Painlevé Analysis, Lax Pairs and New Analytic Solutions for a High-order Boussinesq-Burgers Equation

Jinming Zuo

Abstract—**In this work, a high-order Boussinesq-Burgers equation is investigated. Painlevé analysis and Lax pairs are given out, and an auto-Bäcklund transformation is presented via the truncated Painlevé expansion, a basic Darboux transformation of a spectral problem is considered. Some new solutions are given, including travelling wave solutions, periodic solutions and soliton and so on.**

Index Terms—**high-order Boussinesq-Burgers equation, Painlevé analysis, Lax pairs, Bäcklund transformation, Darboux transformation.**

I. Introduction

It is known that there are many approaches to find the exact solutions for a given partial differential equation in the nonlinear science, such as the symmetry reduction, Homogeneous balance method [1], Hirota's bilinear method [2,3], Bäcklund transformation [4-6], Darboux transformation [7,8] and the variable separation approach, etc. The Painlevé analysis [9-15] plays a very important role because it can be used not only to isolate out integrable models but also to find many other integrable properties such as the Bäcklund transformations, Lax pair, Schwarzian form and more new integrable models.

In this work, we will discuss the following high-order Boussinesq-Burgers equation [16-19]

$$\begin{array}{l} u_t - 3\sigma u^2 u_x + \frac{3}{2}\sigma(uv)_x - \frac{1}{4}\sigma u_{xxx} = 0, \\ v_t + \frac{3}{2}\sigma v v_x - 3\sigma(u^2 v)_x + 3\sigma u_x u_{xx} + \frac{3}{2}\sigma u u_{xxx} \\ -\frac{1}{4}\sigma v_{xxx} = 0. \end{array} \tag{1}$$

where σ is a non-zero arbitrary constant.

Zuo and Zhang [16] first applied the simplified Hirota's method to derive multiple kink solutions, where they used this derivation to claim that Eq. (1) is integrable although other justifications, such as Painlevé analysis and Lax pairs, were not given to confirm this result. Guo et al. [17] applied the homogeneous balance method to find multiple-soliton (kink) solutions of Eq. (1). Jaradat et al. [18] and Wazwaz [19] used function expansion methods to investigate soliton and periodic solutions.

Our aim from this work is two fold. The first goal is to investigate Painlev integrability and Lax pairs of the high-order Boussinesq-Burgers equation (1), which claim that Eq. (1) is integrable. We aim second to find more soliton and periodic solutions based on an auto-Bäcklund transformation and Darboux transformation.

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II. Painlevé integrability and auto-Bäcklund transformation

The Painlevé analysis is a powerful method for testing the integrability of any nonlinear partial differential equations. The Weiss-Tabor-Carnevale (WTC) method [9] and Kruskals simplification method are the most widely applied tools to prove the Painlevé property [9]. More details are given in Refs. [10-15]. To proceed with the Painlevé singularity analysis, we set

$$\begin{array}{l} u = \phi^\alpha \sum\limits_{j=0}^{\infty} u_j \phi^j, \\ v = \phi^\beta \sum\limits_{j=0}^{\infty} v_j \phi^j. \end{array} \tag{2}$$

where we are searching for the singular solution manifold given by $\phi = \phi(x,t)$, and $u_j, v_j (j = 0, 1, 2, \cdots)$ are functions of (x,t), and α and β are negative integers to be determined.

In order to get the leading orders of the solutions of Eq. (1), we suppose they have the forms as below

$$u \sim u_0 \phi^\alpha, \; v \sim v_0 \phi^\beta. \tag{3}$$

Inserting Expressions (3) into Eq. (1) and balancing the highest order derivative terms with the nonlinear terms, we can obtain that $\alpha = -1, \beta = -2$, and the following relations

Case 1: $u_0 = \frac{1}{2}\phi_x, v_0 = \frac{1}{2}\phi_x^2$,
Case 2: $u_0 = -\frac{1}{2}\phi_x, v_0 = \frac{1}{2}\phi_x^2$,
Case 3: $u_0 = \phi_x, v_0 = \phi_x^2$,
Case 4: $u_0 = -\phi_x, v_0 = \phi_x^2$,

For simplicity, we only discuss the Painlevé property in Case 1, the process of which can be extended to the others similarly.

