of n items have failed or when a pre-determined time T, has been reached. In Type-I HCS, the life-testing experiment is terminated at a random time $T_* = \min\{X_{k:n}, T\}$, where, $X_{k:n}$ represents the failure time of the k^{th} item, T is the pre-fixed time allowed for the test, $k\in\{1,2,\cdots,n\}$ and $T\in(0,\infty)$ are determined in advance. In other words, the experiment is terminated as soon as a pre-determined number k out of n items has failed or a pre-fixed time T has been reached.
In Type-II HCS, the life-testing experiment is terminated at a random time $T^* = \max\{X_{k:n}, T\}$, i.e. the experiment is terminated when the later of two stopping rules is reached, this guarantees that at least k failures will be observed.

In this paper, we use Type-I HCS, in which we have one of the following types of censored data
Case I : $\{X_{1:n} < X_{2:n} < \cdots < X_{k:n}\}$, if $X_{k:n} < T$, and pre-specified k number of failure occurred before the censoring time T,

Case II : $\{X_{1:n} < X_{2:n} < \cdots < X_{m:n}\}$, if $X_{m:n} \le T < X_{m+1:n}$, and only $m<k$ number of failure occurred before the pre-specified censoring time T.

We suppose that $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are n failure lifetime observations under Type-I hybrid censored sample from Burr-X distribution. Therefore, the likelihood function for the considered cases is given by

Case I:

$$\begin{aligned} L(\alpha) &= \frac{n!}{(n-k)!}\Big\{1-F(x_{k:n})\Big\}^{n-k}\prod_{i=1}^{k} f(x_{i:n}) \\ &= \frac{n!}{(n-k)!}\Big\{1-\big(1-\exp(-x_{k:n}^2)\big)^{\alpha}\Big\}^{n-k} \\ &\times\prod_{i=1}^{k} 2\alpha x_{i:n}\exp(-x_{i:n}^2)\big(1-\exp(-x_{i:n}^2)\big)^{\alpha-1}. \end{aligned} \tag{5}$$

Case II:

$$\begin{aligned} L(\alpha) &= \frac{n!}{(n-m)!}\Big\{1-F(T)\Big\}^{n-m}\prod_{i=1}^{m} f(x_{i:n}) \\ &= \frac{n!}{(n-m)!}\Big\{1-\big(1-\exp(-T^2)\big)^{\alpha}\Big\}^{n-m} \\ &\times\prod_{i=1}^{m} 2\alpha x_{i:n}\exp(-x_{i:n}^2)\big(1-\exp(-x_{i:n}^2)\big)^{\alpha-1}. \end{aligned} \tag{6}$$

By combining Eqs. (5) and (6), the likelihood function can

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be written as follows:

$$L(\alpha) = \frac{n!}{(n-r)!}\Big\{1-\big(1-\exp(-s^2)\big)^{\alpha}\Big\}^{n-r} \times \prod_{i=1}^{r} 2\alpha x_{i:n}\exp(-x_{i:n}^2)\big(1-\exp(-x_{i:n}^2)\big)^{\alpha-1}, \tag{7}$$

where,

$$r = \begin{cases} k, & \text{for Case I,} \\ m, & \text{for Case II,} \end{cases}$$

and

$$s = \begin{cases} x_{k:n}, & \text{for Case I,} \\ T, & \text{for Case II.} \end{cases}$$

The remainder of the paper is organized as follows. Maximum likelihood estimation is presented in Section II. Bayesian estimation under SEL and LINEX loss functions is described in Section III. In Section IV, E-Bayesian estimation under SEL and LINEX loss functions is introduced. MCMC method is presented in Section V. In Section VI, illustrative examples, real data set and conclusion of the results are reported.

