The Research on the Calculation of Barrier Options under Stochastic Volatility Models Based on the Exact Simulation

Yuting Yang, Junmei Ma*, Yijuan Liang

Abstract— **This work researched the exact simulation problem of the two kinds of stochastic volatility models based on the Broadie and Kaya's work. Rejection sampling technique was deeply discussed in the exact simulation process based on the moment analysis. Then conditional Monte Carlo and antithetic variable techniques were used to reduce the variance of Monte Carlo simulation when calculating the price and Greeks of Barrier Options. The numerical results show that the combination of exact simulation and conditional Monte Carlo method can get unbiased estimation and smaller variance, compared with the crude Monte Carlo with Euler discretization. This algorithm in the paper can be used to solve the calculation of other more complicated products under the stochastic volatility models.**

Index Terms— **Accelerate, Barrier option, conditional Monte Carlo, exact simulation, Greeks, stochastic volatility**

I. Introduction

In 1973, Black and Scholes proposed the famous option pricing formula, then pricing problems of financial derivative instrument became the core research content in the field of financial mathematics. Black-Scholes (BS) model indicated the price of derivative securities as a function of underlying asset price. BS model is simple and easy to calculate, while the volatility of stock price is assumed to be constant which is not consistent with observed results of the real market. Therefore, many research scholars devoted themselves to the improvement of BS model, and stochastic volatility model was one of them, which set the volatility of stock price as another random process. In fact, the concept of stochastic volatility was firstly raised by Hull and White (1987) [6]. Then Scott (1987) [7], Stein-Stein (1987) [8] and Heston (1993) [2] conducted a further development for the relevant research works and came up with different stochastic volatility models.

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Option is a common derivative product in the financial market, and it is also a class of financial contract that grants investors the power of buying or selling a certain underlying asset with the pre-determined price at a time in the future. It possesses a strong flexibility that is capable of offering investors an effective risk management. The type of options can be divided in terms of different modes [9]. In particular, barrier option is a type of exotic option widely applied in the financial market, and its return is depended on whether the path of underlying asset price has reached a pre-determined level. Besides, it belongs to a type of conditional option, of which the effect is decided by whether the underlying asset price encounters barriers within its validity. Barrier option is divided into two categories: knock-in option and knock-out option. Knock-in option refers to that an option begins to function only if the underlying asset price reaches the pre-determined level (barrier price); while knock-out option refers to that an option expires worthless only if the underlying asset price reaches the barrier price. In addition, it can be also divided into up barrier option and down barrier option in accordance with the barrier price that underlying asset price rises to or declines to. Theoretical researches on pricing algorithm of barrier option are very important, and a great number of theories can be generalized for the pricing of credit derivatives product, e.g. CDS and etc. Giary Okten[10] researched the pricing of barrier option under the constant volatility by Monte Carlo simulation.

Along with the rapid development of financial derivative market, category and structure of financial products are becoming increasingly diversified and complicated. A majority of financial products' price cannot be solved under stochastic volatility models, and need to be solved by numerical methods. Under the two stochastic volatility models, this paper will focus on the study of accelerating simulation theory for option pricing and sensitivity estimate, based on the exact simulation sampling algorithm. Calculation for barrier option is taken as an example, by which the theory of algorithms mentioned in this paper can also be further applied in other more complicated calculating study of financial products.

II. Monte Carlo Simulation Method

Monte Carlo method is also referred to as a stochastic simulation approach, and it adopts random numbers to solve practical issues on the basis of probability theory. In practical applications, Monte Carlo method is to build a random model at first, to make variable of the model equal to the solution of the required problem. Next, it calculates statistical characteristics of analogue variable, and finally gives an approximate solution of the model, i.e. an approximate solution to the primary problem, whose simulation precision can be measured by standard error of the estimated value. The law of large numbers ensures the estimated value approaches the correct value along with the increase of simulation times, and at the same time central-limit theorem provides estimation for error limit of limited total times simulation.

To apply Monte Carlo method to solve problems, it is first to establish a probability space, and then determine a statistical magnitude $g(x)$ in this probability space. This statistical magnitude relies on random variable X, and the mathematical expectation of $g(X)$ is:

$$\mathrm{E}(g)=\int g(x)\,dF(x)$$

wherein, $F(x)$ is the distribution function of random variable X, $\mathrm{E}(g)$ is equal to the required value G. According to the distribution of X, it conducts sampling simulation to generate simple sample of the random variable X, by arithmetic mean value of the corresponding statistical magnitude $g(x_1),\dots,g(x_n)$:

$$G_n=\frac{1}{n}\sum_{i=1}^{n}g(x_i)$$

As approximate estimated value of G, G_n is convergent to the required value G in probability, when $\forall\varepsilon>0$, there is:

$$\lim_{n\to\infty}P(|G_n\text{-}G|<\varepsilon)=1$$

And approximate equation is as followed:

$$P\left(|G_n\text{-}G|<\frac{\sigma}{\sqrt{n}}\lambda\right)\cong\frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{+\lambda}e^{\frac{t^2}{2}}dt=1-\alpha$$

If $\sigma\geq 0$, then the error $\boldsymbol{\varepsilon}$ of Monte Carlo method is:

$$\varepsilon=\frac{\sigma}{\sqrt{n}}\lambda$$

It can be seen from the formulas above, both of σ and n decide the error ε of Monte Carlo method, and the error form $\sigma/\sqrt{n}$ is the main characteristic of the method. On the condition that σ is invariable, it requires to increase n by 100 times so as to enhance the accuracy of one-digit number, which greatly increases the computational workload. However, from another perspective, on the condition that σ decreases by 10 times, which can reduce workload by 100 times.