For the purpose of getting the resonances, at which the solutions have arbitrary coefficient functions, we suppose the solutions have the following forms

$$\begin{array}{l} u \sim u_0 \phi^{-1} + u_j \phi^{j-1}, \\ v \sim v_0 \phi^{-2} + v_j \phi^{j-2}. \end{array} \tag{4}$$

Substituting Expressions (4) into Eq. (1) and collecting the terms with the lowest powers of ϕ, we can derive the general recursion relations

$$Q(j)\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix}, \tag{5}$$

where

$$\begin{array}{l} F_j = -\frac{1}{4}(j^3 - 6j^2 + 11j - 6)\sigma\phi_x^3 u_j + \frac{3}{4}(j-3)\sigma\phi_x^2 v_j, \\ G_j = \frac{3}{4}(j^3 - 8j^2 + 19j - 12)\sigma\phi_x^4 u_j \\ \quad -\frac{1}{4}(j^3 - 9j^2 + 26j - 24)\sigma\phi_x^3 v_j. \end{array}$$

Setting

$$\det Q(j) = \tfrac{1}{16}(j^6 - 15j^5 + 82j^4 - 186j^3 + 97j^2 + 201j - 180)\sigma^2\phi_x^6,$$

we find that the resonances occur at $j = -1, 1, 3, 3, 4, 5$ and the resonance $j = -1$ usually corresponds to the arbitrariness of the singular manifold $\phi(x,t) = 0$.

To verify the compatibility conditions of Eq. (1), substituting

$$u \sim \phi^{-1}\sum_{j=0}^{5} u_j\phi^j,\ v \sim \phi^{-2}\sum_{j=0}^{5} v_j\phi^j \tag{6}$$

into Eq. (1), where the upper limit of the sum five means the largest resonance, we get that there are sufficient numbers of arbitrary functions at the non-negative resonances, i.e., u_1 or v_1,u_3, v_3, u_4 or v_4, and u_5 or v_5 are arbitrary, and the following conditions can be obtained

$$v_1 = -\frac{1}{2}\phi_{xx}, \tag{7}$$

$$u_2 = -\frac{12\sigma u_1^2\phi_x + 6\sigma u_1\phi_{xx} + \sigma\phi_{xxx} + 6\sigma\phi_x u_{1,x} - 4\phi_t}{6\sigma\phi_x^2}, \tag{8}$$

$$v_2 = \frac{12\sigma u_1^2\phi_x + 6\sigma u_1\phi_{xx} + \sigma\phi_{xxx} - 4\phi_t}{6\sigma\phi_x} \tag{9}$$

(u_4 and v_5 can be seen in Appendix A). In summary, from the above analysis we can conclude that Eq. (1) has the Painlevé property and hence is expected to be integrable.

At the same time, Bäcklund transformation is a powerful tool in the study on the solutions of the nonlinear evolution equations. The Painlevé truncation provides us a straightforward way to obtain auto-Bäcklund transformation.

To achieve auto-Bäcklund transformation, we must work with the general form $\phi(x,t) = 0$ of the noncharacteristic singularity manifold. With leading-order analysis, we obtain the truncated Painlevé expansion at the constant level term as

$$\begin{aligned} u &= u_0\phi^{-1} + u_1, \\ v &= v_0\phi^{-2} + v_1\phi^{-1} + v_2. \end{aligned} \tag{10}$$

