Implicit-Explicit Time Stepping Scheme Based on the Streamline Diffusion Method for Fluid-fluid Interaction Problems

Lingzhi Qian, and Huiping Cai*

Abstract—**In this paper, we study numerical approximations for the fluid-fluid interaction problems. As a simplified model, the convection-dominated convection-diffusion-reaction equations are coupled by an interface condition. The implicit-explicit time stepping streamline diffusion method for the problem is proposed. The stability analysis and error estimates for the proposed scheme are derived. Computational tests are performed to demonstrate the robustness of this scheme.**

Index Terms—**fluid-fluid interaction problems, implicit-explicit method, streamline diffusion, stability analysis, error estimates.**

I. Introduction

THERE are many problems in which different physical models, different parameter regimes, or different solution behaviors are coupled across interfaces. Monolithic solution methods for solving the coupled problems are alternative. But these methods preclude usage of highly optimized black box subdomain solvers. Decoupling methods have obvious and large advantages in these aspects over monolithic solution methods. Among these decoupling methods, partitioned time stepping scheme is promising for solving the coupled problems. In this scheme, a convenient decoupling strategy for large problems is provided. At each time step, we solve the coupled problems by passing information across interface, then the coupled problems are decoupled into individual subproblems independently. Typical applications in which the partitioned time stepping scheme is highly desirable include atmosphere-ocean coupling and fluid-solid interaction problems [3], [4], [5].

In this paper, we consider a simplified model of two convection-diffusion equations coupled across their common interface through a jump condition. This reduced problem still retains the essential difficulty of the coupled problems. Fig. 1 illustrates the subdomains considered here, and the domain consists of two subdomains Ω_1 and Ω_2 coupled across an interface $I = \partial\Omega_1 \cap \partial\Omega_2$, where $\Omega_i \subset R^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega_i$. We set $\Gamma_i = \partial\Omega_i \backslash I$, for $i = 1, 2$. The problem studied in this paper is: given $\mu_i > 0$ (i=1,2), $k \in R$, find $u_i : \overline{\Omega}_i \times [0, T] \to R$ satisfying

$$\begin{cases} u_{i,t} - \mu_i \Delta u_i + \vec{\beta}_i(x,t) \cdot \nabla u_i + \sigma_i(x,t) u_i = f_i \quad \text{in } \Omega_i, \\ -\mu_i \nabla u_i \cdot \vec{n}_i = k(u_i - u_j) \quad \text{on } I, \; i,j = 1,2, \; i \neq j, \\ u_i\left(x, 0\right) = u_i^0\left(x\right) \quad \text{in } \Omega_i, \\ u_i = 0 \quad \text{on } \Gamma_i, \end{cases} \tag{1}$$

where $u_{i,t} = \frac{\partial u_i}{\partial t}$, $\vec{\beta}_i(x,t) \in L^\infty(0,T; W^{1,\infty}(\Omega_i))$ and $\sigma_i(x,t) \in L^\infty(0,T; L^\infty(\Omega_i))$, $\mu_i \ll |\vec{\beta}_i| = \sqrt{\beta_{i1}^2 + \beta_{i2}^2}$. In the following, it is assumed that there is a positive constant γ_0 such that

$$0 < \gamma_0 \le \sigma_i(x,t) - \frac{1}{2}\text{div } \vec{\beta}_i(x,t) \quad \forall (x,t) \in \Omega_i \times [0,T].$$

This is a standard assumption in the analysis of the problem (1).

Denote $Q_i = \Omega_i \times [0,T]$, $b_0 = \max_i \sup_{Q_i} |\vec{\beta}_i(x,t)|$, $b_1 = \max_i ||\text{div } \vec{\beta}_i(x,t)||_{L^\infty(Q_i)}$, $\mu_0 = \min\{\mu_1, \mu_2\}$, $\sigma_0 = \max_i ||\sigma_i(x,t)||_{L^\infty(Q_i)}$. Let $X_i := \{v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \Gamma_i\}$. For $u_i \in X_i$, set $\boldsymbol{u} = (u_1, u_2)$ and $X := \{(v_1, v_2) : v_i \in H^1(\Omega_i) : v_i = 0 \text{ on } \Gamma_i, \; i = 1, 2\}$. A natural subdomain variational formulation for the problem (1) is to find (for $i, j = 1, 2, i \neq j$) $u_i : [0, T] \to X_i$ satisfying

$$\begin{aligned} (u_{i,t}, v_i)_{\Omega_i} + \mu_i (\nabla u_i, \nabla v_i)_{\Omega_i} + \int_I k(u_i - u_j) v_i ds \\ + (\vec{\beta}_i \cdot \nabla u_i, v_i)_{\Omega_i} + (\sigma_i u_i, v_i)_{\Omega_i} = (f_i, v_i)_{\Omega_i}, \end{aligned} \tag{2}$$

for all $v_i \in X_i$. The natural monolithic variational formulation for (1) is to find $\boldsymbol{u} : [0, T] \to X$ satisfying

$$\begin{aligned} (\boldsymbol{u_t}, \boldsymbol{v}) + \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \int_I k[\boldsymbol{u}][\boldsymbol{v}] ds + (\vec{\beta} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) \\ + (\sigma \boldsymbol{u}, \boldsymbol{v}) = (f, \boldsymbol{v}), \end{aligned} \tag{3}$$

for all $\boldsymbol{v} \in X$, where $[\cdot]$ denotes the jump across the interface I, $(\cdot\,,\cdot)$ is the $L^2(\Omega_1 \cup \Omega_2)$ inner product and $\mu = \mu_i, f = f_i$ in Ω_i.

It is well known that dominating convection feature has a hyperbolic nature. Standard applications of the finite element method to convection-dominated problems usually lead to unstable numerical schemes. To overcome these difficulties, some modified nonstandard finite element methods can be used such as the streamline diffusion (SD) method [10], [13], [14], [15], [16], [17]. The SD method has both stability

Manuscript received October 09, 2017; revised 20 April, 2018; revised second 23 May, 2018. This work was supported in part by the Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems open funded projects (No. 201701), Scientific research starting foundation for the high level talents of shihezi university (No. RCSX201732) and Independently funded research projects of Shihezi university (No. ZZZC201751B).

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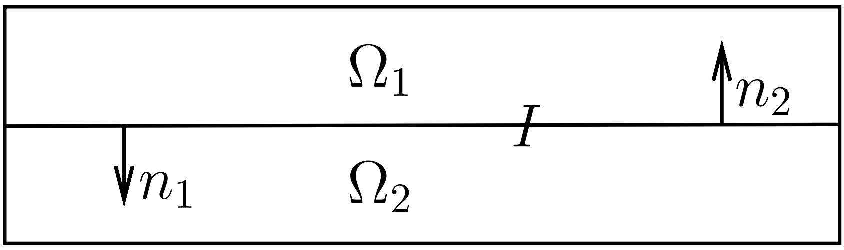

Fig. 1: Example adjoining subdomains

properties and higher accuracy. For time-dependent problems, we use the SD finite element method discrete only in space variables and the finite difference discrete in time direction [12], [21], [22]. This method keeps the essential aspect of the original SD method and simplifies the algorithm structure [1], [6], [8], [9], [18], [20], [23], [24].

In this paper, an implicit-explicit time stepping scheme based on the SD method is proposed for the two domain convection-dominated convection-diffusion-reaction problem. The natural combination of partitioned implicit-explicit time stepping scheme with the SD method retains the best features of both methods, then the proposed method has a number of attractive computational properties for this problem: stable convergence, convenience to decouple large problems, easy implementation of subdomain solvers, parallel computation in decoupled subdomain equations. The stability analysis and error estimates of the proposed method are developed. Finally, some numerical experiments are given to compare this new scheme with the standard implicit-explicit time stepping scheme for this problem.

The remainder of this work is organized as follows: in Section 2, the implicit-explicit time stepping algorithm is described. Stability analysis of the proposed method is presented in Section 3. Convergence results of the proposed method are provided in Section 4, and computations are performed to investigate stability and accuracy of this new algorithm in Section 5. Finally, Section 6 presents the conclusions and future research directions.

II. Preliminaries

Set $L^2(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$. For $\boldsymbol{u}, \boldsymbol{v} \in X$, define the L^2 inner product

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1,2} \int_{\Omega_i} u_i v_i dx,$$

and the H^1 inner product

$$(\boldsymbol{u}, \boldsymbol{v})_X = \sum_{i=1,2} (\int_{\Omega_i} u_i v_i dx + \int_{\Omega_i} \nabla u_i \cdot \nabla v_i dx),$$

and the induced norms $||\boldsymbol{v}|| = (\boldsymbol{v}, \boldsymbol{v})^{\frac{1}{2}}$, and $||\boldsymbol{v}||_X = (\boldsymbol{v}, \boldsymbol{v})_X^{\frac{1}{2}}$.