In order to improve the simulated efficiency of Monte Carlo method, many scholars have put forward various variance reduction techniques [1], such as control variant technique[15], importance sampling technique, conditional Monte Carlo[12], stratify sampling algorithm and etc. This paper does not intend to introduce them in detail here, but mainly introduces the concept of acceleration technique of conditional Monte Carlo.

The major principle of conditional Monte Carlo is conditional variance formula[11]. To calculate the expectation of random variable G, for an arbitrary random variable H, $\mathrm{E}[G|H]$ expresses the conditional expectation of G under the given condition H. This is a random variable about H, there is double expectation formula:

$$\mathrm{E}[G]=\mathrm{E}\big[\mathrm{E}[G|H]\big].$$

Conditional variance formula:

$$\mathrm{var}(G)=\mathrm{var}(\mathrm{E}[G|H])+\mathrm{E}[\mathrm{var}(G|H)],$$

Since $\mathrm{E}[\mathrm{var}(G|H)]\geq 0$, there is:

$$\mathrm{var}(G)\geq\mathrm{var}(\mathrm{E}[G|H])$$

It follows that the variance of the random variable $\mathrm{E}[G|H]$ is smaller than the original random variable G, so that it achieves the objective of variance reduction in case the exact value $\mathrm{E}[G|H]$ is known.

Conditional Monte Carlo can be used to improve the simulation efficiency under stochastic volatility model. To simulate price on the condition of volatility that can reduce problems' dimensions and thereby enhance accuracy. This method is very suitable for calculating path-dependent option with a solution under determined volatility.

III. Stochastic Volatility Model and Exact Simulation Algorithm

This section will discuss two types of Heston SV model and SVCJ model, as well as their exact simulation algorithms. SV and SVCJ models describe a dynamic variation process between underlying asset price and instant variance rate under continuous time case, whilst continuous-time dynamic variation cannot be directly simulated by computer simulation. Generally, it is required to firstly discretize continuous time into discrete time for processing. Euler discretization method is often used approximately to estimate the path of underlying processes at discrete time. However, Euler discretization method usually causes bias, and sometimes reaches to the degree that it cannot be ignored [1] [3]. Monte Carlo accelerating simulation technique is also investigated, it also requires to research exact simulation algorithm of the underlying processes.

Based on the Broadie and Kaya's exact simulation method [3-4], this section will further deeply discuss the applications of rejection sampling technique introduced in D'ippoliti [5], which proposed sampling from Gamma distribution instead. This paper researched the choice of parameters for the new rejection sampling density in detail from the moment analysis, in order to improve the effect of accurate sampling. Then exact simulation algorithms for the two types of stochastic volatility models-----SV and SVCJ, will be proposed, and combined with the application of several kinds of variance reduction in barrier option pricing will be investigated.

A. Accelerating Simulation Algorithm for Barrier Option Pricing under SV model

Introduction of SV Model

Heston Model is a mathematical model for describing the dynamic process of underlying asset under stochastic volatility provided by Steven Heston in 1993 [2]. In risk neutral measure, it assumes that underlying asset S_t and instant variance rate V_t satisfy the stochastic differential equation model as following:

$$dS_t = rS_t dt + \sqrt{V_t}S_t\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right] \quad (1)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^{(1)}, \quad (2)$$

Equation (1) describes the dynamic variation of underlying asset, wherein S_t means the underlying asset price at time t, r means risk neutral drift ratio and $\sqrt{V_t}$ means volatility. As for Equation (2), it describes the dynamic variation of instant variance rate V_t, wherein θ means long-term mean square deviation, κ means regression speed of variance i.e. the speed of V_t regressing to θ, and σ_v means the corresponding volatility of this variance process. $W_t^{(1)}$ and $W_t^{(2)}$ are two independent Brownian motion processes, ρ means the correlation coefficient between stock process and volatility process.