Substituting (10) into Eq. (1) and making the coefficients of like powers of ϕ vanish with symbolic computation, we can derive

$$u_0 = \pm\frac{1}{2}\phi_x, v_0 = \frac{1}{2}\phi_x^2, v_1 = -\frac{1}{2}\phi_{xx}, \tag{11}$$

and u_1 and v_2 have to be a set of solutions of Eq. (1), and

$$\begin{aligned} &\tfrac{1}{2}\phi_{xt} - \tfrac{3}{4}\sigma u_{1,x}\phi_{xx} - \tfrac{3}{2}\sigma u_1^2\phi_{xx} - \tfrac{3}{4}\sigma u_1\phi_{xxx} \\ &+\tfrac{3}{4}\sigma v_2\phi_{xx} + \tfrac{3}{4}\sigma v_{2,x}\phi_x - 3\sigma u_1 u_{1,x}\phi_x \\ &-\tfrac{1}{8}\sigma\phi_{xxxx} = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &-\tfrac{1}{8}\phi_x(8\phi_t - 12\sigma u_1^2\phi_x - 6\sigma u_1\phi_{xx} + 6\sigma v_2\phi_x \\ &-\sigma\phi_{xxx}) = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &-\tfrac{1}{2}\phi_{xxt} + \tfrac{3}{4}\sigma u_{1,xxx}\phi_x + \tfrac{3}{2}\sigma u_{1,x}\phi_{xxx} \\ &+\tfrac{3}{2}\sigma u_{1,xx}\phi_{xx} + \tfrac{3}{4}\sigma u_1\phi_{xxxx} + 3\sigma u_1 u_{1,x}\phi_{xx} \\ &-3\sigma u_1 v_2\phi_{xx} - \tfrac{3}{4}\sigma v_{2,x}\phi_{xx} + \tfrac{3}{2}\sigma u_1^2\phi_{xxx} \\ &-3\sigma v_2 u_{1,x}\phi_x - \tfrac{3}{4}\sigma v_2\phi_{xxx} - 3\sigma u_1 v_{2,x}\phi_x \\ &+\tfrac{1}{8}\sigma\phi_{xxxxx} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\phi_{xt}\phi_x + \tfrac{1}{2}\phi_t\phi_{xx} + \tfrac{3}{4}\sigma v_2\phi_{xx}\phi_x - \tfrac{3}{2}\sigma u_1\phi_{xxx}\phi_x \\ &-\tfrac{3}{4}\sigma u_1\phi_{xx}^2 + 3\sigma u_1 v_2\phi_x^2 - \tfrac{9}{2}\sigma u_1^2\phi_{xx}\phi_x \\ &-\tfrac{3}{2}\sigma u_{1,xx}\phi_x^2 - 3\sigma u_1 u_{1,x}\phi_x^2 - 3\sigma u_{1,x}\phi_{xx}\phi_x \\ &-\tfrac{1}{4}\sigma\phi_{xxxx}\phi_x - \tfrac{1}{8}\sigma\phi_{xxx}\phi_{xx} = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &-\tfrac{1}{4}\phi_x^2(4\phi_t - 6\sigma u_{1,x}\phi_x - 12\sigma u_1^2\phi_x \\ &-6\sigma u_1\phi_{xx} - \sigma\phi_{xxx}) = 0. \end{aligned} \tag{16}$$

Therefore, we obtain an auto-Bäcklund transformation of Eq. (1) as follows

$$\begin{aligned} u &= \pm\tfrac{1}{2}\tfrac{\partial}{\partial x}\ln\phi + u_1, \\ v &= -\tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\ln\phi + v_2. \end{aligned} \tag{17}$$

We take the trivial vacuum solution $u_1 = 0, v_2 = 0$ as the seed solution, then constraint conditions (12)-(16) are simplified to

$$\phi_t - \frac{1}{4}\sigma\phi_{xxx} = 0, \tag{18}$$

and the auto-Bäcklund transformation (17) can be read as

$$\begin{aligned} u &= \pm\tfrac{1}{2}\tfrac{\partial}{\partial x}\ln\phi, \\ v &= -\tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\ln\phi. \end{aligned} \tag{19}$$