Let $\mathcal{T}_i$ be a triangulation of Ω_i and $\mathcal{T}_h = \mathcal{T}_1 \cup \mathcal{T}_2$, h be the mesh parameter of $\mathcal{T}_h$ and $0 < h \le h_0 < 1$. Take $X_{i,h} \subset X_i$ to be conforming finite element spaces for $i = 1,\ 2$, and define $X_h = X_{1,h} \times X_{2,h} \subset X$. It follows that $X_h \subset X$ is a Hilbert space with corresponding inner product and induced norm. For $\boldsymbol{u} \in X$, we define the operators $A,\ B : X \to (X)^{'}$ via the Riesz representation theorem as

$$(A\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1,2} \mu_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i dx \quad \forall \boldsymbol{v} \in X, \tag{4}$$

$$(B\boldsymbol{u}, \boldsymbol{v}) = k \int_I [\boldsymbol{u}][\boldsymbol{v}] ds \quad \forall \boldsymbol{v} \in X. \tag{5}$$

The discrete operators $A_h,\ B_h : X_h \to (X_h)^{'} = X_h$ are defined analogously by restricting (4) and (5) to $\boldsymbol{v}_h \in X_h$.

The partitioned time stepping scheme based on the SD method can be stated as follows: find $u_i : [0,T] \to X_i$, such that

$$\begin{aligned} &(u_{i,t}, v_i + \delta v_{i_{\vec{\beta}_i}})_{\Omega_i} + \mu_i(\nabla u_i, \nabla v_i)_{\Omega_i} \\ &+ \int_I k(u_i - u_j) v_i ds - \mu_i(\Delta u_i, \delta v_{i_{\vec{\beta}_i}})_{\Omega_i} \\ &+ (u_{\vec{\beta}_i} + \sigma_i u_i, v_i + \delta v_{i_{\vec{\beta}_i}})_{\Omega_i} = (f_i, v_i + \delta v_{i_{\vec{\beta}_i}})_{\Omega_i}, \forall v_i \in X_i, \end{aligned} \tag{6}$$

where $v_{i_{\vec{\beta}_i}} \triangleq \vec{\beta}_i \cdot \nabla v_i$ and $\delta > 0$ is an appropriate artificial diffusion parameter. We propose the following choice: restricting $\Delta t \le ah$ and taking

$$\delta = \begin{cases} \delta_1 = a_1 \Delta t, & \text{if } \frac{\mu_0}{b_0} \le h \le h_0, \\ \delta_2 = a_2 \Delta t^2, & \text{if } h < \frac{\mu_0}{b_0}, \end{cases} \tag{7}$$

here a_1 and a_2 are two positive constants. The choice for a_1 and a_2 will be specified in Theorem 3.1. The corresponding monolithic variational formulation is to find $\boldsymbol{u} : [0,T] \to X$ satisfying

$$\begin{aligned} &(\boldsymbol{u}_t, \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}) + (A\boldsymbol{u}, \boldsymbol{v}) + (B\boldsymbol{u}, \boldsymbol{v}) - \mu(\Delta \boldsymbol{u}, \delta \boldsymbol{v}_{\vec{\beta}}) \\ &+ (\boldsymbol{u}_{\vec{\beta}} + \sigma \boldsymbol{u}, \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}) = (f, \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}) \ \forall \ \boldsymbol{v} \in X. \end{aligned} \tag{8}$$

Now the implicit-explicit time stepping scheme based on the SD method is stated as follows:

Let $\Delta t > 0$, for each $M \in \mathcal{N}, M \le \frac{T}{\Delta t}$, given $\boldsymbol{u}^n \in X_h,\ n = 0,\ 1, \cdots, M-1$, find $\boldsymbol{u}^{n+1} \in X_h$ satisfying

$$\begin{aligned} &(\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t}, \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}) + (A_h \boldsymbol{u}^{n+1}, \boldsymbol{v}) + (B_h \boldsymbol{u}^n, \boldsymbol{v}) \\ &- \mu(\Delta \boldsymbol{u}^{n+1}, \delta \boldsymbol{v}_{\vec{\beta}}) + (\boldsymbol{u}^{n+1}_{\vec{\beta}} + \sigma \boldsymbol{u}^{n+1}, \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}) \\ &= (f(t^{n+1}), \boldsymbol{v} + \delta \boldsymbol{v}_{\vec{\beta}}),\ \forall \ \boldsymbol{v} \in X_h. \end{aligned} \tag{9}$$

In order to analysis the stability and convergence, it is necessary to work with norms induced by the operators A and B and relate these norms to $||\cdot||$ and $||\cdot||_X$.

Lemma 2.1: [7] Let $\boldsymbol{v}=(v_1,v_2)\in X$ and $\alpha\geq 0$. Then

$$||\boldsymbol{v}||_{A+\alpha I}= \left\{\sum_{i=1,2}\mu_i\int_{\Omega_i}|\nabla v_i|^2dx+\alpha\sum_{i=1,2}\int_{\Omega_i}|v_i|^2dx\right\}^{1/2} \quad (10)$$

defines a norm on X. Furthermore, there exists a constant $C>0$ such that if $\alpha\in R^+$ satisfies

$$\alpha\geq Ck^2\max\{\mu_1^{-1},\mu_2^{-1}\}, \quad (11)$$

then it follows that

$$||\boldsymbol{v}||_{A+\alpha I-B}=\left\{\sum_{i=1,2}\mu_i\int_{\Omega_i}|\nabla v_i|^2dx + \alpha\sum_{i=1,2}\int_{\Omega_i}|v_i|^2dx-k\int_I|v_1-v_2|^2ds\right\}^{1/2} \quad (12)$$

defines a norm on X. The above norms are equivalent to $||\cdot||_X$.

The following discrete Gronwall lemma will also be utilized in the subsequent analysis.

Lemma 2.2: [11] Let $l,\ m$, and $a_s,\ b_s,\ d_s,\ g_s$, for integers $s\geq 0$, be nonnegative numbers such that

$$a_n+l\sum_{s=0}^{n}b_s\leq l\sum_{s=0}^{n}d_sa_s+l\sum_{s=0}^{n}g_s+m,\quad \forall n\geq 0. \quad (13)$$

Suppose that $ld_s<1$ for all s, and set $\rho_s\equiv(1-ld_s)^{-1}$. Then

$$a_n+l\sum_{s=0}^{n}b_s \leq \exp\left(l\sum_{s=0}^{n}\rho_sd_s\right)\left\{l\sum_{s=0}^{n}g_s+m\right\},\quad \forall n\geq 0. \quad (14)$$

III. Stability analysis

In this section, we will give the stability analysis of the presented scheme. Throughout this paper, C_i denotes positive constant independent of $\mu,\ \Delta t$ and h .

Theorem 3.1: Let $\boldsymbol{u}^{n+1}\in X_h$ satisfy (9) for each $n\in\{0,\ 1,\cdots,\frac{T}{\Delta t}-1\}$, and $0<\Delta t<(2\alpha+b_0^2)^{-1}$ for α satisfying (11). Then there exists nonnegative numbers $C_1(\alpha)$ and $C_2(\alpha)$ such that

$$||\boldsymbol{u}^{n+1}||^2+\Delta t\sum_{k=0}^{n+1}||\boldsymbol{u}^k||_X^2+\frac{\delta\Delta t}{4}\sum_{k=0}^{n}||\boldsymbol{u}^k_{\vec\beta}||^2 \leq C_1(\alpha)e^{C_2(\alpha)T}\left\{||\boldsymbol{u}^0||^2+\Delta t||\boldsymbol{u}^0||_X^2 + \Delta t\sum_{k=0}^{n+1}||f(t^{k+1})||^2\right\}. \quad (15)$$

Proof. Taking $\boldsymbol{v}=\boldsymbol{u}^{k+1}$ in (9), it follows that

$$\begin{aligned}&(\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^k}{\Delta t},\boldsymbol{u}^{k+1}+\delta\boldsymbol{u}^{k+1}_{\vec\beta})+(A_h\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1})\\&+(B_h\boldsymbol{u}^k,\boldsymbol{u}^{k+1})-\mu(\Delta\boldsymbol{u}^{k+1},\delta\boldsymbol{u}^{k+1}_{\vec\beta})\\&+(\boldsymbol{u}^{k+1}_{\vec\beta}+\sigma\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}\\&+\delta\boldsymbol{u}^{k+1}_{\vec\beta})=(f(t^{k+1}),\boldsymbol{u}^{k+1}+\delta\boldsymbol{u}^{k+1}_{\vec\beta}).\end{aligned} \quad (16)$$

