Some New Gronwall-Bellman Type And Volterra-Fredholm Type Fractional Integral Inequalities And Their Applications in Fractional Differential Equations

Bin Zheng*

Abstract—**In this paper, based on the modified Riemann-Liouville fractional derivative and the theory of fractional calculus, some new Gronwall-Bellman type fractional integral inequalities and Volterra-Fredholm type fractional integral inequalities are established. These inequalities provide explicit bounds for unknown functions, and can be used in the research of qualitative as well as quantitative properties for solutions of fractional differential equations with certain forms. As for applications, we apply the results presented to research boundedness for the solutions of one certain fractional differential equation. For further extensions, we also presented corresponding results under the definition of the conformable fractional derivative.**

MSC 2010: 26A33; 26D15

Index Terms—**fractional integral inequality, Gronwall-Bellman type inequality, Volterra-Fredholm type inequality, explicit bound, fractional differential equation**

I. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, and are widely used in various domains including engineering, physics, biology, signal processing, systems identification, control theory, finance, fractional dynamics and so on [1-3]. In particular, the fractional derivative has proved to be very useful in describing the memory and hereditary properties of materials and processes. One of its most important applications is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equation is usually used for modeling this movement. Recently, various aspects for fractional diffusion equations have been researched by many authors. In [4,5], the authors proposed certain methods for finding analytical solutions or numerical solutions of fractional differential equations, while in [6,7], the authors proposed two effective methods for finding numerical solutions of fractional differential equations. In [8-10], qualitative and quantitative properties of solutions of several fractional differential equations were investigated.

In the research of qualitative as well as quantitative properties of solutions of differential equations, difference equations, dynamic equations on time scales as well as fractional differential and integral equations, the Gronwall-Bellman inequality [11,12] and its various generalizations play important roles as such inequalities provide explicit bounds for solutions concerned. During the last few decades, much work has been done for developing various Gronwall-Bellman type inequalities. These inequalities involve Gronwall-Bellman type differential and integral inequalities [13-18], retarded inequalities [19-23], difference inequalities [24-28], Volterra-Fredholm type inequalities [24,27], and dynamic inequalities on time scales [29-33]. Recently, some authors have researched and established some Gronwall-Bellman type fractional differential and integral inequalities [34-37], which have contributed much in the research of qualitative and quantitative properties of solutions of certain fractional differential and integral equations. However, the earlier inequalities established are inadequate in the research of the properties of solutions of some certain fractional differential equations, and it is necessary to establish new Gronwall-Bellman type inequalities so as to fulfill corresponding analysis.

The aim of this paper is to present some new Gronwall-Bellman type fractional integral inequalities and Volterra-Fredholm type fractional integral inequalities for the sake of researching qualitative and quantitative properties of solutions of some fractional differential equations with more complicated form. The fractional derivative used here is defined in the sense of the modified Riemann-Liouville derivative [38,39]. In Section 2, we present the main results, and establish some new Gronwall-Bellman type fractional integral inequalities and Volterra-Fredholm type fractional integral inequalities. Based on these inequalities, explicit bounds for unknown functions are obtained. In Section 3, we apply the present results to research boundedness for the solution of one certain fractional differential equation. In Section 4, we present further extensions for the established fractional integral inequalities in Section 2 under the definition of the conformable fractional derivative. Finally, in Section 5, we present some concluding comments.

Definition 1. The modified Riemann-Liouville derivative of order α is defined as below:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, \quad n\le\alpha<n+1,\ n\ge 1. \end{cases}$$

Definition 2. The Riemann-Liouville fractional integral of order α on the interval $[0,t]$ is defined by

Manuscript received January 28, 2018.

B. Zheng is with the School of Mathematics and Statistics, Shandong University Of Technology, Zibo, Shandong, 255049 China *e-mail: zhengbin2601@126.com

$$I^\alpha f(t)=\frac{1}{\Gamma(1+\alpha)}\int_0^t f(s)(ds)^\alpha=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds.$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral [25] are listed as follows (the interval concerned below is always defined by $[0,t]$):

$$\begin{cases}(i):\ D_t^\alpha t^r=\dfrac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha},\\ (ii):\ D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha f(t)+f(t)D_t^\alpha g(t),\\ (iii):\ D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t)=D_g^\alpha f[g(t)](g'(t))^\alpha,\\ (iv):\ I^\alpha(D_t^\alpha f(t))=f(t)-f(0).\end{cases}$$

II. Main Results

First we consider the following Gronwall-Bellman type inequality:

$$u(t)\le C+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)[u^{2-p}(s)$$
$$+\int_0^s(s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds,\ t\ge 0,\qquad (1)$$

where $\alpha>0$, the functions $u,\ f,\ g$ are nonnegative continuous functions defined on $t\ge 0$, $C,\ p,\ q$ are constants with $0<p\le 1,\ q\ge 0,\ q\ne 1,\ C\ge 1$.

To derive the explicit bound for $u(t)$, we need the following lemmas.

Lemma 1. Let $\alpha>0$, $a,\ b,\ u$ be real-valued nonnegative continuous functions defined on $t\ge 0$, $q\ge 0$ is a constant with $q\ne 1$. if for $t\ge 0$,

$$D_t^\alpha u(t)\le a(t)u(t)+b(t)u^q(t),\qquad (2)$$

then one has

$$u(t)\le \exp[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$
$$\{u^{1-q}(0)+\frac{(1-q)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}b(\tau)\exp[-(1-q)$$
$$\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}^{\frac{1}{1-q}},\qquad (3)$$

provided that $u^{1-q}(0)+\frac{(1-q)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}b(\tau)\exp[-(1-q)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau>0$.

Proof. From (2) one can obtain that

$$u^{-q}(t)D_t^\alpha u(t)\le a(t)u^{1-q}(t)+b(t),\qquad (4)$$

Setting $u^{1-q}(t)=z(t)$, by the first equality of (iii) one has $D_t^\alpha z(t)=(1-q)u^{-q}(t)D_t^\alpha u(t)$.