We are going to find the solutions of Eq. (18) in the form

$$\phi = 1 + p(\xi)e^{\eta} = 1 + p(Kx + Ct)e^{kx+ct}, \tag{20}$$

where K, k, C and c are all constants, whereas function $p(\xi)$ may be sine, cosine, hyperbolic sine, hyperbolic cosine and so on. Now, we consider some special situations

Case 1 $p(\xi) = 1$

In this case, the general solution of Eq. (18) reads

$$\phi_1 = 1 + e^{kx+\frac{\sigma}{4}k^3 t}. \tag{21}$$

Case 2 $p(\xi) = \sin(\xi)$ or $p(\xi) = \cos(\xi)$

After some calculation, we obtain the solutions of Eq. (18)

$$\phi_2 = 1 + \sin\left[Kx + \tfrac{\sigma}{4}K(3k^2 - K^2)t\right] e^{kx+\frac{\sigma}{4}k(k^2-3K^2)t}, \tag{22}$$

and

$$\phi_3 = 1 + \cos\left[Kx + \tfrac{\sigma}{4}K(3k^2 - K^2)t\right] e^{kx+\frac{\sigma}{4}k(k^2-3K^2)t}. \tag{23}$$

Case 3 $p(\xi) = \sinh(\xi)$ or $p(\xi) = \cosh(\xi)$

Similarly, we get the solutions of Eq. (18)

$$\phi_4 = 1 + \sinh\left[Kx + \tfrac{\sigma}{4}K(K^2 + 3k^2)t\right] e^{kx+\frac{\sigma}{4}k(k^2+3K^2)t}, \tag{24}$$

and

$$\phi_5 = 1 + \cosh\left[Kx + \tfrac{\sigma}{4}K(K^2 + 3k^2)t\right] e^{kx+\frac{\sigma}{4}k(k^2+3K^2)t}. \tag{25}$$

The corresponding analytic solutions of Eq. (1) read

$$\begin{aligned} u_i &= \pm\tfrac{1}{2}\tfrac{\partial}{\partial x}\ln\phi_i, \\ v_i &= -\tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\ln\phi_i. \end{aligned} \tag{26}$$

where $i = 1, 2, \cdots, 5$. Because Eq. (18) is linear, some kinds of ϕ solution listed above can be combined appropriately. Then we can get various kinds of analytic solutions. The results read

$$\phi_6 = 1 + \sum_{i=1}^{N} e^{k_i x+\frac{\sigma}{4}k_i^3 t}, \tag{27}$$

$$\phi_7 = 1 + \sum_{i=1}^{N}\sin\left[K_i x + \tfrac{\sigma}{4}K_i(3k_i^2 - K_i^2)t\right] e^{k_i x+\frac{\sigma}{4}k_i(k_i^2-3K_i^2)t}, \tag{28}$$

$$\phi_8 = 1 + \sum_{i=1}^{N} \cos\left[K_i x + \tfrac{\sigma}{4} K_i (3k_i^2 - K_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 - 3K_i^2) t}, \tag{29}$$

$$\phi_9 = 1 + \sum_{i=1}^{N} \sinh\left[K_i x + \tfrac{\sigma}{4} K_i (K_i^2 + 3k_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 + 3K_i^2) t}, \tag{30}$$

and

$$\phi_{10} = 1 + \sum_{i=1}^{N} \cosh\left[K_i x + \tfrac{\sigma}{4} K_i (K_i^2 + 3k_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 + 3K_i^2) t}. \tag{31}$$

where the parameters $k_i, K_i (i = 1, 2, \cdots, N, N > 1)$ are all arbitrary. The solution (26), while ϕ satisfies Eqs. (28)-(31) in turn, had not been given in Refs. [16-19].

To understand the analytic solutions well, we plot the solution (26), while ϕ satisfes Eq. (27) with some special parameters in Fig 1 and Fig 2.