When the time $u < t$, underlying asset price S_u and instant variance rate V_u known, Equation (1) and (2) can be written as followed:

$$S_t = S_u \exp\left[r(t-u) - \frac{1}{2}\int_u^t V_s ds + \rho\int_u^t \sqrt{V_s}dW_s^{(1)} + \sqrt{1-\rho^2}\int_u^t \sqrt{V_s}dW_s^{(2)}\right] \quad (3)$$

$$V_t = V_u + \kappa\theta(t-u) - \kappa\int_u^t V_s ds + \sigma_v\int_u^t \sqrt{V_s}dW_s^{(1)} \quad (4)$$

SV Model and Exact Simulation Algorithm

Exact simulation algorithm suggested by Broadie and Kaya[3-4] is a type of unbiased method aiming at the simulation of sampling the original random process in terms of its accurate distribution, which is able to eliminate the bias caused by continuous-time discretization. Under SV model, the detailed procedures of exact simulation are in order:

Divide time $0 = t_0 < \ldots < t_m = T, 0 \le i < j \le m$, and consider time interval $[t_i, t_j]$, detailed descriptions for each procedure will be conducted:

(1)*Sample from the distribution of V_{t_j} according to V_{t_i}*

In 1984, Cox proposed that when V_{t_i} is known, $(t_i < t_j), V_{t_j}$ follows non-centrally chi-square distribution, and can be expressed as:

$$V_{t_j} = \frac{\sigma_v^2\left(1-e^{-\kappa(t_j-t_i)}\right)}{4\kappa}\chi_d'^2\left(\frac{4\kappa e^{-\kappa(t_j-t_i)}}{\sigma_v^2\left(1-e^{-\kappa(t_j-t_i)}\right)}V_u\right), t_j > t_i \quad (5)$$

$\chi_d'^2(\lambda)$ means random variable of non-centrally chi-square distribution for degree of freedom d and non-centrality parameter λ, wherein:

$$\lambda = \frac{4\kappa e^{-\kappa(t_j-t_i)}}{\sigma_v^2\left(1-e^{-\kappa(t_j-t_i)}\right)}V(t_i), d = \frac{4\theta\kappa}{\sigma_v^2}.$$

Then non-centrally chi-square distribution $\chi_d'^2(\lambda)$ can be expressed by centrally chi-square distribution χ_{d-1}^2 and normal distribution $Z \sim N(0,1)$, which is:

$$\chi_d'^2(\lambda) = \left(Z + \sqrt{\lambda}\right)^2 + \chi_{d-1}^2$$

In case degree of freedom $d > 0$, non-centrally chi-square distribution $\chi_d^2(\lambda)$ and χ_{d+2N}^2 centrally chi-square distribution have the same probability distribution, wherein is the random variable of Poisson distribution for expectation $\frac{1}{2}\lambda$. In this case, it can firstly simulate the random number of Poisson distribution, and then generate non-centrally chi-square distribution on the condition.

(2)*Sample from the distribution of $\int_{t_i}^{t_j} V_s ds$ according to V_{t_i} and V_{t_j}*

After sampling V_{t_j}, it can be obtained $\int_{t_i}^{t_j} V_s ds$ in accordance with V_{t_i} and V_{t_j}. Broadie and Kaya obtained the conditional characteristic function $\Phi(a)$ of $\int_{t_i}^{t_j} V_s ds$ by Laplace transform [3]:

$$\Phi(a) = E\left[\exp\left(ia\int_{t_i}^{t_j} V_s ds\right)\middle| V_{t_i}, V_{t_j}\right]$$

$$= \frac{\gamma(a)e^{-0.5(\gamma(a)-\kappa)(t_j-t_i)}\left(1-e^{-\kappa(t_j-t_i)}\right)}{\kappa\left(1-e^{-\gamma(a)(t_j-t_i)}\right)} \times$$

$$\exp\left\{\frac{V_{t_i}+V_{t_j}}{\sigma^2}\left[\frac{\kappa\left(1+e^{-\kappa(t_j-t_i)}\right)}{1-e^{-\kappa(t_j-t_i)}} - \frac{\gamma(a)\left(1+e^{-\gamma(a)(t_j-t_i)}\right)}{1-e^{-\gamma(a)(t_j-t_i)}}\right]\right\}$$

$$\times \frac{I_{0.5d-1}\left[\sqrt{V_{t_i}V_{t_j}}\frac{4\gamma(a)e^{-0.5\gamma(a)(t_j-t_i)}}{\sigma^2\left(1-e^{-\gamma(a)(t_j-t_i)}\right)}\right]}{I_{0.5d-1}\left[\sqrt{V_{t_i}V_{t_j}}\frac{4\kappa e^{-0.5\kappa(t_j-t_i)}}{\sigma^2\left(1-e^{-\kappa(t_j-t_i)}\right)}\right]}$$

wherein, $\gamma(a) = \sqrt{\kappa^2 - 2\sigma^2 ia}$ and $I_v(x)$ is the first-class modified Bessel function.