If $0\le q<1$, then it holds that

$$D_t^\alpha z(t)\le(1-q)a(t)z(t)+(1-q)b(t),\qquad (5)$$

By use of (i), (ii), (iii) one can obtain that:

$$D_t^\alpha\{z(t)\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\}$$
$$=\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]D_t^\alpha z(t)$$
$$+z(t)D_t^\alpha\{\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\}$$
$$=\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]D_t^\alpha z(t)-$$
$$(1-q)a(t)z(t)\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]$$
$$D_t^\alpha(\frac{t^\alpha}{\Gamma(1+\alpha)})$$
$$=\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]$$
$$[D_t^\alpha z(t)-(1-q)a(t)z(t)]$$
$$\le(1-q)b(t)\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds].(6)$$

Substituting t with τ, fulfilling fractional integral of order α for (3) with respect to τ from 0 to t we deduce that

$$z(t)\exp[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]$$
$$\le z(0)+\frac{(1-q)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}b(\tau)\exp[-(1-q)$$
$$\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau,\qquad (7)$$

which implies

$$z(t)\le\exp[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$
$$\{z(0)+\frac{(1-q)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}b(\tau)$$
$$\exp[-(1-q)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}\qquad (8)$$

So by $u^{1-q}(t)=z(t)$ and $0\le q<1$, (3) can be obtained.

If $q>1$, then it holds that

$$D_t^\alpha z(t)\ge(1-q)a(t)z(t)+(1-q)b(t),\qquad (9)$$

Similar to the process above one can deduce that

$$z(t)\ge\exp[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}(1-q)a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$
$$\{z(0)+\frac{(1-q)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}b(\tau)$$
$$\exp[-(1-q)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} a((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}\qquad (10)$$

On the other hand, as $z(t)=u^{1-q}(t)=\frac{1}{u^{q-1}(t)}$, then (3) can also be obtained. The proof is complete.

Lemma 2. Suppose $a,b,x,\gamma>0$.

(a) If $0<\gamma\le 1$, then $(a+x)^\gamma\le a^\gamma+x^\gamma$.

(b) If $1<\gamma\le 2$, then $(a+x)^\gamma\le\gamma a^{\gamma-1}x+x^\gamma+a^\gamma$.

Proof. (i) Set $f(x)=(a+x)^\gamma-[a^\gamma+x^\gamma]$. Then

$f(0) = 0$, and $f'(x) = \gamma(a+x)^{\gamma-1} - \gamma x^{\gamma-1} \le 0$. So $f(x)$ is nonincreasing for $x > 0$, and then the proof of (i) is complete.

(ii) Set $f(x) = (a+x)^\gamma - [\gamma a^{\gamma-1}x + x^\gamma + a^\gamma]$. Then $f(0) = 0$, and $f'(x) = \gamma(a+x)^{\gamma-1} - [\gamma a^{\gamma-1} + \gamma x^{\gamma-1}] = \gamma\{(a+x)^{\gamma-1} - [a^{\gamma-1} + x^{\gamma-1}\}$. By (i) it holds that $f'(x) \le 0$. So $f(x)$ is nonincreasing for $x > 0$, and then (ii) is proved.

Lemma 3. Suppose $\alpha > 0$, f is a continuous function, then $D^\alpha(I_t^\alpha f(t)) = f(t)$.

The proof of Lemma 3 under the condition $0 < \alpha < 1$ can be referred to [35, Lemma 1], while the proof under the condition $\alpha \ge 1$ can be obtained similarly with reference to the combination of Definition 1 and [35, Lemma 1].

Theorem 4. Suppose the inequality (1) holds. Then we have the following estimate for $u(t)$:

$$u(t) \le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta(s)ds,\ t \ge 0, \tag{11}$$

provided that

$$C^{(2-p)(1-q)} + (1-q)\int_0^t (t-\tau)^{\alpha-1}g(\tau)\exp[-(1-q)$$

$$(2-p)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau > 0,$$

where

$$\theta(t) = f(t)\exp[p(2-p)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$

$$\{C^{(2-p)(1-q)} + (1-q)\int_0^t (t-\tau)^{\alpha-1}g(\tau)\exp[-(1-q)$$

$$(2-p)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}^{\frac{p}{1-q}}. \tag{12}$$

Proof. Denote the right hand side of (1) by $v(t)$. Then we have

$$u(t) \le v(t),\ t \ge 0, \tag{13}$$

and by use of Lemma 3 it holds that

$$D_t^\alpha v(t) = f(t)[u^{2-p}(t) + \int_0^t (t-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p$$

$$\le f(t)[v^{2-p}(t) + \int_0^t (t-\xi)^{\alpha-1}g(\xi)v^q(\xi)d\xi]^p. \tag{14}$$

Denote $m(t) = v^{2-p}(t) + \int_0^t (t-\xi)^{\alpha-1}g(\xi)v^q(\xi)d\xi$. Then we have

$$D_t^\alpha v(t) \le f(t)m^p(t), \tag{15}$$

and

$$v(t) \le v^{2-p}(t) \le m(t),$$

Furthermore, by use of the property (iii) and Lemma 3 one can obtain that

$$D_t^\alpha m(t) = (2-p)v^{1-p}(t)D_t^\alpha v(t) + \Gamma(\alpha)g(t)v^q(t)$$

$$\le (2-p)f(t)m(t) + \Gamma(\alpha)g(t)m^q(t). \tag{16}$$

On the other hand, Since u, f, g is continuous, then there exists a positive number M_i, $i = 1,2,3$ such that $|f(s)| \le M_1$, $|g(\xi)| \le M_2$, $|u(\xi)| \le M_3$ for $\xi \in [0,s]$, $s \in [0,t]$ and $t \in [0,\varepsilon]$, where $\varepsilon > 0$. So for $t \in [0,\varepsilon]$, we have $|\int_0^s (s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi| \le M_2M_3^q\int_0^s (s-\xi)^{\alpha-1}d\xi = \frac{M_2M_3^q s^\alpha}{\alpha}$. Then combining with (a) of Lemma 2 one has