II. Maximum Likelihood Estimation

By taking the logarithm of Eq. (7), the log-likelihood function can be written as follows:

$$\begin{aligned}\ell &= \ln L(\alpha) \\ &= (n-r)\ln\Big\{1-\big(1-\exp(-s^2)\big)^{\alpha}\Big\} \\ &+ r\ln(2\alpha) + \sum_{i=1}^{r}\ln x_{i:n} + \sum_{i=1}^{r}(-x_{i:n}^2) \\ &+ (\alpha-1)\sum_{i=1}^{r}\ln\Big(1-\exp(-x_{i:n}^2)\Big).\end{aligned} \tag{8}$$

The maximum likelihood estimate (MLE) of the unknown parameter α, is obtained by setting the first partial derivative of Eq. (8) to zero with respect to α and solving numerically the following equation.

$$\frac{\partial \ell}{\partial \alpha} = 0, \tag{9}$$

we obtain $\hat{\alpha}$, the MLE of the unknown parameter α. Also, we can obtain MLE $\hat{R(t)}$ of the reliability function $R(t)$ by replacing α by its MLE, $\hat{\alpha}$, in Eq.(3) as follows:

$$\hat{R(t)} = 1-(1-\exp(-t^2))^{\hat{\alpha}}. \tag{10}$$

III. Bayesian Estimation

In this section, we derive Bayesian estimates for the parameter α and the reliability function $R(t)$ of the Burr-X distribution based on Type-I HCS. We consider the prior PDF for the parameter α to be $Gamma(a,b)$ and is written as follows:

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)}\,\alpha^{a-1}\exp(-b\alpha), \;\; a,b>0, \tag{11}$$

from (7) and (11), the posterior PDF of α can be written in the following form:

$$\pi(\alpha|x) = K^{-1}\alpha^{r+a-1}\exp(-b\alpha)\Big\{1-W_s^{\alpha}\Big\}^{n-r} \times \prod_{i=1}^{r} x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}, \tag{12}$$

where $W_s = 1-\exp(-s^2)$, $W_{x_{i:n}} = 1-\exp(-x_{i:n}^2)$ and K is a normalizing constant given by

$$K = \int_0^{\infty}\pi(\alpha|x)d\alpha. \tag{13}$$

A. Bayesian Estimates under Squared Error Loss Function

The Bayesian estimate of the parameter α under SEL function, is the posterior mean, i.e.

$$\begin{aligned}\hat{\alpha}_{BS} &= E[\alpha|x] \\ &= \int_0^{\infty}\alpha\pi(\alpha|x)d\alpha \\ &= K^{-1}\int_0^{\infty}\alpha^{r+a}\exp(-b\alpha)\Big\{1-W_s^{\alpha}\Big\}^{n-r} \\ &\times \prod_{i=1}^{r} x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}d\alpha,\end{aligned} \tag{14}$$

also, the Bayesian estimate of the reliability function $R(t)$ based on SEL function is given by

$$\begin{aligned}\hat{R(t)}_{BS} &= E[R(t)|x] \\ &= \int_0^{\infty}R(t)\pi(\alpha|x)d\alpha \\ &= K^{-1}\int_0^{\infty}1-(1-\exp(-t^2))^{\alpha}\alpha^{r+a-1}\exp(-b\alpha) \\ &\times \Big\{1-W_s^{\alpha}\Big\}^{n-r}\prod_{i=1}^{r} x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}d\alpha.\end{aligned} \tag{15}$$

B. Bayesian Estimates under LINEX Loss Function

Based on LINEX loss function, the Bayesian estimate of the parameter α is given by

$$\begin{aligned}\hat{\alpha}_{BL} &= \frac{-1}{h}\ln E[\exp(-h\alpha)|x] \\ &= \frac{-1}{h}\ln\int_0^{\infty}\exp(-h\alpha)\pi(\alpha|x)d\alpha \\ &= \frac{-1}{Kh}\ln\int_0^{\infty}\alpha^{r+a-1}\exp(-(b+h)\alpha) \\ &\times \Big\{1-W_s^{\alpha}\Big\}^{n-r}\prod_{i=1}^{r} x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}d\alpha, \quad h\neq 0,\end{aligned} \tag{16}$$