It is assumed that on the condition that V_{t_i} and V_{t_j} are known and random variable $V(t_i, t_j)$ and $\int_{t_i}^{t_j} V_s ds$ are identically distributed, then applying Fourier expansion it can solve cumulative distribution function $F(x)$ of $V(t_i, t_j)$ by conditional characteristic function $\Phi(a)$:

$$F(x)\equiv P(V(u,t)\le x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin ux}{u}\Phi(a)da=\frac{2}{\pi}\int_{0}^{+\infty}\frac{\sin ux}{u}\mathrm{Re}[\Phi(a)]da$$

The density function $f(x)$ is:

$$f(x)\equiv\frac{2}{\pi}\int_{0}^{+\infty}\cos(ux)\mathrm{Re}[\Phi(a)]da$$

It is worth noting that density function is an integral from 0 to infinite region, and it cannot be directly solved by Matlab. However, it can be transformed into integral on finite interval by applying Trapezoid formula. In order to effectively use the trapezoid formula, it needs to draw the image of integrand $f(x)\equiv\frac{2}{\pi}\cos(ux)\mathrm{Re}[\Phi(a)]$ and independent variable a by Matlab at first. See Figure 1.

As can be seen from Figure 1, the choice of the integral interval should be related with T. In addition, κ, V_0 and initial quantity also influence the selection of integral interval. Therefore, it is required to consider selecting an appropriate integral domain for different initial values.

For this, when using computer simulation, we can make use of trapezoid formula to calculate the integral value on a finite interval, thus approximately obtain a density function. However, we need to pay special attention to the time division.

Although the density function $f(x)$ of $\int_{t_i}^{t_j}V_s ds$ can be obtained by trapezoid formula, direct sampling is not easy due to the complicated form of $f(x)$. Then rejection sampling technique can be considered to sample points from the distribution of $\int_{t_i}^{t_j}V_s ds$, the procedures are as followed:

(1) Select an appropriate density function $g(x)$*, so that for all constants* $c>1$, $f(x)<cg(x)$ *is satisfied;*

(2)Sample x *from* $g(x)$ *and sample* u *from uniform distribution* $U\sim U(0,1)$*;*

(3)Examine whether $u<\frac{f(x)}{cg(x)}$ *is satisfied.*

If inequality is satisfied, accept x as the sampling from $\int_{t_i}^{t_j}V_s ds$, else refuse x and repeat procedures mentioned above.

In order to select an appropriate $g(x)$, it needs to draw the image of density function $\int_{t_i}^{t_j}V_s ds$ first. According to observation Fig.2, image of $\int_{t_i}^{t_j}V_s ds$'s density function $f(x)$ approaches to GAMMA distribution, and thereby it can be considered that selecting density function $g(x)$ that follows the distribution of GAMMA(α,β).