At first, we estimate the terms of left-hand sides of (16). It is easy to see that

$$\begin{aligned}&(\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^k}{\Delta t},\boldsymbol{u}^{k+1})\geq\frac{1}{2\Delta t}(||\boldsymbol{u}^{k+1}||^2-||\boldsymbol{u}^k||^2), \quad (17)\\&(\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^k}{\Delta t},\delta\boldsymbol{u}^{k+1}_{\vec\beta})=\frac{\delta}{\Delta t}[(\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}_{\vec\beta})-(\boldsymbol{u}^k,\boldsymbol{u}^{k+1}_{\vec\beta})]\\&\geq-\frac12\left\{\frac{\delta}{\Delta t}b_1||\boldsymbol{u}^{k+1}||^2+(\frac{\delta}{\Delta t})^2||\nabla\boldsymbol{u}^{k+1}||^2+b_0^2||\boldsymbol{u}^k||^2\right\}.\end{aligned}$$

For case $\delta=\delta_1$, that is $\frac{\delta}{\Delta t}=a_1$, choose $a_1>0$ such that

$$a_1^2\leq\mu_0\quad\text{and}\quad a_1b_1\leq\frac12\gamma_0.$$

If $\delta=\delta_2$, then $\frac{\delta}{\Delta t}=a_2\Delta t$ and $h<\frac{\mu_0}{b_0},\ \Delta t\leq ah<\frac{a\mu_0}{b_0}$, we can choose $a_2>0$ such that

$$a_2^2a^2\leq\frac{b_0^2}{\mu_0}\quad\text{and}\quad a_2ab_1\frac{\mu_0}{b_0}\leq\frac12\gamma_0.$$

Then in the above two cases, we have

$$(\frac{\boldsymbol{u}^{k+1}-\boldsymbol{u}^k}{\Delta t},\delta\boldsymbol{u}^{k+1}_{\vec\beta})\geq-\frac12\left\{\mu_0||\nabla\boldsymbol{u}^{k+1}||^2+\frac12\gamma_0||\boldsymbol{u}^{k+1}||^2+b_0^2||\boldsymbol{u}^k||^2\right\}. \quad (18)$$

In addition

$$\begin{aligned}&(\boldsymbol{u}^{k+1}_{\vec\beta}+\sigma^{k+1}\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}+\delta\boldsymbol{u}^{k+1}_{\vec\beta})=\delta||\boldsymbol{u}^{k+1}_{\vec\beta}||^2\\&+((\sigma^{k+1}-\frac12\text{div }\vec\beta^{k+1})\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1})+\delta(\sigma^{k+1}\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}_{\vec\beta})\\&\geq\frac{\delta}{2}||\boldsymbol{u}^{k+1}_{\vec\beta}||^2-\frac{\delta}{2}\sigma_0^2||\boldsymbol{u}^{k+1}||^2+\gamma_0||\boldsymbol{u}^{k+1}||^2.\end{aligned}$$

Choosing a_1 and a_2 again, such that

$$a_1ah_0\sigma_0^2\leq\frac12\gamma_0\quad\text{and}\quad a_2a^2(\frac{\mu_0}{b_0})^2\sigma_0^2\leq\frac12\gamma_0,$$

then we have

$$(\boldsymbol{u}^{k+1}_{\vec\beta}+\sigma^{k+1}\boldsymbol{u}^{k+1},\boldsymbol{u}^{k+1}+\delta\boldsymbol{u}^{k+1}_{\vec\beta}) \geq\frac{\delta}{2}||\boldsymbol{u}^{k+1}_{\vec\beta}||^2+\frac34\gamma_0||\boldsymbol{u}^{k+1}||^2, \quad (19)$$

and

$$(\mu\Delta\boldsymbol{u}^{k+1},\delta\boldsymbol{u}^{k+1}_{\vec\beta})\leq\frac{\delta}{4}||\boldsymbol{u}^{k+1}_{\vec\beta}||^2+\delta\mu^2C_0^2h^{-2}||\nabla\boldsymbol{u}^{k+1}||^2.$$

In the case $\delta=\delta_1$, we have $\frac{\mu_0}{b_0}\leq h\leq h_0$, choosing a_1 such that

$$aa_1h_0\mu^2C_0^2h^{-2}\leq aa_1h_0C_0^2b_0^2\leq\frac12\mu_0.$$

In the case $\delta=\delta_2=a_2\Delta t^2\leq a_2a^2h^2$, choosing $a_2>0$, such that

$$a_2a^2h^2C_0^2h^{-2}\mu^2\leq\frac12\mu_0,$$

then we get

$$(\mu\Delta\boldsymbol{u}^{k+1},\delta\boldsymbol{u}^{k+1}_{\vec\beta})\leq\frac{\delta}{4}||\boldsymbol{u}^{k+1}_{\vec\beta}||^2+\frac{\mu_0}{2}||\nabla\boldsymbol{u}^{k+1}||^2. \quad (20)$$

Since $\delta \le \max\{a_1 a h_0, a_2 a^2(\frac{\mu_0}{b_0})^2\}$, combing (16)-(20), we have

$$\frac{1}{2\Delta t}(||\boldsymbol{u}^{k+1}||^2 - ||\boldsymbol{u}^k||^2) + (A_h\boldsymbol{u}^{k+1}, \boldsymbol{u}^{k+1}) + (B_h\boldsymbol{u}^k, \boldsymbol{u}^{k+1}) + \frac{\delta}{4}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 + \frac{\gamma_0}{2}||\boldsymbol{u}^{k+1}||^2 - \frac{b_0^2}{2}||\boldsymbol{u}^k||^2 = (f(t^{k+1}), \boldsymbol{u}^{k+1}) + (f(t^{k+1}), \delta\boldsymbol{u}_{\vec{\beta}}^{k+1}) \le \frac{\gamma_0}{2}||\boldsymbol{u}^{k+1}||^2 + \frac{1}{2\gamma_0}||f(t^{k+1})||^2 + \frac{\delta}{8}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 + \frac{2}{\delta}||f(t^{k+1})||^2 \le \frac{\gamma_0}{2}||\boldsymbol{u}^{k+1}||^2 + \frac{\delta}{8}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 + C_1||f(t^{k+1})||^2. \quad (21)$$

Adding $\alpha(\boldsymbol{u}^{k+1}, \boldsymbol{u}^{k+1})$ to both sides of (21), and applying Lemma 2.1, it follows

$$\frac{1}{2\Delta t}(||\boldsymbol{u}^{k+1}||^2 - ||\boldsymbol{u}^k||^2) + ||\boldsymbol{u}^{k+1}||_{A+\alpha I}^2 + (B_h\boldsymbol{u}^k, \boldsymbol{u}^{k+1}) + \frac{\delta}{8}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 - \frac{b_0^2}{2}||\boldsymbol{u}^k||^2 \le C_1||f(t^{k+1})||^2 + \alpha||\boldsymbol{u}^{k+1}||^2. \quad (22)$$

Note that

$$(B_h\boldsymbol{u}^k, \boldsymbol{u}^{k+1}) \ge -\frac{1}{2}(B_h\boldsymbol{u}^{k+1}, \boldsymbol{u}^{k+1}) - \frac{1}{2}(B_h\boldsymbol{u}^k, \boldsymbol{u}^k), \quad (23)$$

combining (12), (22)-(23), we get

$$\frac{1}{2\Delta t}(||\boldsymbol{u}^{k+1}||^2 - ||\boldsymbol{u}^k||^2) + \frac{1}{2}||\boldsymbol{u}^{k+1}||_{A+\alpha I-B}^2 + \frac{1}{2}(||\boldsymbol{u}^{k+1}||_{A+\alpha I}^2 - ||\boldsymbol{u}^k||_{A+\alpha I}^2) + \frac{1}{2}||\boldsymbol{u}^k||_{A+\alpha I-B}^2 + \frac{\delta}{8}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 \le C_1||f(t^{k+1})||^2 + \alpha||\boldsymbol{u}^{k+1}||^2 + \frac{b_0^2}{2}||\boldsymbol{u}^k||^2. \quad (24)$$

Summing over $k = 0,\ 1, \cdots, n$ for (24) yields

$$\frac{1}{2\Delta t}(||\boldsymbol{u}^{n+1}||^2 - ||\boldsymbol{u}^0||^2) + \frac{1}{2}(||\boldsymbol{u}^{n+1}||_{A+\alpha I}^2 - ||\boldsymbol{u}^0||_{A+\alpha I}^2) + \frac{1}{2}\sum_{k=0}^{n}(||\boldsymbol{u}^{k+1}||_{A+\alpha I-B}^2 + ||\boldsymbol{u}^k||_{A+\alpha I-B}^2) + \frac{\delta}{8}\sum_{k=0}^{n}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 \le \sum_{k=0}^{n}(C_1||f(t^{k+1})||^2 + \alpha||\boldsymbol{u}^{k+1}||^2 + \frac{b_0^2}{2}||\boldsymbol{u}^k||^2). \quad (25)$$