also, the Bayesian estimate of the reliability function $R(t)$, under LINEX loss function, is given by

$$\begin{aligned}\hat{R(t)}_{BL} &= \frac{-1}{h}\ln E[\exp(-hR(t))|x] \\ &= \frac{-1}{h}\ln\int_0^\infty \exp(-hR(t))\pi(\alpha|x)d\alpha \\ &= \frac{-1}{Kh}\ln\int_0^\infty \alpha^{r+a-1}e^{\left(h[(1-\exp(-t^2))^\alpha-1]-b\alpha\right)} \\ &\times\left\{1-W_s^\alpha\right\}^{n-r}\prod_{i=1}^{r}x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}d\alpha, \\ & h\neq 0.\end{aligned} \tag{17}$$

For more details about Byesian approach and reliability, one can see, ([7], [8], and [9]).

IV. E-Bayesian Estimation

According to [10], the prior parameters a and b should be selected to guarantee that $\pi(\alpha)$ is a decreasing function of α. The derivative of $\pi(\alpha)$ with respect to α is given as follows:

$$\frac{d\pi(\alpha)}{d\alpha} = \frac{b^a}{\Gamma(a)}\alpha^{a-2}\exp(-b\alpha)\{(a-1)-b\alpha\}.$$

Thus, for $0 < a < 1$ and $b > 0$, the prior PDF $\pi(\alpha)$ is a decreasing function of α. We assume that the hyper parameters a and b are independent with bi-variate PDF as follows:

$$\pi(a,b) = \pi_1(a)\pi_2(b),$$

the E-Bayesian estimate of the parameter α and the reliability function $R(t)$ are, respectively, given as follows:

$$\hat{\alpha}_{EB} = E[\alpha_B|x] = \int\int \hat{\alpha_B}(a,b)\pi(a,b)dadb, \tag{18}$$

and

$$\hat{R(t)}_{EB} = E[R(t)_B|x] = \int\int \hat{R(t)}_B\pi(a,b)dadb. \tag{19}$$

where $\hat{\alpha}_B(a,b)$ and $\hat{R(t)}_B$ are the Bayesian estimate of the parameter α and the reliability function $R(t)$, respectively, under SEL and LINEX loss functions. In recent years, there has been an increasing interest in this approach of estimation, see for example, ([11], [12], [13], [14], [15], [16] and [17]).

A. E-Bayesian Estimate of α

To derive the E-Bayesian estimate of α, we consider three different distributions of a and b to illustrate the effect of these prior PDFs on the E-Bayesian estimates. The prior PDFs of a and b are given as follows:

$$\left.\begin{aligned}\pi_1(a,b) &= \frac{2a}{c}, && 0<a<1, 0<b<c, \\ \pi_2(a,b) &= \frac{2b}{c^2}, && 0<a<1, 0<b<c, \\ \pi_3(a,b) &= \frac{3b^2}{c^3}, && 0<a<1, 0<b<c,\end{aligned}\right\} \tag{20}$$

the E-Bayesian estimate of α under SEL function can be obtained from (14), (18) and (20) as follows:

$$\hat{\alpha}_{EBS_j} = \int\int \hat{\alpha}_{BS}(a,b)\pi_j(a,b)dadb, \qquad j=1,2,3, \tag{21}$$

also, the E-Bayesian estimate of α under LINEX loss function can be obtained from (16), (18) and (20) as follows:

$$\hat{\alpha}_{EBL_j} = \int\int \hat{\alpha}_{BL}(a,b)\pi_j(a,b)dadb, \qquad j=1,2,3. \tag{22}$$