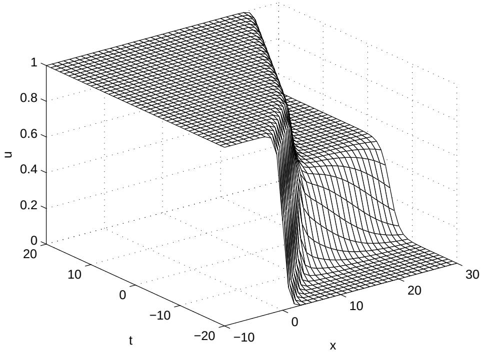

Fig. 1 Spatial structure of two-travelling wave solution (26), while ϕ satisfes Eq. (27) with $N = 2, k_1 = 1, k_2 = 2$ and $\sigma = -1$.

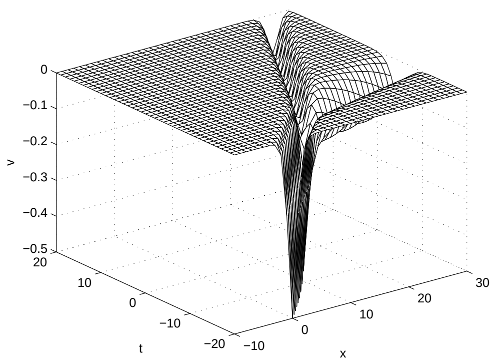

Fig. 2 Spatial structure of two-soliton (26), while ϕ satisfes Eq. (27) with $N = 2, k_1 = 1, k_2 = 2$ and $\sigma = -1$.

Moreover, different kinds of solutions ϕ given in Eq. (18) can be combined to form some new kinds of analytic solutions. For example

$$\begin{aligned}\phi_{11} &= 1 + e^{kx + \frac{\sigma}{4} k^3 t} \\ &+ \sin\left[Kx + \tfrac{\sigma}{4} K (3k^2 - K^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 - 3K^2) t} \\ &+ \sinh\left[Kx + \tfrac{\sigma}{4} K (K^2 + 3k^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 + 3K^2) t},\end{aligned} \tag{32}$$

$$\begin{aligned}\phi_{12} &= 1 + e^{kx + \frac{\sigma}{4} k^3 t} \\ &+ \cos\left[Kx + \tfrac{\sigma}{4} K (3k^2 - K^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 - 3K^2) t} \\ &+ \cosh\left[Kx + \tfrac{\sigma}{4} K (K^2 + 3k^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 + 3K^2) t},\end{aligned} \tag{33}$$

$$\begin{aligned}\phi_{13} &= 1 + e^{kx + \frac{\sigma}{4} k^3 t} \\ &+ \sin\left[Kx + \tfrac{\sigma}{4} K (3k^2 - K^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 - 3K^2) t} \\ &+ \cos\left[Kx + \tfrac{\sigma}{4} K (3k^2 - K^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 - 3K^2) t} \\ &+ \sinh\left[Kx + \tfrac{\sigma}{4} K (K^2 + 3k^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 + 3K^2) t} \\ &+ \cosh\left[Kx + \tfrac{\sigma}{4} K (K^2 + 3k^2) t\right] e^{kx + \frac{\sigma}{4} k (k^2 + 3K^2) t},\end{aligned} \tag{34}$$

and

$$\begin{aligned}\phi_{14} &= 1 + \sum_{i=1}^{N} e^{k_i x + \frac{\sigma}{4} k_i^3 t} \\ &+ \sum_{i=1}^{N} \sin\left[K_i x + \tfrac{\sigma}{4} K_i (3k_i^2 - K_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 - 3K_i^2) t} \\ &+ \sum_{i=1}^{N} \cos\left[K_i x + \tfrac{\sigma}{4} K_i (3k_i^2 - K_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 - 3K_i^2) t} \\ &+ \sum_{i=1}^{N} \sinh\left[K_i x + \tfrac{\sigma}{4} K_i (K_i^2 + 3k_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 + 3K_i^2) t} \\ &+ \sum_{i=1}^{N} \cosh\left[K_i x + \tfrac{\sigma}{4} K_i (K_i^2 + 3k_i^2) t\right] e^{k_i x + \frac{\sigma}{4} k_i (k_i^2 + 3K_i^2) t}.\end{aligned} \tag{35}$$

and so on, where the parameters k, K and $k_i, K_i (i = 1, 2, \cdots, N, N > 1)$ are all arbitrary.