After multiplying by $2\Delta t$ and rearranging terms of (25), we obtain

$$||\boldsymbol{u}^{n+1}||^2 + \Delta t||\boldsymbol{u}^{n+1}||_{A+\alpha I}^2 + \Delta t\sum_{k=0}^{n}\Big(||\boldsymbol{u}^{k+1}||_{A+\alpha I-B}^2 + ||\boldsymbol{u}^k||_{A+\alpha I-B}^2\Big) + \frac{\delta}{4}\Delta t\sum_{k=0}^{n}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 \le ||\boldsymbol{u}^0||^2 + \Delta t||\boldsymbol{u}^0||_{A+\alpha I}^2 + C_1\Delta t\sum_{k=0}^{n}||f(t^{k+1})||^2 + (2\alpha + b_0^2)\Delta t\sum_{k=0}^{n}||\boldsymbol{u}^{k+1}||^2. \quad (26)$$

Taking $d_n \equiv 2\alpha + b_0^2$ and using Lemma 2.2 for (26), it follows that

$$||\boldsymbol{u}^{n+1}||^2 + \Delta t||\boldsymbol{u}^{n+1}||_{A+\alpha I}^2 + \Delta t\sum_{k=0}^{n}(||\boldsymbol{u}^{k+1}||_{A+\alpha I-B}^2 + ||\boldsymbol{u}^k||_{A+\alpha I-B}^2) + \frac{\delta}{4}\Delta t\sum_{k=0}^{n}||\boldsymbol{u}_{\vec{\beta}}^{k+1}||^2 \le e^{C_2(\alpha)T}\left\{||\boldsymbol{u}^0||^2 + \Delta t||\boldsymbol{u}^0||_{A+\alpha I}^2 + \Delta t\sum_{k=0}^{n}||f(t^{k+1})||^2\right\},$$

where $C_2(\alpha) = (2\alpha+b_0^2)(1-(2\alpha+b_0^2)\Delta t)^{-1}$. From Lemma 2.1, the norms of $||\cdot||_{A+\alpha I}$ and $||\cdot||_{A+\alpha I-B}$ are equivalent to $||\cdot||_X$, we can determine $C_1(\alpha)$ and derive the final result.

IV. Convergence results

Theorem 4.1: Let $\boldsymbol{u}(t;x) \in X$ for all $t \in (0, T)$ solve the problem (1) such that $\boldsymbol{u}_t \in L^2(0, T; X)$ and $\boldsymbol{u}_{tt} \in L^2(\Omega)$. There exists nonnegative numbers $C_3(\alpha)$ and $C_4(\alpha)$, for any $n \in \{0,\ 1, \cdots, \frac{T}{\Delta t} - 1\}$, and $0 < \Delta t < (2\alpha + b_0^2 + 2)^{-1}$ for α satisfying (11), the solution $\boldsymbol{u}^{n+1} \in X_h$ of (9) satisfies

$$||\boldsymbol{u}(t^{n+1}) - \boldsymbol{u}^{n+1}||^2 + \Delta t||\boldsymbol{u}(t^{n+1}) - \boldsymbol{u}^{n+1}||_X^2 + \frac{\Delta t}{2}\sum_{k=0}^{n}||\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}||_X^2 + \frac{\delta\Delta t}{12}\sum_{k=0}^{n}||\boldsymbol{u}_{\vec{\beta}}(t^{k+1}) - \boldsymbol{u}_{\vec{\beta}}^k||^2 \le C_3(\alpha)e^{C_4(\alpha)T}\Big\{||\boldsymbol{u}(0) - \boldsymbol{u}^0||^2 + \Delta t||\boldsymbol{u}(0) - \boldsymbol{u}^0||_X^2 + (\Delta t)^2||\boldsymbol{u}_t||_{L^2(0,T;X)}^2 + (4\delta + 1)(\Delta t)^2||\boldsymbol{u}_{tt}||_{L^2(0,T;L^2(\Omega))}^2 + \inf_{\boldsymbol{v}^0 \in X_h}\left\{||\boldsymbol{u}(0) - \boldsymbol{v}^0||^2 + \Delta t||\boldsymbol{u}(0) - \boldsymbol{v}^0||_X^2\right\} + (12\delta + 1)\inf_{\boldsymbol{v}\in X_h}||(\boldsymbol{u}(0) - \boldsymbol{v})_t||^2 + T\max_{k=0,1,\cdots,n+1}\inf_{\boldsymbol{v}^k\in X_h}||\boldsymbol{u}(t^k) - \boldsymbol{v}^k||_X^2 + \mu^2\delta T\max_{k=0,1,\cdots,n+1}\inf_{\boldsymbol{v}^k\in X_h}||\Delta(\boldsymbol{u}(t^k) - \boldsymbol{v}^k)||^2\Big\}. \quad (27)$$

Proof. Restricting test functions to X_h, subtracting (9) from (8) yields the error equation

$$(\boldsymbol{u}_t(t^{k+1}) - \frac{\boldsymbol{u}^{k+1} - \boldsymbol{u}^k}{\Delta t}, \boldsymbol{v} + \delta\boldsymbol{v}_{\vec{\beta}}) + (A(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}), \boldsymbol{v}) + (B(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^k), \boldsymbol{v}) - \mu(\Delta(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}), \delta\boldsymbol{v}_{\vec{\beta}}) + ((\boldsymbol{u}_{\vec{\beta}} + \sigma\boldsymbol{u})(t^{k+1}) - (\boldsymbol{u}_{\vec{\beta}}^{k+1} + \sigma^{k+1}\boldsymbol{u}^{k+1}), \boldsymbol{v} + \delta\boldsymbol{v}_{\vec{\beta}}) = 0. \quad (28)$$

Define $\boldsymbol{r}^{k+1} = \boldsymbol{u}_t(t^{k+1}) - \frac{\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}(t^k)}{\Delta t}$ and rearrange terms

$$(\boldsymbol{r}^{k+1}, \boldsymbol{v} + \delta\boldsymbol{v}_{\vec{\beta}}) + (\frac{\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}}{\Delta t} - \frac{\boldsymbol{u}(t^k) - \boldsymbol{u}^k}{\Delta t}, \boldsymbol{v} + \delta\boldsymbol{v}_{\vec{\beta}}) + (A(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}), \boldsymbol{v}) + (B(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^k), \boldsymbol{v}) - \mu(\Delta(\boldsymbol{u}(t^{k+1}) - \boldsymbol{u}^{k+1}), \delta\boldsymbol{v}_{\vec{\beta}}) + ((\boldsymbol{u}_{\vec{\beta}} + \sigma\boldsymbol{u})(t^{k+1}) - (\boldsymbol{u}_{\vec{\beta}}^{k+1} + \sigma^{k+1}\boldsymbol{u}^{k+1}), \boldsymbol{v} + \delta\boldsymbol{v}_{\vec{\beta}}) = 0. \quad (29)$$

Define for each $k = 0, 1, \cdots$ the functions $(\boldsymbol{u}(t^k) - \boldsymbol{v}^k) + (\boldsymbol{v}^k - \boldsymbol{u}^k) = \eta^k + \xi^k$, where $\boldsymbol{v}^k \in X_h$ is arbitrary. Then by adding and subtracting $\boldsymbol{v}^k$, (29) may be rewritten as

$$\begin{aligned}
&\frac{1}{\Delta t}(\xi^{k+1}-\xi^k,\boldsymbol{v}+\delta\boldsymbol{v}_{\vec\beta})+(A\xi^{k+1},\boldsymbol{v})\\
&+(B(\boldsymbol{u}(t^{k+1})-\boldsymbol{u}^k),\boldsymbol{v})-(\mu\Delta\xi^{k+1},\delta\boldsymbol{v}_{\vec\beta})\\
&=-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\boldsymbol{v}+\delta\boldsymbol{v}_{\vec\beta})-(A\eta^{k+1},\boldsymbol{v})\\
&-(\boldsymbol{r}^{k+1},\boldsymbol{v}+\delta\boldsymbol{v}_{\vec\beta})+\mu(\Delta\eta^{k+1},\delta\boldsymbol{v}_{\vec\beta}).\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\boldsymbol{v}+\delta\boldsymbol{v}_{\vec\beta}) \qquad (30)
\end{aligned}$$