B. E-Bayesian Estimate of The Reliability Function $R(t)$

The E-Bayesian estimate of the reliability function $R(t)$ under SEL function can be obtained from (15), (19) and (20) as follows:

$$\hat{R(t)}_{EBS_j} = \int\int \hat{R(t)}_{BS}\pi_j(a,b)dadb, \qquad j=1,2,3, \tag{23}$$

also, the E-Bayesian estimate of the reliability function $R(t)$ by using LINEX loss function can be obtained from (17), (19) and (20) and given by

$$\hat{R(t)}_{EBL_j} = \int\int \hat{R(t)}_{BL}\pi_j(a,b)dadb, \qquad j=1,2,3. \tag{24}$$

It is noticed that the Bayesian and E-Bayesian estimators can not be expressed in an explicit form. Therefore, we use the MCMC method to derive Bayesian and E-Bayesian estimates of the parameter α and the reliability function $R(t)$.

V. MCMC Method

This section describes the MCMC method that has been used to derive Bayesian and E-Bayesian estimates of the parameter α and the reliability function $R(t)$ of Burr-X distribution. We consider the Metropolis-Hastings algorithm, to generate posterior sample for the parameter α from the full conditional posterior PDF given in the following form.

$$\pi^*(\alpha|x) = \alpha^{r+a-1}\exp(-b\alpha)\left\{1-W_s^\alpha\right\}^{n-r}\prod_{i=1}^{r}W_{x_{i:n}}^\alpha. \tag{25}$$

It is noticed from (25) that the full conditional posterior PDF of the parameter α cannot be reduced to a well -known distribution, so, we use normal distribution as a proposal distribution.

Algorithm:

Step 1: Start with initial guess of α say $\alpha^{(0)} = \hat{\alpha}_{MLE}$.

Step 2: At iteration j generate $\alpha^{(*)}$ from a normal distribution as a proposal distribution.

Step 3: Generate a sample u from the uniform distribution $U(0,1)$ and take $z = logu$.

Step 4: Calculate the acceptance probability

$$r(\alpha^{j-1}|\alpha^{(*)}) = min\Big[1, \frac{\pi^*(\alpha^{(*)}|x)}{\pi^*(\alpha^{(j-1)}|x)}\Big] \tag{26}$$

Step 5: If $z < r$ accept $\alpha^{(*)}$ as $\alpha^{(j)}$, otherwise, $\alpha^{(j)} = \alpha^{(j-1)}$

Step 6: Compute $R(t)$ as follows:

$$R^{(j)}(t) = 1-(1-\exp(-t^2))^{\alpha^{(j)}}. \tag{27}$$

Step 7: Repeat $(3-6)$ N times to obtain $\alpha^{(j)}$ and $R^{(j)}(t)$, $j = M+1, \cdots, N$.

Step 8: Bayesian estimates for the parameter α and the reliability function $R(t)$ under SEL function are given, respectively, by

$$\hat{\alpha}_{BS} = \frac{1}{N-M} \sum_{j=M+1}^{N} \alpha^{(j)}. \tag{28}$$

$$\hat{R(t)}_{BS} = \frac{1}{N-M} \sum_{j=M+1}^{N} R^{(j)}(t). \tag{29}$$

where M is an optional burn-in period.

Step 9: Bayesian estimates of the parameter α and the reliability function $R(t)$, under LINEX loss function are, respectively, given by

$$\hat{\alpha}_{BL} = \frac{-1}{h} \ln\left[\frac{1}{N-M} \sum_{j=M+1}^{N} e^{-h\alpha^{(j)}}\right], \quad h \neq 0, \tag{30}$$

$$\hat{R(t)}_{BL} = \frac{-1}{h} \ln\left[\frac{1}{N-M} \sum_{j=M+1}^{N} e^{-hR^{(j)}(t)}\right], \quad h \neq 0. \tag{31}$$