$$|v(t)| = |C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)[u^{2-p}(s)$$

$$+ \int_0^s (s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds|$$

$$\le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|M_1[u^{2-p}(s)$$

$$+ \int_0^s (s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p|ds$$

$$\le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}M_1(M_3^{2-p} + \frac{M_2M_3^q s^\alpha}{\alpha})^p ds$$

$$\le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}M_1(M_3^{p(2-p)} + \frac{M_2^pM_3^{pq}s^{\alpha p}}{\alpha^p})ds$$

$$= C + \frac{M_1M_3^{p(2-p)}t^\alpha}{\alpha\Gamma(\alpha)} + \frac{M_1M_2^pM_3^{pq}}{\alpha^p\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha p}ds$$

$$= C + \frac{M_1M_3^{p(2-p)}t^\alpha}{\alpha\Gamma(\alpha)} + \frac{M_1M_2^pM_3^{pq}B(\alpha p+1,\alpha)t^{(p+1)\alpha}}{\alpha^p\Gamma(\alpha)},$$

where $B(.,.)$ denotes the Beta function.

From above one can see $v(0) = C$. Similarly, one can deduce that $m(0) = v^{2-p}(0)$. So an application of Lemma 1 to (16) yields that

$$m(t) \le \exp[(2-p)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$

$$\{C^{(2-p)(1-q)} + (1-q)\int_0^t (t-\tau)^{\alpha-1}g(\tau)\times\exp[-(1-q)$$

$$(2-p)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}^{\frac{1}{1-q}}, \tag{17}$$

Combining (15) and (17) one can deduce that

$$D_t^\alpha v(t) \le f(t)\exp[p(2-p)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$

$$\{C^{(2-p)(1-q)} + (1-q)\int_0^t (t-\tau)^{\alpha-1}g(\tau)\exp[-(1-q)$$

$$(2-p)\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}^{\frac{p}{1-q}} = \theta(t), \tag{18}$$

where $\theta(t)$ is defined as in (12).

By (iv), fulfilling fractional integral of order α for (18) with respect to s from 0 to t one can obtain that

$$v(t) \le v(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta(s)ds$$

$$= C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta(s)ds. \tag{19}$$

Combining (13) and (19) we can derive the desired result.

Remark 1. If we set $\alpha = 1$, then the inequality (1) reduces to the inequality (2.28) in [16, Theorem 2.5], and the estimate obtained in Theorem 4 reduces to the result

(2.29) in [16, Theorem 2.5] correspondingly.

Based on the inequality (1), we now study one Volterra-Fredholm type inequality as follows:

$$u(t) \le C+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)[u^{2-p}(s)+\int_0^s(s-\xi)^{\alpha-1} g(\xi)u^q(\xi)d\xi]^p ds + \frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)[u^{2-p}(s) + \int_0^s(s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds,\ t\in[0,T], \quad (20)$$

where $T>0$ is a constant, $\alpha,\ u,\ f,\ g,\ C,\ p,\ q$ are defined as in the inequality (1).

Theorem 5. Suppose the inequality (20) holds. Then we have the following estimate for $u(t)$:

(a) If $0\le q<1,\ p+q<1$, then

$$u(t)\le M_T+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)[M_T+H_2^{\frac{p}{1-q}}(s)]ds \quad t\in[0,T], \quad (21)$$

provided that $\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds<1$, where

$$\begin{cases} M_T=\dfrac{C+\frac{1}{\Gamma(\alpha)}\displaystyle\int_0^T(T-s)^{\alpha-1}H_1(s)H_2^{\frac{p}{1-q}}(s)ds}{1-\frac{1}{\Gamma(\alpha)}\displaystyle\int_0^T(T-s)^{\alpha-1}H_1(s)ds},\\ H_1(t)=f(t)\exp[p(2-p)\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}}f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds],\\ H_2(t)=(1-q)\int_0^t(t-\tau)^{\alpha-1}g(\tau)\exp[-(1-q)(2-p) \int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau. \end{cases} \quad (22)$$

(b) If $q>1$, then

$$u(t)\le C+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)ds,\ t\in[0,T], \quad (23)$$

provided that $C^{(2-p)(1-q)}+H_2(t)>0$, where $H_1(t),\ H_2(t)$ are defined as in (22).

Proof. Denote the right-hand side of (20) by $v(t)$. Then we have

$$u(t)\le v(t),\ t\in[0,T],$$

By the similar analysis as in Theorem 4 one can obtain

$$v(0)=C+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)[u^{2-p}(s) +\int_0^s(s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds.$$

So

$$v(t)=v(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)[u^{2-p}(s) +\int_0^s(s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds.$$

Also by the similar process as in (14)-(19)we deduce that

$$v(t)\le v(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\theta(s)ds,\ t\in[0,T]. \quad (24)$$

where $\theta(t)=H_1(t)\{[v(0)]^{(2-p)(1-q)}+H_2(t)\}^{\frac{p}{1-q}}$, and $H_1(t),\ H_2(t)$ are defined as in (22).

(a) If $0\le q<1,\ p+q<1$, then $\frac{p}{1-q}<1$, and by use of (a) of Lemma 2, considering $p(2-p)\le 1$, one can obtain that

$$\theta(t)\le H_1(t)\{[v(0)]^{p(2-p)}+H_2^{\frac{p}{1-q}}(t)\} \le H_1(t)[v(0)+H_2^{\frac{p}{1-q}}(t)]. \quad (25)$$

So

$$v(t)\le v(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)[v(0)+H_2^{\frac{p}{1-q}}(s)]ds = v(0)[1+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)ds] +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)H_2^{\frac{p}{1-q}}(s)ds,\ t\in[0,T].$$

On the other hand, we have

$$2v(0)-C=v(T) \le v(0)[1+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds] +\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)H_2^{\frac{p}{1-q}}(s)ds,$$

which implies

$$v(0)\le\frac{C+\frac{1}{\Gamma(\alpha)}\displaystyle\int_0^T(T-s)^{\alpha-1}H_1(s)H_2^{\frac{p}{1-q}}(s)ds}{1-\frac{1}{\Gamma(\alpha)}\displaystyle\int_0^T(T-s)^{\alpha-1}H_1(s)ds}, \quad (26)$$

provided that $\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds<1$.