Note that

$$B\boldsymbol{u}(t^{k+1})-B\boldsymbol{u}^k=B(\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k))+B\eta^k+B\xi^k,$$

hence by choosing $\boldsymbol{v}=\xi^{k+1}$, we have

$$\begin{aligned}
&\frac{1}{\Delta t}(\xi^{k+1}-\xi^k,\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})+(A\xi^{k+1},\xi^{k+1})\\
&+(B\xi^k,\xi^{k+1})-(\mu\Delta\xi^{k+1},\delta\xi^{k+1}_{\vec\beta})\\
&+(\xi^{k+1}_{\vec\beta}+\sigma^{k+1}\xi^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})\\
&=-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})\\
&-(\boldsymbol{r}^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})-(A\eta^{k+1},\xi^{k+1})\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})\\
&+\mu(\Delta\eta^{k+1},\delta\xi^{k+1}_{\vec\beta})-(B\eta^k,\xi^{k+1})\\
&-(B(\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)),\xi^{k+1}). \qquad (31)
\end{aligned}$$

The terms on the left-hand side of (31) are bounded below as in the proof of Theorem 3.1. Adding $\alpha||\xi^{k+1}||^2_{A+\alpha I}$ to both sides, and applying $||\xi^{k+1}||^2_A+\alpha||\xi^{k+1}||^2=||\xi^{k+1}||^2_{A+\alpha I}$, it follows

$$\begin{aligned}
&\frac{1}{2\Delta t}(||\xi^{k+1}||^2-||\xi^k||^2)+||\xi^{k+1}||^2_{A+\alpha I}+(B\xi^k,\xi^{k+1})\\
&+\frac{\delta}{4}||\xi^{k+1}_{\vec\beta}||^2+\frac{\gamma_0}{2}||\xi^{k+1}||^2-\frac{b_0^2}{2}||\xi^k||^2\\
&\le-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})\\
&-(\boldsymbol{r}^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})-(A\eta^{k+1},\xi^{k+1})\\
&+\mu(\Delta\eta^{k+1},\delta\xi^{k+1}_{\vec\beta})-(B\eta^k,\xi^{k+1})\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})\\
&-(B(\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)),\xi^{k+1})+\alpha||\xi^{k+1}||^2. \qquad (32)
\end{aligned}$$

The error terms involving the operator B must be absorbed into the $A+\alpha I$ norms. Using similar technique in Theorem 3.1, we have

$$\begin{aligned}
&\frac{1}{2\Delta t}(||\xi^{k+1}||^2-||\xi^k||^2)+\frac{1}{2}||\xi^{k+1}||^2_{A+\alpha I-B}\\
&+\frac{1}{2}(||\xi^{k+1}||^2_{A+\alpha I}-||\xi^k||^2_{A+\alpha I})+\frac{1}{2}||\xi^k||^2_{A+\alpha I-B}\\
&+\frac{\delta}{4}||\xi^{k+1}_{\vec\beta}||^2+\frac{\gamma_0}{2}||\xi^{k+1}||^2-\frac{b_0^2}{2}||\xi^k||^2\\
&\le-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})-(A\eta^{k+1},\xi^{k+1})\\
&-(\boldsymbol{r}^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})+\mu(\Delta\eta^{k+1},\delta\xi^{k+1}_{\vec\beta})\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\xi^{k+1}+\delta\xi^{k+1}_{\vec\beta})-(B\eta^k,\xi^{k+1})\\
&-(B(\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)),\xi^{k+1})+\alpha||\xi^{k+1}||^2. \qquad (33)
\end{aligned}$$

Now we estimate the terms of the right-hand side in (33), it is easy to see

$$\begin{aligned}
&-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\xi^{k+1})-(\boldsymbol{r}^{k+1},\xi^{k+1})\\
&\le\frac{1}{2}||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2+\frac{1}{2}||\boldsymbol{r}^{k+1}||^2+||\xi^{k+1}||^2,\\
&-\frac{1}{\Delta t}(\eta^{k+1}-\eta^k,\delta\xi^{k+1}_{\vec\beta})-(\boldsymbol{r}^{k+1},\delta\xi^{k+1}_{\vec\beta})\\
&\le6\delta||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2+6\delta||\boldsymbol{r}^{k+1}||^2+\frac{\delta}{12}||\xi^{k+1}_{\vec\beta}||^2,\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\xi^{k+1})\\
&=(\eta^{k+1},\xi^{k+1}_{\vec\beta})-((\sigma^{k+1}-div\vec\beta^{k+1})\eta^{k+1},\xi^{k+1})\\
&\le\frac{\delta}{24}||\xi^{k+1}_{\vec\beta}||^2+\frac{6}{\delta}||\eta^{k+1}||^2+C_3||\eta^{k+1}||^2+\frac{\gamma_0}{2}||\xi^{k+1}||^2,\\
&-(\eta^{k+1}_{\vec\beta}+\sigma^{k+1}\eta^{k+1},\delta\xi^{k+1}_{\vec\beta})\le\frac{\delta}{24}||\xi^{k+1}_{\vec\beta}||^2+C_4||\eta^{k+1}||^2_X,\\
&\mu(\Delta\eta^{k+1},\delta\xi^{k+1}_{\vec\beta})\le\frac{\delta}{24}||\xi^{k+1}_{\vec\beta}||^2+6\mu^2\delta||\Delta\eta^{k+1}||^2.
\end{aligned}$$

The remaining three terms of the right hand in (33) require special treatment. Note that

$$\begin{aligned}
&-(A\eta^{k+1},\xi^{k+1})=\sum_{i=1,2}\left\{\mu_i\int_{\Omega_i}\nabla\eta_i^{k+1}\cdot\nabla\xi_i^{k+1}dx\right\}\\
&\le\sum_{i=1,2}\mu_i\left\{\int_{\Omega_i}|\nabla\eta_i^{k+1}|^2dx\right\}^{1/2}\left\{\int_{\Omega_i}|\nabla\xi_i^{k+1}|^2dx\right\}^{1/2}\\
&\le C(\mu_1,\mu_2)||\eta^{k+1}||_X||\xi^{k+1}||_X,
\end{aligned}$$

and $||\xi^{k+1}||_X\le C||\xi^{k+1}||_{A+\alpha I-B}$, applying Lemma 2.1, and using Young's inequality yield

$$-(A\eta^{k+1},\xi^{k+1})\le C||\eta^{k+1}||^2_X+\frac{1}{12}||\xi^{k+1}||^2_{A+\alpha I-B}.$$

The two remaining terms of the right hand in (33) are treated in the same way. In general, for $\phi=(\phi_1,\phi_2)\in X$ and $\psi=(\psi_1,\psi_2)\in X_h$, we can bound the term $-(B\phi,\psi)$ as follows

$$\begin{aligned}
&-2(B\phi,\psi)=k\int_I(\phi_1-\phi_2)(\psi_1-\psi_2)ds\\
&\le k\left\{\int_I|\phi_1-\phi_2|^2ds\right\}^{1/2}\left\{\int_I|\psi_1-\psi_2|^2ds\right\}^{1/2}\\
&\le C(k,\Omega_1,\Omega_2)||\phi||_X||\psi||_X\\
&\le C||\phi||_X||\psi||_{A+\alpha I-B}\\
&\le C||\phi||^2_X+\frac{1}{12}||\psi||^2_{A+\alpha I-B}. \qquad (34)
\end{aligned}$$

Hence taking $\psi=\xi^{k+1}$, and $\phi=\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)$ or $\phi=\eta^{k+1}$ in (34), provides the needed bounds for (33).