A. Simulation Study

- We choose the values n, r, T, h.
- We generate a and b from (20).
- We generate α from Gamma(a,b).
- We generate a random sample of size n from $U(0, 1)$.
- We generate Type-I hybrid censored sample from Burr-X distribution using inverse function method as follows:

$$X = \{-\ln(1 - U^{\frac{1}{\alpha}})\}^{\frac{1}{2}}.$$

- The $MLEs$ of the unknown parameter α and the reliability function $R(t)$ are obtained from (9), (10), respectively.
- Using Metropolis-Hastings, we generate a Markov chain with 11000 observations, discarding the first 1000 observations as a "burn-in".
- Using MCMC samples, we compute Bayesian estimates $\hat{\alpha}_{BS}$ and $\hat{R(t)}_{BS}$ under SEL function from (28) and (29), respectively.
- We derive Bayesian estimates $\hat{\alpha}_{BL}$ and $\hat{R(t)}_{BL}$ under LINEX loss function from (30) and (31), respectively.
- We compute E-Bayesian estimates $\hat{\alpha}_{EBS}$ and $\hat{R(t)}_{EBS}$ under SEL function from (21) and (23), respectively.
- We compute E-Bayesian estimates $\hat{\alpha}_{EBL}$ and $\hat{R(t)}_{EBL}$ under LINEX loss function from (22) and (24), respectively.
- We compute $(\hat{\alpha} - \alpha)^2$ where $\hat{\alpha}$ is the estimate of α.
- We compute $(\hat{R(t)} - R(t))^2$ where $\hat{R(t)}$ is the estimate of $R(t)$.
- We compute mean squared error (MSE) of estimates α and $R(t)$, respectively, by

$$MSE(\hat{\alpha}) = \frac{1}{1000} \sum (\hat{\alpha} - \alpha)^2$$

.

$$MSE(\hat{R(t)}) = \frac{1}{1000} \sum (\hat{R(t)} - R(t))^2$$

.

- The numerical results are displayed in Tables(I-IV).

VI. Illustrative Examples

In this section, we consider a life test when 20 units of lifetimes following Burr-X distribution are put under the test with changing the value of T, the maximum allowable time of the test, we can observe the following:

- When $n = 20, r = 15, \alpha = 1.83675$ and $T = 0.95$. In this case, we obtain the following data: 0.421457, 0.534205, 0.596791, 0.638163, 0.770691, 0.772107, 0.778328, 0.883723, 1.06571, 1.11015, 1.20905, 1.30329, 1.3112, 1.33565 and 1.36371, hence the test is terminated at $T_* = min\{X_{15:20}, T\} = min\{1.36371, 0.95\} = 0.95$. That is only 8 items fail at random time T_*=0.95.

- When $n = 20, r = 15, \alpha = 1.83675$ and $T = 1.5$. In this case, we obtain the following data: 0.582787, 0.699181, 0.776085, 0.786884, 0.830979, 1.00813, 1.0599, 1.07425, 1.07554, 1.08599, 1.24903, 1.25451, 1.32985, 1.33519 and 1.37724, hence the test is terminated at $T_* = min\{X_{15:20}, T\} = min\{1.37724, 1.5\} = 1.37724$. That is only 15 items fail at random time $T_* = 1.37724$

A. Example (Real Data Set)

We consider a numerical example of real testing data set to illustrate the performance of the proposed methods in a practical application. These data were reported by [18], representing minority electron mobility for p-type $Ga_{1-x}Al_xAs$ with seven different values of mole fraction. Only one data set related to the mole fractions 0.25 is considered. This data set was used by [19] , who proved that Burr-X distribution gives a good fit for this data set. The data set is 21 observations given as follows:

Data Set (belongs to mole fraction 0.25): 3.051, 2.779, 2.604, 2.371, 2.214, 2.045, 1.715, 1.525, 1.296, 1.154, 1.016, 0.7948, 0.7007, 0.6292, 0.6175, 0.6449, 0.8881, 1.115, 1.397, 1.506, and 1.528.