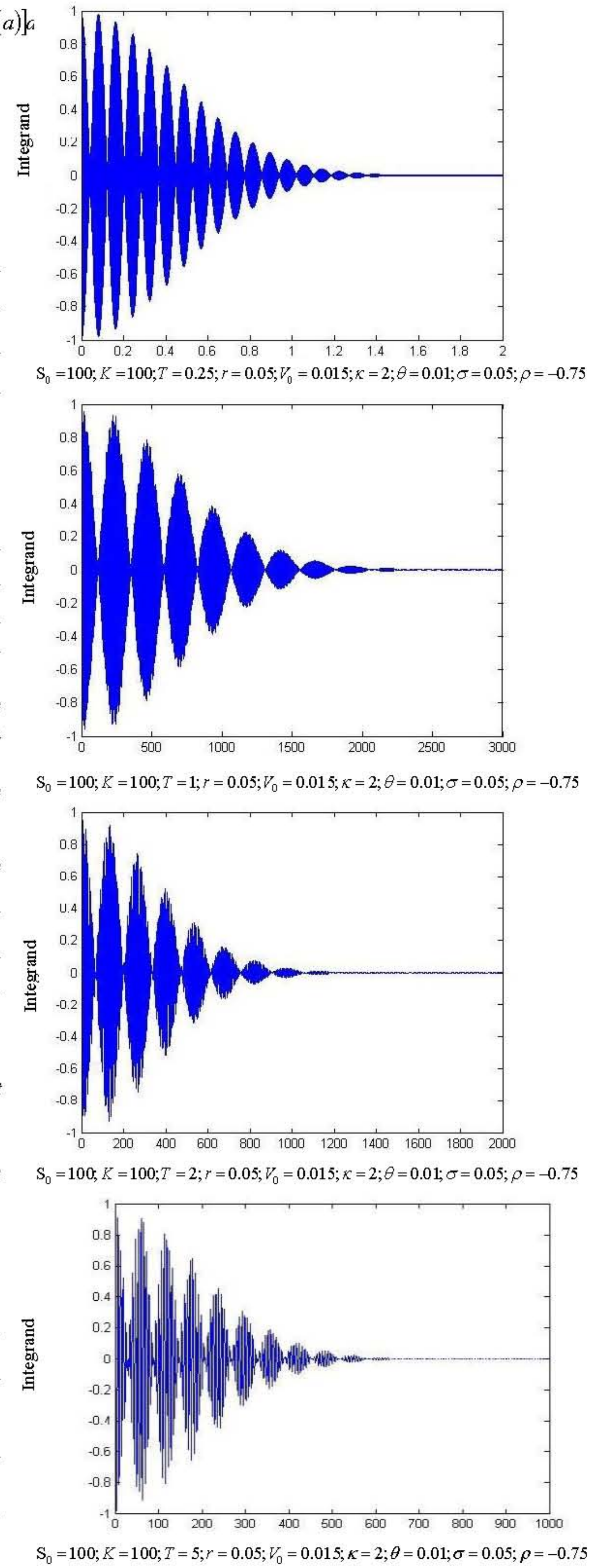

Fig 1. Relationship between integrand f(x) and variable a

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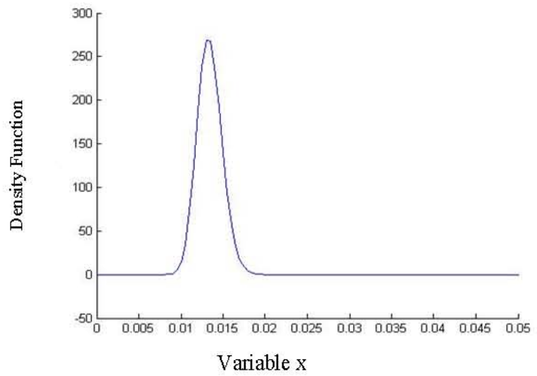


$S_0 = 100; K = 100; T = 1; r = 0.05; V_0 = 0.015; \kappa = 2; \theta = 0.01; \sigma = 0.05; \rho = -0.75$

Fig 2. Image of density function f(x)

$$g(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

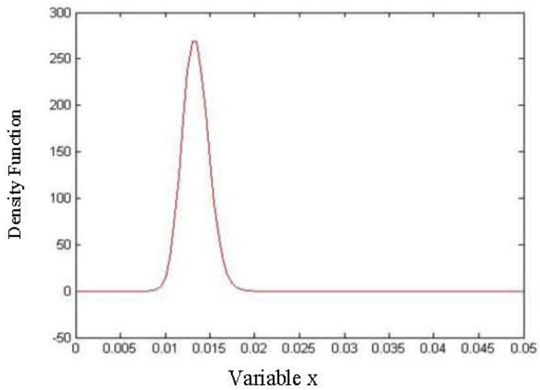
As the more similar the original density, the rejection sample density, the more efficient rejection sampling is. Their corresponding parameters α, β can be obtained by making their first two moments equal to make sure these two density functions more alike. The moments of $V(t_i, t_j)$ is attained from the characteristic function $\Phi(a)$ of $\int_{t_i}^{t_j} V_s ds$ as followed:

$$E[X] = \frac{\Phi'(0)}{i}, E[X^2] = -\Phi''(0)$$

So,

$$\alpha = -\frac{[\Phi'(0)]^2}{[\Phi'(0)]^2 - \Phi''(0)}, \beta = \frac{\Phi'(0)}{[[\Phi'(0)]^2 - \Phi''(0)] \cdot i}$$

To draw the image of $g(x)$:



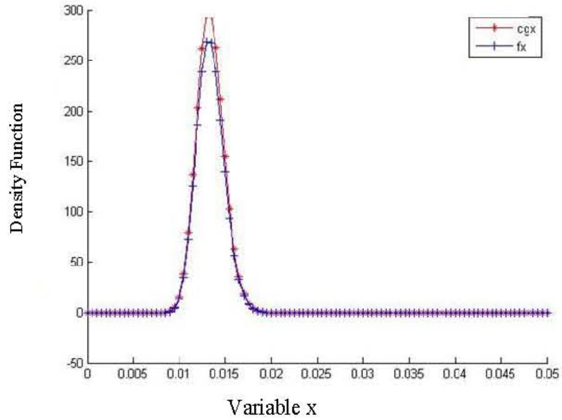
$S_0 = 100; K = 100; T = 1; r = 0.05; V_0 = 0.015; \kappa = 2; \theta = 0.01; \sigma = 0.05; \rho = -0.75$

Fig 3. Image of rejection sampling density function g(x)

When selecting a small $c = 1.1$, the images of $f(x)$ *and* $cg(x)$ are as follows in Fig.4. So $c = 1.1$ satisfies the condition of rejection sampling method, which can be used to sample much effectively the corresponding sampling points of $\int_{t_i}^{t_j} V_s ds$. A smaller c a better sampling efficiency.

(3) Calculate the value of $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$ ***according to*** V_{t_i}, V_{t_j} ***and*** $\int_{t_i}^{t_j} V_s ds$

When V_{t_i}, V_{t_j} and $\int_{t_i}^{t_j} V_s ds$ are obtained, we can sample



$S_0 = 100; K = 100; T = 1; r = 0.05; V_0 = 0.015; \kappa = 2; \theta = 0.01; \sigma = 0.05; \rho = -0.75, c = 1.1$

Fig 4. Image of rejection sampling density g(x) and original density f(x)

$\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$ from the following equation:

$$\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)} = \frac{1}{\sigma_v}\left(V_{t_j} - V_{t_i} - \kappa\theta(t_j - t_i) + \kappa\int_{t_i}^{t_j} V_s ds\right)$$

Since random process V_t and Brownian movement $W_t^{(2)}$ are independent, when the path generated by V_{t_j} is known, which follows the normal distribution of $N\left(0, \int_{t_i}^{t_j} V_s ds\right)$.