Combining the above results, we get

$$\begin{aligned}
&\frac{1}{2\Delta t}(||\xi^{k+1}||^2-||\xi^k||^2)+\frac{1}{2}||\xi^{k+1}||^2_{A+\alpha I-B}\\
&+\frac{1}{2}(||\xi^{k+1}||^2_{A+\alpha I}-||\xi^k||^2_{A+\alpha I})+\frac{1}{2}||\xi^k||^2_{A+\alpha I-B}\\
&+\frac{\delta}{4}||\xi^{k+1}_{\vec{\beta}}||^2+\frac{\gamma_0}{2}||\xi^{k+1}||^2-\frac{b_0^2}{2}||\xi^k||^2\\
&\le(6\delta+\frac{1}{2})||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2+\frac{5\delta}{24}||\xi^{k+1}_{\vec{\beta}}||^2\\
&+(6\delta+\frac{1}{2})||\boldsymbol{r}^{k+1}||^2+\frac{\gamma_0}{2}||\xi^{k+1}||^2+(C_3+\frac{6}{\delta})||\eta^{k+1}||^2\\
&+(C+C_4)||\eta^{k+1}||^2_X+6\mu^2\delta||\Delta\eta^{k+1}||^2\\
&+\frac{1}{4}||\xi^{k+1}||^2_{A+\alpha I-B}+(\alpha+1)||\xi^{k+1}||^2. \qquad (35)
\end{aligned}$$

After multiplying by $2\Delta t$ and summing over $k=0,1,\cdots,n$, it follows that

$$\begin{aligned}
&||\xi^{n+1}||^2+\Delta t||\xi^{n+1}||^2_{A+\alpha I}+\frac{\Delta t}{2}\sum_{k=0}^{n}||\xi^{k+1}||^2_{A+\alpha I-B}\\
&+\Delta t\sum_{k=0}^{n}||\xi^k||^2_{A+\alpha I-B}+\frac{\delta}{12}\Delta t\sum_{k=0}^{n}||\xi^{k+1}_{\vec{\beta}}||^2\\
&\le||\xi^0||^2+\Delta t||\xi^0||^2_{A+\alpha I}+2(\alpha+1)\Delta t\sum_{k=0}^{n}||\xi^{k+1}||^2\\
&+(12\delta+1)\Delta t\sum_{k=0}^{n}\left\{||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2+||\boldsymbol{r}^{k+1}||^2\right\}\\
&+b_0^2\Delta t\sum_{k=0}^{n}||\xi^k||^2+C\Delta t\sum_{k=0}^{n}\Big\{||\eta^{k+1}||^2_X+||\eta^{k+1}||^2\\
&+||\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)||^2_X+\mu^2\delta||\Delta\eta^{k+1}||^2\Big\}. \qquad (36)
\end{aligned}$$

The discrete Gronwall lemma may be applied to (36). We give a simplified bound as follows

$$\begin{aligned}
&||\xi^{n+1}||^2+\Delta t||\xi^{n+1}||^2_{A+\alpha I}+\frac{\Delta t}{2}\sum_{k=0}^{n}||\xi^{k+1}||^2_{A+\alpha I-B}\\
&+\frac{\delta}{12}\Delta t\sum_{k=0}^{n}||\xi^{k+1}_{\vec{\beta}}||^2\le e^{C_4(\alpha)T}\Big\{||\xi^0||^2+\Delta t||\xi^0||^2_{A+\alpha I}\\
&+C\Delta t\sum_{k=0}^{n+1}||\eta^k||^2_X+\mu^2\delta\Delta t\sum_{k=0}^{n}||\Delta\eta^{k+1}||^2\\
&+(12\delta+1)\Delta t\sum_{k=0}^{n}\left\{||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2+||\boldsymbol{r}^{k+1}||^2\right\}\\
&+C\Delta t\sum_{k=0}^{n}\left\{||\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)||^2_X\right\}\Big\}, \qquad (37)
\end{aligned}$$

with $C_4(\alpha)=(2\alpha+b_0^2+2)(1-\Delta t(2\alpha+b_0^2+2))^{-1}$. Bounds for the last three terms in (37) can be derived using well known arguments [7]. Indeed, the following inequalities hold

$$\begin{aligned}
&\Delta t\sum_{k=0}^{n}||\frac{\eta^{k+1}-\eta^k}{\Delta t}||^2\\
&\le\int_0^{t^{n+1}}||\eta_t||^2dt\le||\eta_t||^2_{L^2(0,T;L^2(\Omega))},\\
&\Delta t\sum_{k=0}^{n}||\boldsymbol{u}(t^{k+1})-\boldsymbol{u}(t^k)||^2_X\\
&\le\Delta t^2\int_0^{t^{n+1}}||\boldsymbol{u}_t||^2_X\le\Delta t^2||\boldsymbol{u}_t||^2_{L^2(0,T;X)}, \qquad (38)\\
&\Delta t\sum_{k=0}^{n}||\boldsymbol{r}^{k+1}||^2\\
&\le\frac{\Delta t^2}{3}\int_0^{t^{n+1}}||\boldsymbol{u}_{tt}||^2dt\le\frac{\Delta t^2}{3}||\boldsymbol{u}_{tt}||^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Applying the triangle inequality, recalling $\eta^k=\boldsymbol{u}(t^k)-\boldsymbol{v}^k$ for any $\boldsymbol{v}^k\in X_h$, taking the infimum over $\boldsymbol{v}^k\in X_h$, and combining three inequalities in (38), it follows that

$$\begin{aligned}
&||\xi^{n+1}||^2+\Delta t||\xi^{n+1}||^2_{A+\alpha I}+\frac{\Delta t}{2}\sum_{k=0}^{n+1}||\xi^{k+1}||^2_{A+\alpha I-B}\\
&+\frac{\delta}{12}\Delta t\sum_{k=0}^{n}||\xi^{k+1}_{\vec{\beta}}||^2\le C_3(\alpha)e^{C_4(\alpha)T}\Big\{||\boldsymbol{u}(0)-\boldsymbol{u}^0||^2\\
&+\Delta t||\boldsymbol{u}(0)-\boldsymbol{u}^0||^2_{A+\alpha I}+\Delta t\inf_{\boldsymbol{v}^k\in X_h}\sum_{k=0}^{n+1}||\eta^k||^2_X\\
&+\mu^2\delta\Delta t\inf_{\boldsymbol{v}^k\in X_h}\sum_{k=0}^{n}||\Delta\eta^{k+1}||^2+\Delta t^2||\boldsymbol{u}_t||^2_{L^2(0,T;X)}\\
&+(4\delta+1)\Delta t^2||\boldsymbol{u}_{tt}||^2_{L^2(0,T;L^2(\Omega))}\\
&+\inf_{\boldsymbol{v}^0\in X_h}\{||\eta^0||^2+\Delta t||\eta^0||^2_{A+\alpha I}\}\\
&+(12\delta+1)\inf_{\boldsymbol{v}\in X_h}||\eta_t||^2_{L^2(0,T;L^2(\Omega))}\Big\}.
\end{aligned}$$

Now we can replace all norms of type $||\cdot||_{A+\alpha I-B}$ and $||\cdot||_{A+\alpha I}$ with the norm $||\cdot||_X$ by using Lemma 2.1.

Furthermore, it is obvious that

$$\Delta t\inf_{\boldsymbol{v}^k\in X_h}\sum_{k=0}^{n+1}||\eta^k||^2_X\le T\max_{k=0,1,\cdots,n+1}\inf_{\boldsymbol{v}^k\in X_h}||\eta^k||^2_X,$$

and

$$\begin{aligned}
&\mu^2\delta\Delta t\inf_{\boldsymbol{v}^k\in X_h}\sum_{k=0}^{n}||\Delta\eta^{k+1}||^2\\
&\le\mu^2\delta T\max_{k=0,1,\cdots,n+1}\inf_{\boldsymbol{v}^k\in X_h}||\Delta\eta^k||^2.
\end{aligned}$$

Finally we obtain the convergence results by one more application of the triangle inequality and rearranging constants.

Remark 4.2. Let $X_h\subset X$ be a finite element space corresponding to continuous piecewise polynomials of degree k. If $\boldsymbol{u}(\cdot,t)$ is a solution of (1) satisfying the assumptions of Theorem 4.1 and $\boldsymbol{u}^0$ approximates $\boldsymbol{u}(\cdot,0)$ such that

$$||\boldsymbol{u}(\cdot,0)-\boldsymbol{u}^0||=O(h^k),$$

then the corresponding approximations (9) converge at the rate $O(\Delta t + h^k)$ in the norm

$$\left\{\Delta t\sum_{k=0}^{n}||\boldsymbol{u}(t^k) - \boldsymbol{u}^k||_X^2\right\}^{\frac{1}{2}}.$$

V. Numerical experiments

In this section, we present some numerical experiments to illustrate the theoretical results obtained in the previous section and show the efficiency of the new method.