By applying Type-I HCS on these uncensored data, we observe the following cases:

- When $n = 21, r = 15, \alpha = 2.15$ and $T = 1.6$. In this case, we obtained the following data: 0.6175, 0.6292, 0.6449, 0.7007, 0.7948, 0.8881, 1.016, 1.115, 1.154, 1.296, 1.397 1.506, 1.525, 1.528, 1.715, hence the test is terminated at $T_* = min\{X_{15:21}, T\} = min\{1.715, 1.6\} = 1.6$. That is only 14 items fail out of 21, at a random time $T_* = 1.6$.
- When $n = 21, r = 15, \alpha = 2.15$ and $T = 2$. In this case, we obtained the following data: 0.6175, 0.6292, 0.6449, 0.7007, 0.7948, 0.8881, 1.016, 1.115, 1.154, 1.296, 1.397, 1.506, 1.525, 1.528, 1.715, hence the test is terminated at $T_* = min\{X_{15:21}, T\} = min\{1.715, 2\} = 1.715$. That is only 15 items fail out of 21, at a random time $T_* = 1.715$.
  All estimates of the distribution parameter and the reliability function with respect to the real data set are obtained based on the same loss functions and procedures. The numerical results are displayed in Tables (V-VI).

TABLE I: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for α when $T = 0.95$, $\alpha = 1.83675$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
			$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$	$\hat{\alpha}_{EBL1}$	$\hat{\alpha}_{EBL2}$	$\hat{\alpha}_{EBL3}$
20	10	Mean	1.0646	2.2257	2.1477	1.9942	1.7449	1.8322	1.9317	1.6971	1.7794
		MSE	0.63830	0.19384	0.14055	0.05896	0.03458	0.02885	0.04419	0.04638	0.03294
20	15	Mean	1.0768	2.2343	2.1560	2.0019	1.7517	1.839	1.9392	1.7037	1.7863
		MSE	0.61799	0.19792	0.14282	0.05928	0.03172	0.02701	0.04329	0.04277	0.03020
30	20	Mean	1.0264	2.2967	2.2365	2.0578	1.8006	1.8906	2.0096	1.7637	1.8499
		MSE	0.68002	0.25219	0.20194	0.08148	0.02625	0.03041	0.06361	0.03107	0.02859
30	25	Mean	1.0336	2.3122	2.2522	2.0717	1.8127	1.9034	2.0237	1.7760	1.8629
		MSE	0.66857	0.26583	0.21395	0.08715	0.02502	0.03139	0.06802	0.02892	0.02854
40	30	Mean	1.0021	2.3458	1.8673	2.1018	1.8391	1.9310	2.0619	1.8085	1.8974
		MSE	0.71331	0.29569	0.24853	0.09962	0.02247	0.03367	0.08084	0.02381	0.02908
40	35	Mean	1.0100	2.3451	2.2953	2.1012	1.8386	1.9305	2.0613	1.8080	1.8968
		MSE	0.70043	0.29322	0.24617	0.09785	0.02134	0.03233	0.07917	0.02277	0.02783

TABLE II: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for $R(t)$ when $T = 0.95$, $\alpha = 1.83675$, $R(1.42351) = 0.228654$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
			$\hat{R}_{MLE}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{R}_{EBS1}$	$\hat{R}_{EBS2}$	$\hat{R}_{EBS3}$	$\hat{R}_{EBL1}$	$\hat{R}_{EBL2}$	$\hat{R}_{EBL3}$
20	10	Mean	0.13935	0.26848	0.26764	0.240558	0.21048	0.22101	0.23988	0.20997	0.22044
		MSE	0.00858	0.00204	0.00198	0.00051	0.00061	0.00037	0.00050	0.00063	0.00038
20	15	Mean	0.14084	0.26938	0.26855	0.24137	0.2112	0.22176	0.24070	0.21068	0.22119
		MSE	0.00829	0.00209	0.00203	0.00051	0.00057	0.00034	0.00049	0.00059	0.00035
30	20	Mean	0.13485	0.27605	0.27542	0.24734	0.21642	0.22725	0.24684	0.21604	0.22682
		MSE	0.00914	0.00267	0.00262	0.00069	0.00041	0.00029	0.00068	0.00042	0.00029
30	25	Mean	0.13573	0.27764	0.27702	0.24877	0.21767	0.22856	0.24827	0.21729	0.22813
		MSE	0.00898	0.00282	0.00276	0.00074	0.00038	0.00028	0.00072	0.00039	0.00028
40	30	Mean	0.13193	0.28123	0.28072	0.25198	0.22048	0.23151	0.25157	0.22017	0.23116
		MSE	0.00960	0.00314	0.00309	0.00085	0.00030	0.00026	0.00083	0.00030	0.00026
40	35	Mean	0.13289	0.28118	0.28066	0.25194	0.22044	0.23147	0.25152	0.22013	0.23112
		MSE	0.00942	0.00312	0.00307	0.00083	0.00029	0.00025	0.00081	0.00029	0.03117