By a combination of (24)-(26) we can deduce the desired estimate (21).

(b) If $q>1$, then $\frac{p}{1-q}<0$, and $H_2(t)<0$. So under the condition $C^{(2-p)(1-q)}+H_2(t)>0$ we have $\theta(t)\le H_1(t)$. Furthermore,

$$v(t)\le v(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H_1(s)ds,\ t\in[0,T]. \quad (27)$$

Similar to (a), it holds that

$$2v(0)-C=v(T)\le v(0)+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds,$$

which means

$$v(0)\le C+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}H_1(s)ds.$$

Combining with (27) we can deduce the desired result (23). The proof is complete.

Now we study the Gronwall-Bellman type inequality with the following form:

$$u(t)\le a(t)+\frac{b(t)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)[u(s)$$

$$+\int_0^s g(\xi)u(\xi)d\xi]^p ds,\ t\geq 0, \tag{28}$$

where $\alpha>0,\ 0<p\leq 1,\ u,\ a,\ b,\ f,\ g$ are nonnegative continuous functions defined on $t\geq 0$.

Lemma 6 [**17**]. Assume that $a\geq 0, p\geq q\geq 0$, and $p\neq 0$, then for any $K>0$,

$$a^{\frac{q}{p}}\leq \frac{q}{p}K^{\frac{q-p}{p}}a+\frac{p-q}{p}K^{\frac{q}{p}}. \tag{29}$$

Lemma 7 [**36, Lemma 4**]. Let $\alpha>0,\ a,\ b,\ u$ be continuous functions defined on $t\geq 0$. Then for $t\geq 0$,

$$D_t^\alpha u(t)\leq a(t)+b(t)u(t) \tag{30}$$

implies

$$u(t)\leq u(0)\exp[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} b((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]+\frac{1}{\Gamma(\alpha)}$$

$$\int_0^t (t-\tau)^{\alpha-1}a(\tau)\exp[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} b((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau. \tag{31}$$

Theorem 8. Suppose the inequality (28) holds. Then we have the following explicit estimate for $u(t)$:

$$u(t)\leq a(t)+b(t)\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\theta_1(\tau)$$

$$\exp[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}}\theta_2((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\},\ t\geq 0, \tag{32}$$

where

$$\theta_1(t)=f(t)\{pK^{1-p}[a(t)+\int_0^t g(\xi)a(\xi)d\xi]+(1-p)K^p\}, \tag{33}$$

$$\theta_2(t)=pK^{1-p}f(t)[b(t)+\int_0^t g(\xi)b(\xi)d\xi], \tag{34}$$

and $K>0$ is an arbitrarily constant.

Proof. Denote

$$v(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)[u(s)+\int_0^s g(\xi)u(\xi)d\xi]^p ds.$$

Then we have

$$u(t)\leq a(t)+b(t)v(t),\ t\geq 0. \tag{35}$$

Similar to the analysis as in Theorem 4 one can see $v(0)=0$. On the other hand, there also exists a nonnegative number N such that $|f(s)[u(s)+\int_0^s g(\xi)u(\xi)d\xi]^p|\geq N$ for $s\in[0,t]$. So combining with the uniformly convergence of the integral denoted by $v(t)$ one can obtain that $v'(t)\geq \frac{N}{\Gamma(\alpha)}t^{\alpha-1}\geq 0$, which implies $v(t)$ is nondecreasing. So by use of Lemmas 3, 6 one can obtain that

$$D_t^\alpha v(t)=f(t)[u(t)+\int_0^t g(\xi)u(\xi)d\xi]^p$$

$$\leq f(t)[a(t)+b(t)v(t)+\int_0^t g(\xi)(a(\xi)+b(\xi)v(\xi))d\xi]^p$$

$$\leq f(t)\{a(t)+\int_0^t g(\xi)a(\xi)d\xi+[b(t)+\int_0^t g(\xi)b(\xi)d\xi]v(t)\}^p$$

$$\leq f(t)\{pK^{1-p}\{a(t)+\int_0^t g(\xi)a(\xi)d\xi+[b(t)$$

$$+\int_0^t g(\xi)b(\xi)d\xi]v(t)\}+(1-p)K^p\},$$

$$=\theta_1(t)+\theta_2(t)v(t),$$

where $\theta_1(t),\ \theta_2(t)$ are defined as in (33), (34), and $K>0$ is an arbitrary constant.

By use of Lemma 7 one has

$$v(t)\leq\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\theta_1(\tau)\exp[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}}$$

$$\theta_2((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau. \tag{36}$$

Combining (35) and (36) we can obtain the desired result (32).

Remark 2. If we take $\alpha=1,\ b(t)\equiv 1$, then the inequality (28) in Theorem 8 reduces to the inequality (2.73) in [16, Theorem 2.9]. However, the estimate obtained in Theorem 8 is different from that in [16, Theorem 2.9].

Finally we consider the following Volterra-Fredholm type inequality:

$$u(t)\leq C+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)[u(s)$$

$$+\int_0^s g(\xi)u(\xi)d\xi]^p ds+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s)[u(s)$$

$$+\int_0^s g(\xi)u(\xi)d\xi]^p ds,\ t\in[0,T], \tag{37}$$

where $C,\ T,\ p$ are positive constants with $C\geq 1,\ p<\frac{1}{2}$, $\alpha,\ u,\ f,\ g$ are defined as in the inequality (28).