A. Example 1

We first consider the experiment to test the convergence rates for the problem $\Omega_1 = [0,1] \times [0,1]$, and $\Omega_2 = [0,1] \times [-1,0]$, so I is the portion of the x-axis from 0 to 1. Then $\boldsymbol{n}_1 = [0,-1]^T$ and $\boldsymbol{n}_2 = [0,1]^T$. The right-hand side function $\boldsymbol{f}$ is chosen to ensure that

$$\begin{cases} u_1(t,x,y) = x(1-x)(1-y)e^{-t}, \\ u_2(t,x,y) = x(1-x)(c_1 + c_2y + c_3y^2)e^{-t}. \end{cases}$$

u_1 and u_2 satisfy (1) with $\vec{\beta_i} = (1,1)^T, \sigma_i = 1, (i = 1,2)$. The constants $c_1,\ c_2$ and c_3 are determined from the interface conditions and boundary conditions for u_2. Obviously, it is very important for the selection of parameters $\mu_1,\ \mu_2,\ \delta_1,\ \delta_2$ and k from the stability and convergence analysis. In the first test problems, we choose $k = 1,\ \mu_1 = 10^{-1}$ and $\mu_2 = 10^{-1}$. Simultaneously, we compare our method with the standard implicit-explicit method without based on the SD method. For test problems, computations are performed with the finite element spaces consisting of continuous piecewise polynomials of degree 1. Although the analysis does not require the meshes on Ω_1 and Ω_2 to match on the interface I, the meshes used for tests herein are chosen to match on the interface I. By choosing $\Delta t = h$ the expected convergence rate of $O(\Delta t)$ is achieved by our new scheme and the standard implicit-explicit method. In our proposed method, we choose $\delta_1 = h^2/(\frac{1}{\Delta t}h^2 + 6\mu_1)$ and $\delta_2 = h^2/(\frac{1}{\Delta t}h^2 + 6\mu_2)$. In the following tests, the norm $||\boldsymbol{u}||$ is taken the discrete $L^2(0,T;H^1(\Omega))$ norm given by

$$||\boldsymbol{u}|| = \left(\sum_{n=1}^{N}\Delta t|\boldsymbol{u}(t_n)|_{H^1(\Omega)}\right)^{1/2},$$

where $N = T/\Delta t$.

The domain is partitioned into triangles with the mesh size $h = \frac{1}{N}$ for $N = 2,\ 4,\ 8,\ 16,\ 32$ respectively. We first compare the proposed method with the standard implicit-explicit method, the errors of $u_1(t^n),\ u_2(t^n),\ \boldsymbol{u}(t^n)$ and corresponding convergence rate of these two methods are shown in Tables I-II.

Similarly, in order to demonstrate our method prefer to smaller viscosity coefficient, we change $\mu_1 = \mu_2 = 10^{-3}$ and $k = 0.5$, the rest of the parameters retains the same, the standard implicit-explicit method does not converge, but our proposed method can achieve the expected convergence rate. Furthermore, even the values of μ_1 and μ_2 are further reduced to 10^{-7}, our proposed method is still effective. The results are presented in Tables III-IV, respectively.

In Tables I-II, we show the convergence of our method and the standard implicit-explicit time stepping scheme respectively, which agrees with our theoretical results in Theorem 4.2. And these two methods are more effective for moderate viscosity coefficient $\mu_1 = \mu_2 = 10^{-1}$. From Tables III-IV, we find for the smaller viscosity coefficient $\mu_1 = \mu_2 = 10^{-3}$ and $\mu_1 = \mu_2 = 10^{-7}$, the proposed method is still effective and retains the convergent order of approximation accuracy. But the standard implicit-explicit time stepping scheme is failed.

In Fig. 2, a plot of $||\boldsymbol{u}||$ computed by each of the solution methods and exact solution for decreasing time step size is given. In this figure, standard IMEX stands for the implicit-explicit time stepping without the SD method, IMEXSD represents the proposed method. From these plots, $\mu_1 = \mu_2 = 10^{-2}$, as the size of k grows, it is observed that the stability of the standard IMEX method decreases, but the IMEXSD method is more exact and effective.

In Fig. 3, the isovalues of $\boldsymbol{u}$ are compared to exaction isovalues by the standard IMEX and IMEXSD methods respectively. In these plots, we choose $k = 0.5$ and $\mu_1 = \mu_2 = 10^{-2}$.

In summary, these experiments confirm the efficiency of our proposed method.

B. Example 2

We consider a coupled system with $\Omega_1 = (0,10) \times (0,1)$ and $\Omega_2 = \{(x,y) : \frac{16(x-5)^4}{10^4} - 1 \le y \le 0,\ x \in (0,10)\}$. Let

$$\begin{cases} u_1 = x(10-x)(1-y)exp(-t), \\ u_2 = x(10-x)(c_1 + c_2y + c_3y^2)exp(-t), \end{cases}$$

where $c_1,\ c_2$ and c_3 are determined by the interface condition. In this example, we choose $k = 1$, $\mu_1 = \mu_2 = 10^{-3}$, $\Delta t = 0.0002$ and $T = 0.2$ (i.e., 1000 time steps). The mesh and isovalue of $\boldsymbol{u}$ for exact solution and numerical solution are shown in Fig. 4.

C. Example 3

In this example we assume that there is a submarine mountain (i.e., the subdomain Ω_2 is nonconvex) whereas Ω_1 is the same as in Example 2. Ω_2 is given by

$$\begin{aligned}\Omega_2 = &\{(x,y) : 0 \ge y \ge \alpha - 0.175(2x-10)\sin(0.35(2x-10)), \\ &x \in (0,10)\},\end{aligned}$$

where $\alpha = 1.75\sin(3.5)$. The boundary and initial conditions are the same as Example 2. In this example, we choose $k = 1$, $\mu_1 = \mu_2 = 10^{-3}$, $\Delta t = 0.0002$ and $T = 0.2$ (i.e., 1000 time steps). The mesh and isovalue of $\boldsymbol{u}$ for exact solution and numerical solution are shown in Fig. 5. We can notice that the presence of the submarine mountain dramatically affects the flow in Ω_2. Indeed, the flow slows down before arriving to the straitness and accelerates again after crossing it.

VI. Conclusion

We present a new implicit-explicit time stepping scheme based on the SD method for the two domain convection-dominated convection-diffusion-reaction problem. The proposed method provides a convenient decoupling strategy for

TABLE I: Numerical results of the standard implicit-explicit method with $\mu_1 = \mu_2 = 10^{-1}$

$h = \Delta t$	$\|\|Err(u_1)\|\|$	Convergence rate	$\|\|Err(u_2)\|\|$	Convergence rate	$\|\|Err(\boldsymbol{u})\|\|$	Convergence rate
$h = \frac{1}{2}$	0.01905	-	0.06499	-	0.06773	-
$h = \frac{1}{4}$	0.02466	-	0.02644	1.30	0.03616	0.91
$h = \frac{1}{8}$	0.04221	-	0.02982	-	0.05168	-
$h = \frac{1}{16}$	0.00501	3.07	0.00605	2.30	0.00785	2.72
$h = \frac{1}{32}$	0.00229	1.13	0.00267	1.18	0.00352	1.16

TABLE II: Numerical results of the proposed method with $\mu_1 = \mu_2 = 10^{-1}$

$h = \Delta t$	$\|\|Err(u_1)\|\|$	Convergence rate	$\|\|Err(u_2)\|\|$	Convergence rate	$\|\|Err(\boldsymbol{u})\|\|$	Convergence rate
$h = \frac{1}{2}$	0.01811	-	0.07986	-	0.08189	-
$h = \frac{1}{4}$	0.03929	-	0.03510	1.19	0.05268	0.64
$h = \frac{1}{8}$	0.02865	0.46	0.02343	0.58	0.03701	0.51
$h = \frac{1}{16}$	0.00444	2.70	0.00585	2.00	0.00734	2.33
$h = \frac{1}{32}$	0.00228	0.96	0.00263	1.15	0.00348	1.08

TABLE III: Numerical results of the proposed method with $\mu_1 = \mu_2 = 10^{-3}$

$h = \Delta t$	$\|\|Err(u_1)\|\|$	Convergence rate	$\|\|Err(u_2)\|\|$	Convergence rate	$\|\|Err(\boldsymbol{u})\|\|$	Convergence rate
$h = \frac{1}{2}$	0.02127	-	0.06213	-	0.06567	-
$h = \frac{1}{4}$	0.02300	-	0.02706	1.20	0.03551	0.89
$h = \frac{1}{8}$	0.01114	1.05	0.01843	0.55	0.02153	0.72
$h = \frac{1}{16}$	0.00783	0.51	0.01059	0.80	0.01318	0.71
$h = \frac{1}{32}$	0.00395	0.99	0.00469	1.18	0.00613	1.10

TABLE IV: Numerical results of the proposed method with $\mu_1 = \mu_2 = 10^{-7}$

$h = \Delta t$	$\|\|Err(u_1)\|\|$	Convergence rate	$\|\|Err(u_2)\|\|$	Convergence rate	$\|\|Err(\boldsymbol{u})\|\|$	Convergence rate
$h = \frac{1}{2}$	0.02131	-	0.06193	-	0.06550	-
$h = \frac{1}{4}$	0.02298	-	0.02709	1.20	0.03552	0.88
$h = \frac{1}{8}$	0.01138	1.01	0.01872	0.55	0.02191	0.70
$h = \frac{1}{16}$	0.00897	0.34	0.01162	0.80	0.01468	0.58
$h = \frac{1}{32}$	0.00452	0.99	0.00531	1.18	0.00697	1.07