(4) ***Sample*** S_{t_i} ***based on the condition of*** $\int_{t_i}^{t_j} V_s ds$ ***and*** $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$

From the procedures above, it can be obtained $\int_{t_i}^{t_j} V_s ds$ *and* $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$, then taking these two results as conditions, we can obtain S_{t_i} that follows logarithmic normal distribution.

Define the variance of mean between t_i, t_j as:

$$\overline{\sigma}_j^2 = \frac{(1-\rho^2)\int_{t_i}^{t_j} V_s ds}{t_j - t_i};$$

Define the auxiliary variable

$$\xi_j = \exp\left(-\frac{\rho^2}{2}\int_{t_i}^{t_j} V_s ds + \rho\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}\right);$$

If variance path and the underlying asset price S_{t_i} at t_i are known, it can be obtained the expression of S_{t_j} as follows:

$$S_{t_j} = S_{t_i} \xi_j \exp\left[\left(r - \frac{\overline{\sigma}_J^2}{2}\right)(t_j - t_i) + \overline{\sigma}_J \sqrt{t_j - t_i} Z\right]$$

Thus, paths of underlying asset price and variance can be simulated under SV model.

Conditional Monte Carlo Algorithm for Barrier Option Pricing under SV Model

Considering a down-and-out call option, whose execution date is T, exercise price is K, initial underlying asset price is S_0, volatility is constant σ, and barrier price is H, its analytical solution can be written as [12]:

$$BS(S_0, \sigma) - \left(\frac{S_0}{H}\right)^{1-\frac{2}{\sigma^2}r} BS\left(\frac{H^2}{S_0}, \sigma\right);$$

SV model refers to that both the asset price and the variance process meet the following stochastic model:

$$dS_t = rS_t dt + \sqrt{V_t} S_t \left[\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right],$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(1)},$$

wherein $\int_0^T V_s ds$ and $\int_0^T \sqrt{V_s} dW_s^{(1)}$ can be sampled in the first three steps of the exact simulation method under SV model. Willard [11] put forward that when taking them as conditions, S_T accords with logarithm normal distribution:

Define the variance of mean between 0 and T as:

$$\overline{\sigma} = \sqrt{\frac{(1-\rho^2)\int_0^T V_s ds}{T}};$$

Define the auxiliary variable:

$$\xi = \exp\left(-\frac{\rho^2}{2}\int_0^T V_s ds + \rho \int_0^T \sqrt{V_s} dW_s^{(1)}\right);$$

Here, S_T satisfies:

$$S_T = S_0 \xi \exp\left[\left(r - \frac{\overline{\sigma}^2}{2}\right)T + \overline{\sigma}\sqrt{T} Z\right].$$

Then, it can be obtained that the down-and-out call option price under SV model by conditional Monte Carlo as:

$$\begin{aligned} C &= E\left[e^{-rT}(S(T)-K)^+ 1\{\min(S_t) > H\}\right] \\ &= E\left[E\left[e^{-rT}(S(T)-K)^+ 1\{\min(S_t) > H\} \middle| \int_0^T V_s ds, \int_0^T \sqrt{V_s} dW_s^{(1)}\right]\right] \\ &= E\left[S_0\xi \cdot N(d_1) - Ke^{-rT} \cdot N(d_2) - \left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r}\left(\frac{H^2}{S_0\xi} \cdot N(d_3) - Ke^{-rT} \cdot N(d_4)\right); \overline{\sigma}\right] \\ &= E\left[BS(S_0\xi, \overline{\sigma}) - \left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r} BS\left(\frac{H^2}{S_0\xi}, \overline{\sigma}\right); \overline{\sigma}\right] \end{aligned}$$

Where

$$d_{1,2} = \frac{\ln\frac{S_0\xi}{K} + \left(r \pm \frac{1}{2}\overline{\sigma}^2\right)T}{\overline{\sigma}\sqrt{T}};$$

$$d_{3,4} = \frac{\ln\frac{H^2}{S_0\xi K} + \left(r \pm \frac{1}{2}\overline{\sigma}^2\right)T}{\overline{\sigma}\sqrt{T}};$$

B. Accelerating Simulation Algorithm for Barrier Option Pricing under SVCJ model

Exact Simulation Algorithm of SVCJ model

SVCJ model adds the jump process of variance on the basis of Heston SV model. It assumes underlying asset S_t and instantaneous variance V_t satisfy the following dynamic stochastic model:

$$dS_t = (r - \lambda\overline{\mu})S_t dt + \sqrt{V_t} S_t \left[\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right] + S_t(J^s - 1)dN_t, \quad (6)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(1)} + J^v dN_t, \quad (7)$$