Theorem 9. Suppose the inequality (37) holds. Then we have the following estimate for $u(t)$:

$$u(t)\leq\{[\frac{C+[\frac{(1-p)}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}}}{1-\frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau}]^{1-p}$$

$$+\frac{(1-p)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau\}^{\frac{1}{1-p}},\ t\in[0,T], \tag{38}$$

provided that $\frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau<1$, where $\widetilde{f}(t)=f(t)[1+\int_0^t g(\xi)d\xi]^p$.

Proof. Denote the right-hand side of (37) by $v(t)$. Then we have

$$u(t)\leq v(t),\ t\in[0,T], \tag{39}$$

By the similar analysis as in Theorem 8 one can obtain that $v(t)$ is nondecreasing, and

$$v(0)=C+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s)[u^{2-p}(s)$$

$$+\int_0^s (s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds.$$

So

$$v(t)=v(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)[u(s)$$

$$+\int_0^s g(\xi)u(\xi)d\xi]^p ds.$$

Then

$$D_t^\alpha v(t) = f(t)[u(t) + \int_0^t g(\xi)u(\xi)d\xi]^p$$
$$\leq f(t)[v(t) + \int_0^t g(\xi)v(\xi)d\xi]^p$$
$$\leq f(t)[1 + \int_0^t g(\xi)d\xi]^p[v(t)]^p = \widetilde{f}(t)[v(t)]^p,$$

where $\widetilde{f}(t) = f(t)[1 + \int_0^t g(\xi)d\xi]^p$.

Using Lemma 1 we obtain that

$$v(t) \leq \{v^{1-p}(0) + \frac{(1-p)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau\}^{\frac{1}{1-p}},\ t \in [0,T]. \quad (40)$$

Under the condition $p < \frac{1}{2}$, by use of (b) of Lemma 2 one can deduce that

$$v(t) \leq \frac{v^p(0)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau + [\frac{(1-p)}{\Gamma(\alpha)}$$
$$\int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}} + v(0)$$
$$\leq v(0)[1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau] + [\frac{(1-p)}{\Gamma(\alpha)}$$
$$\int_0^t (t-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}},\ t \in [0,T].$$

So

$$2v(0) - C = v(T)$$
$$\leq v(0)[1 + \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau] + [\frac{(1-p)}{\Gamma(\alpha)}$$
$$\int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}},$$

which implies

$$v(0) \leq \frac{C + [\frac{(1-p)}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}}}{1 - \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1}\widetilde{f}(\tau)d\tau}, \quad (41)$$

provided that $\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}H_1(s)ds < 1$.

Combining (40) and (41) we can deduce the desired result.

III. Applications

In this section, we consider the following IVP of fractional differential equation:

$$\begin{cases} D_t^{0.8}u(t) = F(t, u(t), \int_0^t (t-\xi)^{-0.2}G(\xi, u(\xi))d\xi),\ t \geq 0, \\ u(0) = u_0, \end{cases} \quad (42)$$

where $u \in C([0,\infty), R)$, $G \in C(R \times R, R)$, $F \in C([0,\infty) \times R^2, R)$.

We will apply the results established above to research the qualitative and quantitative properties for solutions of the IVP (42).

Theorem 10. Suppose $u(t)$ is a solution of the IVP (42), and $|u_0| \geq 1$. If $|F(t,x,y)| \leq f(t)(|x|^{\frac{3}{2}} + |y|)^{\frac{1}{2}}$, and $|G(t,z)| \leq g(t)|z|^{\frac{1}{2}}$, where $f,\ g$ are nonnegative continuous functions on $[0,\infty)$, then we have the following estimate for $u(t)$:

$$|u(t)| \leq |u_0| + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}\theta(s)ds,\ t \geq 0, \quad (43)$$

where

$$\theta(t) = f(t)\exp[\frac{3}{4} \int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]\times$$
$$\{|u_0|^{\frac{3}{4}} + \frac{1}{2} \int_0^t (t-\tau)^{\alpha-1}g(\tau)\exp[-\frac{3}{4} \int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}$$
$$f((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\}.$$

Proof. Fulfilling fractional integral of order 0.8 to (42), by use of the property (iv), one can obtain the equivalent integral form of the IVP (42) as follows:

$$u(t) = u_0 + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}$$
$$F(s, u(s), \int_0^s (s-\xi)^{-0.2}G(\xi, u(\xi))d\xi)ds. \quad (44)$$

So

$$|u(t)| \leq |u_0| + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}$$
$$|F(s, u(s), \int_0^s (s-\xi)^{-0.2}G(\xi, u(\xi))d\xi)|ds$$
$$\leq |u_0| + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}f(s)[|u(s)|^{\frac{3}{2}}$$
$$+|\int_0^s (s-\xi)^{-0.2}G(\xi, u(\xi))d\xi|]^{\frac{1}{2}}ds$$
$$\leq |u_0| + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}f(s)[|u(s)|^{\frac{3}{2}}$$
$$+ \int_0^s |(s-\xi)^{-0.2}G(\xi, u(\xi))|d\xi]^{\frac{1}{2}}$$
$$\leq |u_0| + \frac{1}{\Gamma(0.8)} \int_0^t (t-s)^{-0.2}f(s)[|u(s)|^{\frac{3}{2}}$$
$$+ \int_0^s (s-\xi)^{-0.2}g(\xi)|u(\xi)|^{\frac{1}{2}}d\xi]^{\frac{1}{2}}ds. \quad (45)$$

After applying Theorem 4 to (45) (with $\alpha = 0.8,\ p = q = \frac{1}{2}$) we can obtain the desired estimate (43).