TABLE III: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for α when $T = 1.5$, $\alpha = 1.83675$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
			$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$	$\hat{\alpha}_{EBL1}$	$\hat{\alpha}_{EBL2}$	$\hat{\alpha}_{EBL3}$
20	10	Mean	1.24672	2.17574	2.09575	1.94947	1.70578	1.79107	1.88526	1.65664	1.73688
		MSE	0.432209	0.145423	0.097591	0.037196	0.035903	0.022759	0.026905	0.051275	0.030726
20	15	Mean	1.58361	2.088	2.00982	1.87084	1.63699	1.71884	1.80801	1.58882	1.66575
		MSE	0.194715	0.0980	0.063566	0.029161	0.061341	0.037537	0.02797	0.082374	0.052241
30	20	Mean	1.41277	2.16236	2.10092	1.93748	1.69529	1.78006	1.88812	1.65748	1.73837
		MSE	0.247176	0.135264	0.098427	0.033621	0.037983	0.023029	0.025711	0.049865	0.029198
30	25	Mean	1.58984	2.10379	2.04372	1.8850	1.64937	1.73184	1.8367	1.61235	1.69104
		MSE	0.137916	0.103753	0.074085	0.028373	0.05505	0.03299	0.025214	0.069761	0.042587
40	30	Mean	1.49899	2.15241	2.10286	1.92856	1.68749	1.77187	1.88874	1.65696	1.73822
		MSE	0.17176	0.12793	0.098181	0.031139	0.039665	0.023379	0.024765	0.049288	0.028384
40	35	Mean	1.58899	2.12745	2.07854	1.9062	1.66792	1.75132	1.86688	1.63778	1.7181
		MSE	0.112224	0.113601	0.086543	0.028179	0.046385	0.027014	0.02355	0.057005	0.033247

TABLE IV: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for $R(t)$ when $T = 1.5$, $\alpha = 1.83675$, $R(1.42351) = 0.228654$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
			$\hat{R}_{MLE}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{R}_{EBS1}$	$\hat{R}_{EBS2}$	$\hat{R}_{EBS3}$	$\hat{R}_{EBL1}$	$\hat{R}_{EBL2}$	$\hat{R}_{EBL3}$
20	10	Mean	0.160869	0.263347	0.262477	0.235959	0.206464	0.216787	0.23526	0.20592	0.21619
		MSE	0.005739	0.00153	0.00148	0.00032	0.00069	0.00036	0.00031	0.00072	0.00038
20	15	Mean	0.19952	0.25413	0.2532	0.22770	0.19924	0.20920	0.22700	0.19870	0.20861
		MSE	0.00245	0.00103	0.00099	0.00031	0.00110	0.00064	0.00031	0.00113	0.00066
30	20	Mean	0.18046	0.26222	0.26155	0.23495	0.20558	0.21586	0.23441	0.20517	0.2154
		MSE	0.00320	0.00144	0.00140	0.00029	0.00072	0.0003	0.0002	0.00074	0.00039
30	25	Mean	0.20065	0.25608	0.25541	0.22945	0.20077	0.2108	0.22891	0.2003	0.21035
		MSE	0.00174	0.0011	0.00107	0.00028	0.00099	0.00056	0.00028	0.0010	0.00057
40	30	Mean	0.19047	0.26136	0.26082	0.23418	0.20491	0.21515	0.23375	0.20457	0.21479
		MSE	0.00219	0.00137	0.00134	0.00027	0.00075	0.00039	0.00027	0.00076	0.00040
40	35	Mean	0.20075	0.25876	0.25822	0.2318	0.20286	0.21301	0.23141	0.20253	0.21264
		MSE	0.00141	0.00122	0.00119	0.00026	0.0008	0.0004	0.00026	0.00087	0.00047