Equation (6) manifests underlying asset's dynamic change, wherein S_t indicates the underlying asset price at the time t, r indicates risk-neutral drift rate, $\sqrt{V_t}$ indicates volatility; equation (7) manifests instantaneous variance V_t's dynamic change, wherein θ indicates long-term mean-square deviation, κ indicates the regression speed of variance, namely the speed at which V_t regresses to θ, and σ_v is the volatility corresponding to this variance process; $W_t^{(1)}$ and $W_t^{(2)}$ are two independent Brownian motion processes, ρ indicates the correlation coefficient between stock process and volatility process. N_t is a Poisson process of parameter λ_J, and it is independent from the Poisson process; J^s represents underlying asset price's relative jump range, J^v represents variance's jump range. In particular, underlying asset price is correlative to variance's jump range, which means that they have a simultaneous jump; if they have a jump at time t, then $S_{t+} = S_{t-}J^s$, $V_{t+} = V_{t-} + J^v$, and there is a correlation coefficient ρ_J between them. Jump range J^v follows the exponential distribution of expectation μ_v. When J^v is given, J^s accords with lognormal distribution, $\log J^s \sim N(\mu_s + \rho_J J^v, \sigma_s^2)$, here parameters μ_s and $\overline{\mu}$ are correlated and satisfy the expression

$$\mu_s = \log[(1+\overline{\mu})(1-\rho_J \mu_v)] - \frac{1}{2}\sigma_s^2$$

Likewise, in order to sample the final underlying asset price, we can first sample the change process of variance and subsequently simulate the underlying asset price. Specific steps are as follows:

Divide time as $0 = t_0 < ... < t_m = T, 0 \le i < j \le m$, and consider time interval $[t_i, t_j]$:

(1)Simulate variance jump numbers n_j interval $[t_i, t_j]$, and update time as $t_i < t_{jump\_1} < ... < t_{jump\_n_j} < t_j$;

(2)Generate sample $V_{t_{jump\_1}}, ..., V_{t_{jump\_n_j}}, V_{t_j}$ from V_{t_j} according to V_{t_i};

(3)Generate $\int_{t_i}^{t_{jump\_1}} V_s ds, \int_{t_{jump\_1}}^{t_{jump\_2}} V_s ds, ..., \int_{t_{jump\_n_j}}^{t_j} V_s ds$ according to V_{t_i} and $V_{t_{jump\_1}}, ..., V_{t_{jump\_n_j}}, V_{t_j}$;

(4)Simulate each jump size $J^v_{jump\_1}, ..., J^v_{jump\_n_j}$, and update $\widetilde{V}_{t_{jump\_i}} = V_{t_{jump\_i}} + J^v_{t_{jump\_i}}, 1 \le i \le n_j$;

(5) Compute the value of $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$ according to V_{t_i}, V_{t_j} and $\int_{t_i}^{t_j} V_s ds$;

(6)Determine S_{t_i} on conditions of $\int_{t_i}^{t_j} V_s ds$, $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$, jump numbers n_j and jump size $J^v_{jump\_1}, ..., J^v_{jump\_n_j}$;

The first five-step stimulations are basically identical to SV model, the sixth step will be explained in detail.

On the premise that $\int_{t_i}^{t_j} V_s ds$, $\int_{t_i}^{t_j} \sqrt{V_s} dW_s^{(1)}$, jump numbers n_j and jump size $J^v_{jump\_1}, ..., J^v_{jump\_n_j}$ during temporal interval $[t_i, t_j]$ are given, we can sample S_{t_i}, which conforms to lognormal distribution.

Define the variance of mean between t_i, t_j as:

$$\overline{\sigma}_J^2 = \frac{n_j \sigma_s^2 + (1-\rho^2)\int_{t_i}^{t_j} V_s ds}{t_j - t_i};$$

Define the auxiliary variable:

$$\xi_j = \exp\left(\sum_{k=1}^{n_j}\left(\mu_s + J_k^v \rho_J + \frac{\sigma_s^2}{2}\right) - \lambda\overline{\mu}(t_j - t_i) - \frac{\rho^2}{2}\int_{t_i}^{t_j} V_s ds + \rho\int_{t_i}^{t_j}\sqrt{V_s} dW_s^{(1)}\right).$$

On the condition that variance path and the underlying asset price S_{t_i} at time t_i are known, it can be concluded that the expression of the underlying asset price S_{t_j} at t_j as:

$$S_{t_j} = S_{t_i}\xi_j \exp\left[\left(r - \frac{\overline{\sigma}_J^2}{2}\right)(t_j - t_i) + \overline{\sigma}_J\sqrt{t_j - t_i}Z\right].$$

Thereby it can simulate the underlying asset price path of SVCJ model.

Conditional Monte Carlo Algorithm for Barrier Option Pricing under SVCJ Model

In order to compute a down-and-out call option price with the acceleration technique of conditional Monte Carlo under SVCJ model, it is required to determine the down-and-out call option price under the constant-volatility and jump-diffusion model. We adopt the probabilistic method to deduce the option price's analytical solution, which is shown as Theorem 1. The detailed proof can be seen in the appendix.