III. Lax pairs and Darboux transformations

Based on the AKNS procedure [10,20], Lax pairs of Eq. (1) are obtained as below

$$\Phi_x = M\Phi, M = \begin{pmatrix} \lambda + u & u_x + v \\ 1 & -\lambda - u \end{pmatrix}, \tag{36}$$

$$\Phi_t = N\Phi, N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{37}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

$$A = \frac{1}{4}\sigma\Big[4\lambda^3 - 2(u_x + v)\lambda - v_x + 4uu_x - 2uv + 4u^3\Big],$$

$$\begin{aligned}B = \tfrac{1}{4}\sigma\Big[&4(u_x + v)\lambda^2 + 2(u_x + v - u^2)_x \lambda - 4uv\lambda \\ &+ (u_x + v)_{xx} - 4uu_{xx} + 4u^2 u_x - 4u_x^2 - 4uv_x \\ &-6u_x v + 4u^2 v - 2v^2\Big],\end{aligned}$$

$$C = \frac{1}{2}\sigma\Big(2\lambda^2 - 2u\lambda + 2u^2 - v\Big).$$

where λ is a spectral parameter independent of x and t. The compatibility condition $\Phi_{xt} = \Phi_{tx}$ yields a zero curvature equation

$$M_t - N_x + [M, N] = 0, \tag{38}$$

which leads to Eq. (1) by the direct computation.

Now we consider a Darboux transformation

$$\overline{\phi} = T\phi, \tag{39}$$

where T is defined by

$$T_x + TM = \overline{M}T. \tag{40}$$

A new spectral problem reads

$$\overline{\phi}_x = \overline{M\phi}, \tag{41}$$

where $\overline{M}$ has the same form as M, except replacing u and v with $\overline{u}$ and $\overline{v}$. We assume

$$T = a_0\begin{pmatrix} \lambda + a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \tag{42}$$

where $a_0, a_1, b_1, c_1,$ and d_1 are functions of x and t. Substituting (42) into (40), we compare the coefficients of λ, $j = 2, 1, 0$. The case of $j = 2$ is trivial. For the case of $j = 1$, we have

$$a_0 u - a_0\overline{u} + a_{0x} = 0, \tag{43}$$

$$b_1 = \frac{1}{2}(u_x + v), \tag{44}$$

$$c_1 = \frac{1}{2}, \tag{45}$$

For the case of $j = 0$, using (43)-(45) we can derive

$$a_{1x} = \frac{1}{2}(\overline{u}_x + \overline{v}) - \frac{1}{2}(u_x + v), \tag{46}$$

$$d_1(\overline{u}_x + \overline{v}) = (a_1 - u)(u_x + v) + \frac{1}{2}(u_x + v)_x, \tag{47}$$

$$d_1 = a_1 - \overline{u}, \tag{48}$$

$$d_{1x} = 2d_1(u - \overline{u}). \tag{49}$$

On the other hand, from relation $\mathrm{Tr}M = \mathrm{Tr}\overline{M} = 0$. The solutions $\phi, \overline{\phi}$ of (36) and (41) are two 2×2 matrices. Thus $\det\phi = \det\overline{\phi} =$ constant, it means that there is a constant $\lambda = \lambda_1$ and a solution $\phi = (\phi_1, \phi_2)^T$ of (36), which satisfy

$$\begin{aligned} &a_0(\lambda_1 + a_1)\phi_1 + a_0 b_1\phi_2 = 0,\\ &a_0 c_1\phi_1 + a_0 d_1\phi_2 = 0. \end{aligned}$$