TABLE V: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for α when $\alpha = 2.15$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	T	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
				$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$	$\hat{\alpha}_{EBL1}$	$\hat{\alpha}_{EBL2}$	$\hat{\alpha}_{EBL3}$
21	15	1.6	Mean	2.73145	2.70043	2.54784	2.41959	2.11714	2.22299	2.29614	2.02183	2.11823
			MSE	0.338078	0.30311	0.15838	0.07279	0.00117	0.00542	0.02144	0.0165	0.00108
		2	Mean	2.75234	2.68742	2.53546	2.40793	2.10694	2.21228	2.28499	2.01202	2.10794
			MSE	0.362808	0.2889	0.14863	0.06659	0.0019	0.00393	0.01827	0.01908	0.00181

TABLE VI: Average estimates and MSEs of MLEs, Bayesian and E-Bayesian estimates for $R(t = 1.25)$ when $\alpha = 2.15$, $c = 1$, $a = 0.8$, $b = 0.7$ and $h = 1$.

n	r	T	Criteria	MLE	Bayesian Estimates		E-Bayesian Estimates					
				$\hat{R}_{MLE}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{R}_{EBS1}$	$\hat{R}_{EBS2}$	$\hat{R}_{EBS3}$	$\hat{R}_{EBL1}$	$\hat{R}_{EBL2}$	$\hat{R}_{EBL3}$
21	15	1.6	Mean	0.474034	0.46542	0.46294	0.41702	0.36489	0.38313	0.41502	0.36336	0.38145
			MSE	0.005943	0.00469	0.00436	0.0004	0.00103	0.00019	0.00033	0.00113	0.00024
		2	Mean	0.476613	0.4638	0.46131	0.41557	0.36362	0.3818	0.41357	0.36209	0.38011
			MSE	0.006347	0.00447	0.00414	0.00035	0.00111	0.00023	0.00028	0.00122	0.00028

B. Conclusion

This paper considers the maximum likelihood, Bayesian and E-Bayesian estimation methods for estimating the parameter and the reliability function of Burr-X distribution based on Type-I HCS. All estimates are derived by using SEL and LINEX loss functions. MCMC method is used to obtain Bayesian and E-Bayesian estimates of the parameter and the reliability function of Burr-X distribution. Also, the proposed methods are applied to a real testing data set for the purpose of illustration. Based on the results are shown in Tables (I-IV), we observe the following:

- Bayesian and E-Bayesian methods are better than the maximum likelihood method in terms of MSEs.

- Generally, the MSE of E-Bayesian estimates of α and $R(t)$ are the smallest comparing to the MSE of Bayesian estimates and MLE.
- The E-Bayesian estimator is more efficient than Bayesian and maximum likelihood estimators in terms of MSEs.
- The MSEs of MLE, Bayesian and E-Bayesian estimates increase when n increases in Tables (I-II). But the MSEs of MLE, Bayesian and E-Bayesian estimates decrease with increasing n and r when T becomes larger as in Tables (III-IV).
- The results, as seen in Tables (V-VI), indicate that the proposed methods are convenient for application.
- The proposed methods behave efficiently in the practical performance.
- Also, the E-Bayesian method is the best when compared with the maximum likelihood and the Bayesian estimation methods.
- It has shown from Tables (I-VI), that the E-Bayesian method is more effective and practical than the maximum likelihood and the Bayesian estimation methods.

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