Corollary 11. In Theorem 10, if under the conditions $|F(t,x,y)| \leq f(t)(|x| + |y|)^{\frac{1}{2}}$ and $|G(t,z)| \leq g(t)|z|$ instead, then one can deduce the following bound for $u(t)$:

$$|u(t)| \leq |u_0| + \{\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\theta_1(\tau)$$
$$\exp[-\int_{\frac{\tau^\alpha}{\Gamma(1+\alpha)}}^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \theta_2((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})ds]d\tau\},\ t \geq 0,$$

where $\theta_1(t) = \frac{1}{2}K^{\frac{1}{2}}f(t)\{|u_0|[1 + \int_0^t g(\xi)d\xi] + 1\}$, $\theta_2(t) = \frac{1}{2}K^{\frac{1}{2}}f(t)[1 + \int_0^t g(\xi)d\xi]$, and $K > 0$ is an arbitrarily constant.

The proof of Corollary 11 can be completed by the similar process as in Theorem 10 and the suitable application of Theorem 8.

IV. Further extensions

In this section, we present further extensions for the established fractional integral inequalities in Section 2 under the definition of another fractional derivative named the conformable fractional derivative.

Definition 12 [40, Definition 2.1]. The conformable fractional derivative of order α is defined by
$$D_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}.$$

Definition 13 [40, Definition 3.1]. The conformable fractional integral of order α on the interval $[0,t]$ is defined by

$$I^\alpha f(t)=\int_0^t s^{\alpha-1}f(s)ds.$$

The following properties can be easily proved due to the definition of the conformable fractional derivative and the conformable fractional integral:

(i) $D_t^\alpha[af(t)+bg(t)]=aD^\alpha f(t)+bD^\alpha g(t)$.

(ii) $D_t^\alpha(t^\gamma)=\gamma t^{\gamma-\alpha}$.

(iii) $D_t^\alpha[f(t)g(t)]=f(t)D_t^\alpha g(t)+g(t)D_t^\alpha f(t)$.

(iv) $D_t^\alpha C=0$, where C is a constant.

(v) $D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t)$.

(vi) $D_t^\alpha(\frac{f}{g})(t)=\frac{g(t)D_t^\alpha f(t)-f(t)D_t^\alpha g(t)}{g^2(t)}$.

(vii) $D_t^\alpha f(t)=t^{1-\alpha}f'(t)$.

$(viii)$ $D_t^\alpha(I^\alpha f(t))=f(t)$

(ix) $I_t^\alpha(D_t^\alpha f(t))=f(t)-f(0)$ on the interval $[0,t]$.

Based on the properties of the conformable fractional derivative and the conformable fractional integral, one can obtain the following results on fractional integral inequalities similar to Theorems 4, 5, 8, 9. The prove process for them are similar to those for Theorems 4, 5, 8, 9, which are omitted here.

Theorem 14. Suppose the following inequality holds:

$$u(t)\le C+\int_0^t s^{\alpha-1}f(s)[u^{2-p}(s)+\int_0^s \xi^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds,\ t\ge 0, \tag{46}$$

where $\alpha,\ u,\ f,\ g,\ C,\ p,\ q$ are defined as in (1), then we have the following estimate for $u(t)$:

$$u(t)\le C+\int_0^t s^{\alpha-1}\theta(s)ds,\ t\ge 0, \tag{47}$$

provided that

$$C^{(2-p)(1-q)}+(1-q)\int_0^t \tau^{\alpha-1}g(\tau)\exp[-(1-q)(2-p)\int_0^{\frac{\tau^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]d\tau>0,$$

where

$$\theta(t)=f(t)\exp[p(2-p)\int_0^{\frac{t^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]\times \{C^{(2-p)(1-q)}+(1-q)\int_0^t \tau^{\alpha-1}g(\tau)\exp[-(1-q)(2-p)\int_0^{\frac{\tau^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]d\tau\}^{\frac{p}{1-q}}. \tag{48}$$

Theorem 15. Suppose the following Volterra-Fredholm type inequality holds:

$$u(t)\le C+\int_0^t s^{\alpha-1}f(s)[u^{2-p}(s)+\int_0^s (s-\xi)^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds+\int_0^T s^{\alpha-1}f(s)[u^{2-p}(s)+\int_0^s \xi^{\alpha-1}g(\xi)u^q(\xi)d\xi]^p ds,\ t\in[0,T], \tag{49}$$

where $\alpha,\ u,\ f,\ g,\ C,\ p,\ q,\ T$ are defined as in (20), then we have the following estimate for $u(t)$:

(a) If $0\le q<1,\ p+q<1$, then

$$u(t)\le M_T+\int_0^t s^{\alpha-1}H_1(s)[M_T+H_2^{\frac{p}{1-q}}(s)]ds\quad t\in[0,T], \tag{50}$$
provided that $\int_0^T s^{\alpha-1}H_1(s)ds<1$, where

$$\begin{cases} M_T=\dfrac{C+\displaystyle\int_0^T s^{\alpha-1}H_1(s)H_2^{\frac{p}{1-q}}(s)ds}{1-\displaystyle\int_0^T s^{\alpha-1}H_1(s)ds},\\ H_1(t)=f(t)\exp[p(2-p)\int_0^{\frac{t^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds],\\ H_2(t)=(1-q)\int_0^t \tau^{\alpha-1}g(\tau)\exp[-(1-q)(2-p)\\ \quad\int_0^{\frac{\tau^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]d\tau. \end{cases} \tag{51}$$

(b) If $q>1$, then

$$u(t)\le C+\int_0^T s^{\alpha-1}H_1(s)ds+\int_0^t s^{\alpha-1}H_1(s)ds,\quad t\in[0,T], \tag{52}$$

provided that $C^{(2-p)(1-q)}+H_2(t)>0$, where $\theta(t)$, $H_1(t),\ H_2(t)$ are defined as in (51).