Then, we have

$$\begin{aligned} a_1 &= -\tfrac{1}{2}(u_x + v)\tfrac{\phi_2}{\phi_1} - \lambda_1,\\ d_1 &= -\tfrac{1}{2}\tfrac{\phi_1}{\phi_2}. \end{aligned} \tag{50}$$

If u and v are given, (ϕ_1, ϕ_2) is a solution of the Eq. (36) with $\lambda = \lambda_1$, then seven unknown functions $a_0, a_1, b_1, c_1, d_1, \overline{u}$ and $\overline{v}$ can be defined by seven relations (43)-(45) and (48)-(50). The relations (46) and (47) can be proved to be satisfied automatically. Especially $\overline{u}$ and $\overline{v}$ can be expressed as follows:

$$\begin{aligned} \overline{u} &= -\tfrac{1}{2}(u_x + v)\tfrac{\phi_2}{\phi_1} + \tfrac{1}{2}\tfrac{\phi_1}{\phi_2} - \lambda_1,\\ \overline{v} &= -\tfrac{1}{2}(u_x + v)_x\tfrac{\phi_2}{\phi_1} - \tfrac{1}{2}(u_x + v)(\tfrac{\phi_2}{\phi_1})_x - \tfrac{1}{2}(\tfrac{\phi_1}{\phi_2})_x\\ &\quad + u_x + v. \end{aligned} \tag{51}$$

At the same time, substituting the transformation (42) into $\overline{\phi}_t = \overline{N\phi}$, where $\overline{N}$ has the same form as N in (37) except changing u, v into $\overline{u}, \overline{v}$, the compatibility condition $\overline{\phi}_{xt} = \overline{\phi}_{tx}$ holds, i.e., $\overline{M}_t - \overline{N}_x + [\overline{M}, \overline{N}] = 0$, so that $(\overline{u}, \overline{v})$ is a new solution of Eq (1).

When u and v are constants and $v \neq 0$, (u, v) is a solution of Eq (1). Let us take this solution as our 'seed'. The result (ϕ_1, ϕ_2) can be expressed as follows:

$$\begin{aligned} &\phi_1 = (c_1 + \lambda_1 + u)e^{\xi_1} + (-c_1 + \lambda_1 + u)e^{-\xi_1},\\ &\phi_2 = e^{\xi_1} + e^{-\xi_1}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} &\xi_1 = c_1[x + \sigma(\lambda_1^2 - u\lambda_1 + u^2 - \tfrac{v}{2})t],\\ &c_1 = \sqrt{(\lambda_1 + u)^2 + v},\\ &\tfrac{\phi_1}{\phi_2} = (\lambda_1 + u) + c_1\tanh(\xi_1). \end{aligned}$$

From (51), we get a new solution

$$\begin{aligned} \overline{u} &= \tfrac{1}{2}\tfrac{(\lambda_1 + u + c_1\tanh(\xi_1))^2 - v}{\lambda_1 + u + c_1\tanh(\xi_1)} - \lambda_1,\\ \overline{v} &= \tfrac{1}{2}\tfrac{c_1^2 v\,\mathrm{sech}^2(\xi_1)}{(\lambda_1 + u + c_1\tanh(\xi_1))^2} - \tfrac{1}{2}c_1^2\,\mathrm{sech}^2(\xi_1) + v. \end{aligned} \tag{53}$$

IV. Discussion

In this work, with symbolic computation, we have performed the Painlevé analysis for Eq. (1), and Lax pairs be obtained following the AKNS procedure. We have given the auto-Bäcklund transformation via the truncated Painlevé expansion and a basic Darboux transformation of spectral problem for Eq. (1). Some analytic solutions are given, including the travelling wave solutions, soliton solutions and periodic solutions and so on.