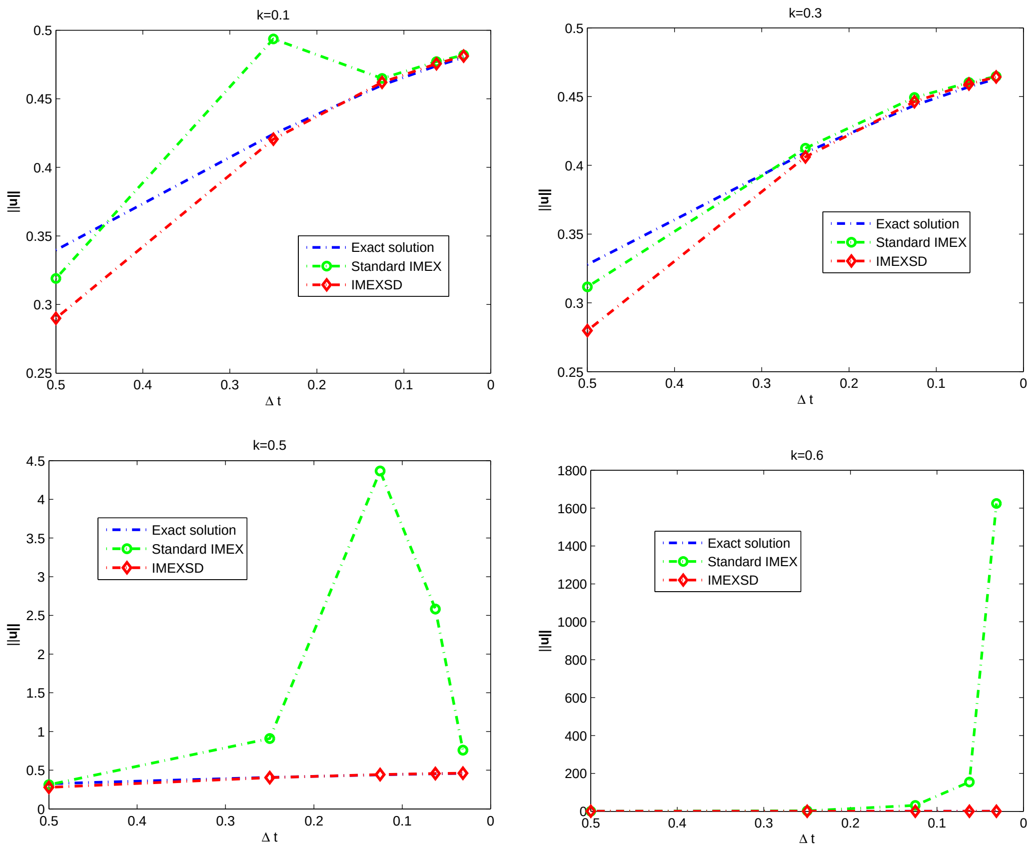

Fig. 2: Stability of $\boldsymbol{u}$ as $\Delta t \to 0$, different values of k

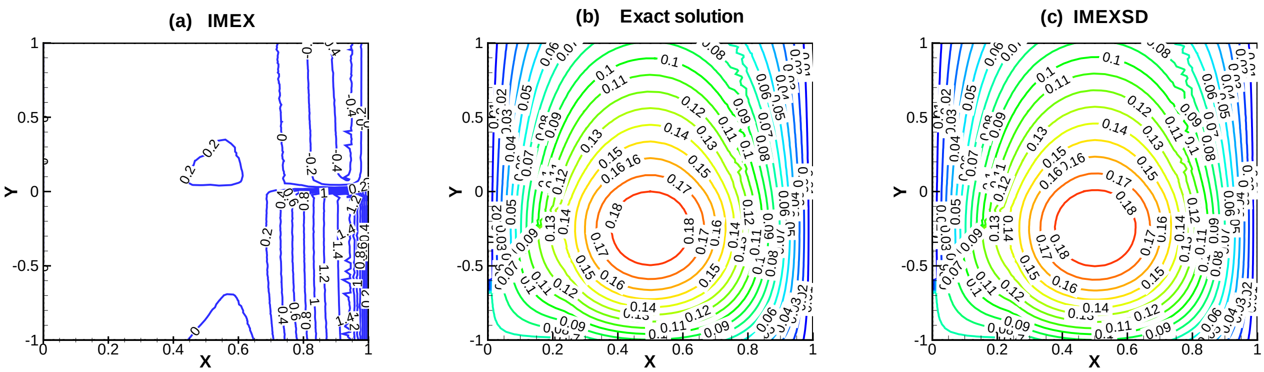

Fig. 3: (a) Isovalue of $\boldsymbol{u}$ by standard IMEX ;(b) Isovalue of $\boldsymbol{u}$ by exact solution; (c) Isovalue of $\boldsymbol{u}$ by IMEXSD

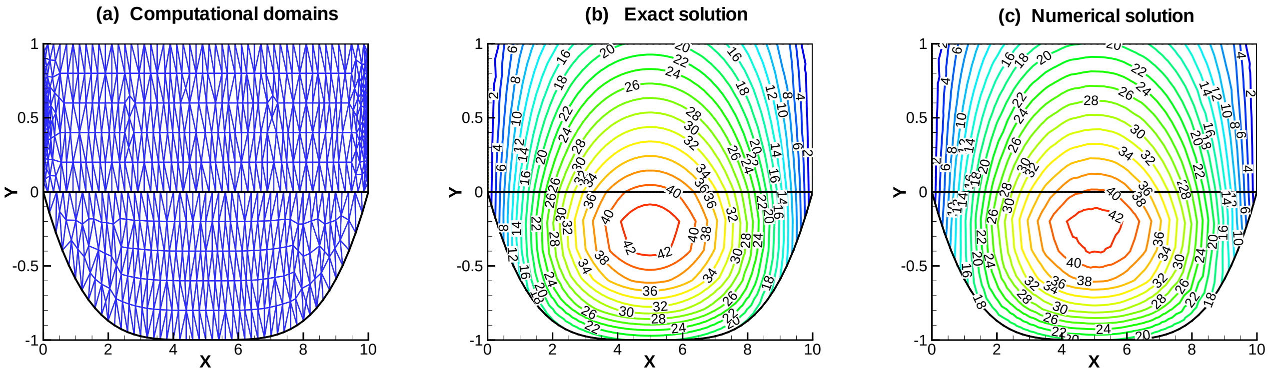

Fig. 4: (a) Mesh ; (b) Isovalue of $\boldsymbol{u}$ for exact solution; (c) Isovalue of $\boldsymbol{u}$ for numerical solution

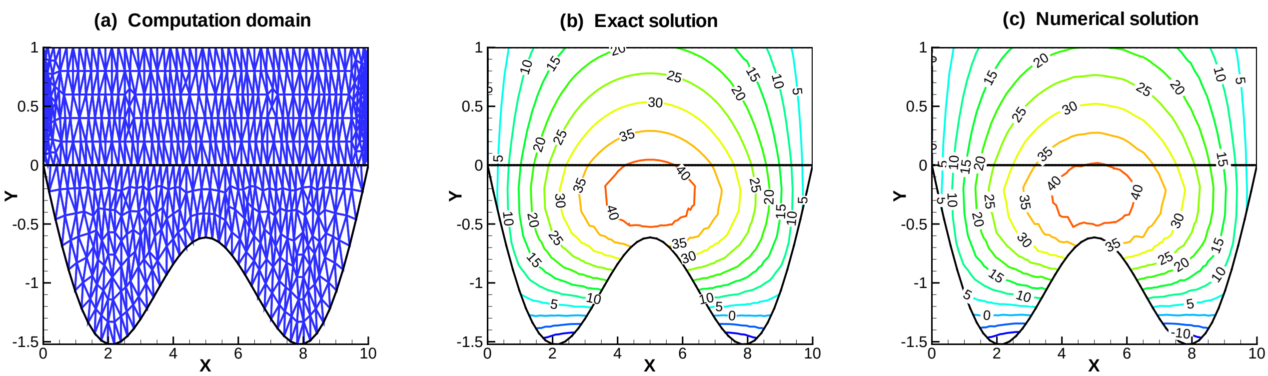

Fig. 5: (a) Mesh ; (b) Isovalue of $\boldsymbol{u}$ for exact solution; (c) Isovalue of $\boldsymbol{u}$ for numerical solution

large problems, allowing easy implementation of subdomain solvers. At each time step data is explicitly passed across the interface and the decoupled subdomain equations are then solved in parallel. In addition, stability and convergence are maintained. Further study is underway to improve the simulation by extending to more realistic problems.

Acknowledgment

The authors would like to thank the editor and anonymous reviewers for their valuable comments and helpful suggestions on the quality improvement of our present paper.

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