Theorem 16. Suppose the following inequality holds:

$$u(t)\le a(t)+b(t)\int_0^t s^{\alpha-1}f(s)[u(s)+\int_0^s g(\xi)u(\xi)d\xi]^p ds,\ t\ge 0, \tag{53}$$

where $\alpha,\ u,\ f,\ g,\ a,\ b,\ p$ are defined as in (28), then we have the following explicit estimate for $u(t)$:

$$u(t)\le a(t)+b(t)\{\int_0^t \tau^{\alpha-1}\theta_1(\tau)\exp[-\int_{\frac{\tau^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}}\theta_2((s\alpha)^{\frac{1}{\alpha}})ds]d\tau\},\ t\ge 0, \tag{54}$$
where $\theta_1(t),\ \theta_2(t)$ are defined as in (33)-(34), and $K>0$ is an arbitrarily constant.

Theorem 17. Suppose the following inequality holds:

$$u(t)\le C+\int_0^t s^{\alpha-1}f(s)[u(s)+\int_0^s g(\xi)u(\xi)d\xi]^p ds$$

$$+\int_0^T s^{\alpha-1}f(s)[u(s)+\int_0^s g(\xi)u(\xi)d\xi]^p ds,\ t\in[0,T], \quad (55)$$

where $\alpha,\ u,\ f,\ g,\ C,\ p,\ T$ are defined as in (37), then we have the following estimate for $u(t)$:

$$u(t)\le\{[\frac{C+[(1-p)\int_0^T \tau^{\alpha-1}\widetilde{f}(\tau)d\tau]^{\frac{1}{1-p}}}{1-\int_0^T \tau^{\alpha-1}\widetilde{f}(\tau)d\tau}]^{1-p}$$
$$+(1-p)\int_0^t \tau^{\alpha-1}\widetilde{f}(\tau)d\tau\}^{\frac{1}{1-p}},\ t\in[0,T], \quad (56)$$

provided that $\int_0^T \tau^{\alpha-1}\widetilde{f}(\tau)d\tau<1$, where $\widetilde{f}(t)=f(t)[1+\int_0^t g(\xi)d\xi]^p$.

For applications of the presented results above, we consider the following IVP of fractional differential equation:

$$\begin{cases} D_t^{0.5}u(t)=F(t,u(t),\int_0^t(t-\xi)^{-0.5}G(\xi,u(\xi))d\xi),\ t\ge 0,\\ u(0)=u_0, \end{cases} \quad (57)$$

where $u\in C([0,\infty),R)$, $G\in C(R\times R,R)$, $F\in C([0,\infty)\times R^2,R)$.

Similar to the proof of Theorem 10 and Corollary 11, by use of Theorems 14 and 16, one can prove the following results.

Theorem 18. Under the conditions of Theorem 10, we have the following estimate for $u(t)$:

$$|u(t)|\le|u_0|+\int_0^t(t-s)^{-0.5}\theta(s)ds,\ t\ge 0,$$

where

$$\theta(t)=f(t)\exp[\frac{3}{4}\int_0^{\frac{t^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]\{|u_0|^{\frac{3}{4}}+\frac{1}{2}\int_0^t \tau^{\alpha-1}$$
$$g(\tau)\exp[-\frac{3}{4}\int_0^{\frac{\tau^\alpha}{\alpha}} f((s\alpha)^{\frac{1}{\alpha}})ds]d\tau\}.$$

Corollary 19. Under the conditions $|F(t,x,y)|\le f(t)(|x|+|y|)^{\frac{1}{2}}$ and $|G(t,z)|\le g(t)|z|$, one can obtain the following bound for $u(t)$ with $t\ge 0$:

$$|u(t)|\le|u_0|+\{\int_0^t \tau^{\alpha-1}\theta_1(\tau)\exp[-\int_{\frac{\tau^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}}\theta_2((s\alpha)^{\frac{1}{\alpha}})ds]d\tau\},$$

where $\theta_1,\ \theta_2$ are defined as in Corollary 11.

V. Conclusions

In this paper, we have established some new Gronwall-Bellman type inequalities as well as Volterra-Fredholm type inequalities, which provide explicit bounds for unknown functions lying in these inequalities. As for applications, we apply the present results to research the boundedness for the IVP of one certain fractional differential equation. For further extensions, we also presented corresponding results under the definition of the conformable fractional derivative. By use of the method in the main theorems, one can construct Gronwall-Bellman type inequalities with more complicated and general forms, such as inequalities in 2D case, discrete case, or inequalities with impulse items, which are expected to further research.

References

[1] A. Kilbas, H. Srivastava and J. Trujillo, "Theory and Applications of Fractional Differential Equations," *Elsevier*, Boston, 2006.

[2] A. M. Bijura, " Systems of Singularly Perturbed Fractional Integral Equations II," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 4, pp. 198-203, 2012.

[3] A. Bouhassoun, " Multistage Telescoping Decomposition Method for Solving Fractional Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 1, pp. 10-16, 2013.

[4] Q.H. Feng, " Jacobi Elliptic Function Solutions For Fractional Partial Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 121-129, 2016.

[5] H.M. Jaradat, S. Al-Shar'a, Q. J.A. Khan, M. Alquran and K. Al-Khaled, " Analytical Solution of Time-Fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 1, pp. 64-70, 2016.

[6] H. Song, M. Yi, J. Huang and Y. Pan, " Bernstein Polynomials Method for a Class of Generalized Variable Order Fractional Differential Equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 437-444, 2016.

[7] S.X. Zhou, F.W. Meng, Q.H. Feng and L. Dong, " A Spatial Sixth Order Finite Difference Scheme for Time Fractional Sub-diffusion Equation with Variable Coefficient," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 2, pp. 175-181, 2017.

[8] Z. Yan and F. Lu, " Existence of A New Class of Impulsive Riemann-Liouville Fractional Partial Neutral Functional Differential Equations with Infinite Delay," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 300-312, 2015.

[9] B. Rebiai and K. Haouam, " Nonexistence of Global Solutions to a Nonlinear Fractional Reaction-Diffusion System," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 259-262, 2015.