Appendix A

$$\begin{aligned} u_4 = &\tfrac{v_4}{2\phi_x} - \tfrac{1}{36\sigma^2\phi_x^6}\big(54\sigma^2 u_3\phi_x^4\phi_{xx} - 18\sigma^2 v_3\phi_x^3\phi_{xx}\\ &+216\sigma^2 u_1^3\phi_x^2\phi_{xx} + 180\sigma^2 u_1^2\phi_x\phi_{xx}^2 - 144\sigma^2 u_1 u_{1xx}\phi_x^3\\ &-84\sigma^2 u_1^2\phi_x^2\phi_{xxx} - 36\sigma^2 u_1\phi_x^2\phi_{xxxx} + 72\sigma u_1\phi_x^2\phi_{xt}\\ &-42\sigma^2 u_{1x}\phi_x^2\phi_{xxx} - 20\sigma\phi_x\phi_{xxx}\phi_t - 18\sigma^2 u_{1xx}\phi_x^2\phi_{xx}\\ &+6\sigma^2\phi_x\phi_{xx}\phi_{xxxx} - 24\sigma\phi_x\phi_{xx}\phi_{xt} + 36\sigma^2 u_{1x}\phi_x\phi_{xx}^2\\ &+72\sigma^2 u_1 u_3\phi_x^5 - 36\sigma^2 u_1 v_3\phi_x^4 - 432\sigma^2 u_1^2 u_{1x}\phi_x^3\\ &-96\sigma u_1^2\phi_x^2\phi_t + 24\sigma u_{1x}\phi_x^2\phi_t + 16\phi_x\phi_t^2 - 12\sigma^2 u_{1xxx}\phi_x^3\\ &+4\sigma^2\phi_x\phi_{xxx}^2 - 36\sigma^2 u_1\phi_{xx}^3 - 6\sigma^2\phi_{xx}^2\phi_{xxx} + 24\sigma\phi_{xx}^2\phi_t\\ &-3\sigma^2\phi_x^2\phi_{xxxxx} + 12\sigma\phi_x^2\phi_{xxt} - 24\sigma u_{1t}\phi_x^3 - 18\sigma^2 v_{3x}\phi_x^4\\ &+36\sigma^2 u_{3x}\phi_x^5 + 144\sigma^2 u_1^4\phi_x^3 + 90\sigma^2 u_1\phi_x\phi_{xx}\phi_{xxx}\\ &-108\sigma^2 u_{1x}^2\phi_x^3 - 108\sigma^2 u_1 u_{1x}\phi_x^2\phi_{xx} - 144\sigma u_1\phi_x\phi_{xx}\phi_t\big), \end{aligned}$$

and

$$\begin{aligned} v_5 = &4u_5\phi_x - \tfrac{1}{6\sigma\phi_x^2}\big(24u_1 u_2 u_{1x} + 24\sigma u_1 u_2^2\phi_x\\ &+12\sigma u_1 u_{3x}\phi_x + 24\sigma u_1^2 u_3\phi_x + 12\sigma u_2 u_{2x}\phi_x\\ &+12\sigma u_3 u_{1x}\phi_x - 12\sigma u_2 v_3\phi_x - 12\sigma u_3 v_2\phi_x - 12\sigma u_1 v_4\phi_x\\ &+9\sigma u_{3x}\phi_{xx} + 5\sigma u_3\phi_{xxx} - 3\sigma v_4\phi_{xx} + 6\sigma u_{3xx}\phi_x\\ &+6\sigma u_2^2\phi_{xx} - 4u_{2t} + 24\sigma u_4\phi_x\phi_{xx} + 12\sigma u_1 u_3\phi_{xx}\\ &+24\sigma u_1 u_4\phi_x^2 + \sigma u_{2xxx} + 24\sigma u_2 u_3\phi_x^2 - 8u_3\phi_t\\ &-6\sigma v_2 u_{2x} - 3\sigma v_{4x}\phi_x - 6\sigma u_1 v_{3x} - 6\sigma u_2 v_{2x}\\ &+18\sigma u_{4x}\phi_x^2 - 6\sigma v_3 u_{1x} + 12\sigma u_1^2 u_{2x}\big). \end{aligned}$$

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