[10] M. A. Abdellaoui, Z. Dahmani, and N. Bedjaoui, " Applications of Fixed Point Theorems for Coupled Systems of Fractional Integro-Differential Equations Involving Convergent Series," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 273-278, 2015.

[11] T.H. Gronwall, "Note on the derivatives with respect to a parameter of solutions of a system of differential equations," *Ann. of Math.*, vol. 20, pp. 292–296, 1919.

[12] R. Bellman, "The stability of solutions of linear differential equations," *Duke Math. J.*, vol. 10, pp. 643–647, 1943.

[13] W.S. Cheung, "Some new nonlinear inequalities and applications to boundary value problems," *Nonlinear Anal. TMA*, vol. 64, pp. 2112–2128, 2006.

[14] A. Gallo and A.M. Piccirillo, "About some new generalizations of Bellman-Bihari results for integro-functional inequalities with discontinuous functions and applications," *Nonlinear Anal.*, vol. 71, pp. e2276–e2287, 2009.

[15] B.G. Pachpatte, "On Some New Inequalities Related to a Certain Inequality Arising in the Theory of Differential Equations," *J. Math. Anal. Appl.*, vol. 251, pp. 736–751, 2000.

[16] H. El-Owaidy, A. A. Ragab, W. Abuelela and A. A. El-Deeb, " On some new nonlinear integral inequalities of Gronwall-Bellman type," *KKYUNGPOOK Math. J.*, vol. 54, pp. 555-575, 2014.

[17] F.C. Jiang and F. W. Meng, "Explicit bounds on some new nonlinear integral inequality with delay," *J. Comput. Appl. Math.*, vol. 205, pp. 479–486, 2007.

[18] B.G. Pachpatte, "Inequalities for Differential and Integral Equations," *Academic Press*, New York, 1998.

[19] O. Lipovan, " Integral inequalities for retarded Volterra equations," *J. Math. Anal. Appl.*, vol. 322, pp. 349-358, 2006.

[20] W.S. Wang, "A class of retarded nonlinear integral inequalities and its application in nonlinear differential-integral equation," *J. Inequal. Appl.*, vol. 2012:154, pp. 1–10, 2012.

[21] R.A.C. Ferreira and D.F.M. Torres, " Generalized retarded integral inequalities," *Appl. Math. Lett.*, vol. 22, pp. 876-881, 2009.

[22] R.P. Agarwal, S.F. Deng and W.N. Zhang, " Generalization of a retarded Gronwall-like inequality and its applications," *Appl. Math. Comput.*, vol. 165, pp. 599-612, 2005.

[23] W.S. Wang, "Some retarded nonlinear integral inequalities and their applications in retarded differential equations," *J. Inequal. Appl.*, vol. 2012:75, pp. 1–8, 2012.

[24] Q.H. Ma, "Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications," *J. Comupt. Appl. Math.*, vol. 233, pp. 2170–2180, 2010.

[25] B. Zheng, " Qualitative and quantitative analysis for solutions to a class of Volterra-Fredholm type difference equation," *Adv. Diff. Equ.*, vol. 2011, Article 30, pp. 1-16, 2011.

[26] S.F. Deng, "Nonlinear discrete inequalities with two variables and their applications," *Appl. Math. Comput.*, vol. 217, pp. 2217–2225, 2010.

[27] Q.H. Feng, F.W. Meng and B.S. Fu, " Some new generalized Volterra-Fredholm type finite difference inequalities involving four iterated sums," *Appl. Math. Comput.*, vol. 219, pp. 8247-8258, 2013.

[28] W.S. Cheung and J.L. Ren, " Discrete nonlinear inequalities and applications to boundary value problems," *J. Math. Anal. Appl.*, vol. 319, pp. 708-724, 2006.
[29] S.H. Saker, " Some nonlinear dynamic inequalities on time scales," *Math. Inequal. Appl.*, vol. 14, pp. 633-645, 2011.
[30] B. Zheng, Q.H. Feng, F.W. Meng and Y.M. Zhang, "Some new Gronwall-Bellman type nonlinear dynamic inequalities containing integration on infinite intervals on time scales," *J. Inequal. Appl.*, vol. 2012:201, pp. 1–20, 2012.
[31] R. Agarwal, M. Bohner and A. Peterson, " Inequalities on time scales: a survey," *Math. Inequal. Appl.*, vol. 4, no 4, pp. 535-557, 2001.
[32] Q.H. Feng, F.W. Meng and B. Zheng, " Gronwall-Bellman type nonlinear delay integral inequalities on time scales," *J. Math. Anal. Appl.*, vol. 382, pp. 772-784, 2011.
[33] W.N. Li, " Some delay integral inequalities on time scales," *Comput. Math. Appl.*, vol. 59, pp. 1929-1936, 2010.
[34] H.P. Ye, J.M. Gao and Y.S. Ding, " A generalized Gronwall inequality and ins application to a fractional differential equation," *J. Math. Anal. Appl.*, vol. 328, pp. 1075-1081, 2007.
[35] B. Zheng, " Explicit bounds derived by some new inequalities and applications in fractional integral equations," *J. Inequal. Appl.*, vol. 2014, Article 4, pp. 1-12, 2014.
[36] B. Zheng and Q.H. Feng, " New Gronwall-Bellman Type Inequalities and Applications in the Analysis for Solutions to Fractional Differential Equations," *Abst. Appl. Anal.*, vol. 2013, Article 705126, pp. 1-13, 2013.
[37] Q.H. Feng, " Some New Gronwall-Bellmann Type Discrete Fractional Inequalities Arising in the Theory of Discrete Fractional Calculus," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 527-534, 2016.
[38] G. Jumarie, " Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *Comput. Math. Appl.*, vol. 51, pp. 1367-1376, 2006.
[39] G. Jumarie, " Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions," *Appl. Math. Lett.*, vol. 22, pp. 378-385, 2009.
[40] Khalil R, Al-Horani M, Yousef A, Sababheh M: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65-70 (2014).