Theorem 1 When underlying asset price satisfies the following stochastic process under risk-neutral measure Q :

$$dS_t = (r - \lambda\overline{\mu})S_t dt + \sigma S_t W_t + S_t(J^s - 1)dN_t, \quad (8)$$

wherein N_t is a Poisson process with parameter λ_J, and it is independent from the Brownian Motion W_t; J^s represents underlying asset price's relative jump range. In particular, if it has a jump at time t, then $S_{t+} = S_{t-}J^s$. J^s accords with lognormal distribution, namely $\log J^s \sim N(\mu_s + \rho_J J^v, \sigma_s^2)$; therein, parameters μ_s is related with $\overline{\mu}$, and satisfy the expression: $\mu_s = \log(1+\overline{\mu}) - \frac{1}{2}\sigma_s^2$.

The analytical solution of the down-and-out call option price under the model (8) can be obtained as:

$$C = \sum_{n=0}^{\infty}\frac{e^{-\lambda' T}(\lambda' T)^n}{n!}\cdot\left(\left[S_0 N(d_{n_1}) - Ke^{-r_x T}\cdot N(d_{n_3})\right] - \left[S_0\cdot e^{b+\frac{2br_x}{\sigma_x^2}} N(d_{n_2}) - Ke^{-r_x T}\cdot e^{\frac{2br_x}{\sigma_x^2}-b}\cdot N(d_{n_4})\right]\right)$$
$$= \sum_{n=0}^{\infty}\frac{e^{-\lambda' T}(\lambda' T)^n}{n!}\cdot\left(BS(S_0, \sigma_n) - \left(\frac{S_0}{H}\right)^{1-\frac{2}{\sigma_x^2}r_x} BS\left(\frac{H^2}{S_0}, \sigma_n\right)\right).$$

Therein:

$$d_{n_1, n_3} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_n \pm \frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$d_{n_2, n_4} = \frac{\ln\left(\frac{H^2}{S_0 K}\right) + \left(r_n \pm \frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$r_n = r - \lambda\overline{\mu} + \frac{n}{T}\left(\mu_s + \frac{1}{2}\sigma^2\right); \sigma_n^2 = \sigma^2 + \frac{n}{T}\sigma_s; \lambda' = \lambda(1+\overline{\mu})$$

As known that $\int_0^T V_s ds$ and $\int_0^T \sqrt{V_s} dW_s^{(1)}$ can be sampled through the exact simulation method of SVCJ model. Similar to the deduction of SV, it can be obtained that the barrier option price under SVCJ model by conditional Monte

$$C = E\left[e^{-rT}(S(T)-K)^{+}1\{\min(S_i)>H\}\right]$$
$$= E\left[E\left[e^{-rT}(S(T)-K)^{+}1\{\min(S_i)>H\}\Big|\int_0^T V_s ds, \int_0^T \sqrt{V_s}dW_s^{(1)}\right]\right]$$
$$= E\left[\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}(\lambda' T)^n}{n!}\cdot\left(\begin{array}{l}\left[S_0\xi\cdot N(d_{n_1}) - Ke^{-r_nT}\cdot N(d_{n_3})\right] \\ -\left[S_0\xi\cdot e^{b+\frac{2br_n}{\sigma_n^2}}N(d_{n_2}) - Ke^{-r_nT}\cdot e^{\frac{2br_n}{\sigma_n^2}-b}\cdot N(d_{n_4})\right]\end{array}\right);\overline{\sigma}\right]$$
$$= E\left[\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}(\lambda' T)^n}{n!}\cdot\left(BS(S_0\xi,\sigma_n,r_n) - \left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\sigma_n^2}r_n}BS\left(\frac{H^2}{S_0\xi},\sigma_n,r_n\right)\right);\overline{\sigma}\right]$$

Therein:

$$d'_{n_1,n_3} = \frac{\ln\left(\frac{S_0\xi}{K}\right)+\left(r_n \pm \frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$d'_{n_2,n_4} = \frac{\ln\left(\frac{H^2}{S_0\xi\cdot K}\right)+\left(r_n \pm \frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$\lambda' = \lambda(1+\overline{\mu});$$

$$\sigma_n^2 = \overline{\sigma}^2 + \frac{n\sigma_S^2}{T};$$

$$r_n = r - \lambda\overline{\mu} + \frac{n}{T}\left(\mu_s + J_k^v\rho_J + \frac{\sigma_s^2}{2}\right);$$

Here, define the average volatility between 0 and T as:

$$\overline{\sigma} = \sqrt{\frac{(1-\rho^2)\int_0^T V_s ds}{T}}.$$

And define the auxiliary variable:

$$\xi = \exp\left(-\frac{\rho^2}{2}\int_0^T V_s ds + \rho\int_0^T \sqrt{V_s}dW_s^{(1)}\right).$$

IV. Numerical Simulation Results of Option Pricing

In this section, it will respectively examine the acceleration effect that crude Monte Carlo method, antithetic variable method and conditional Monte Carlo method exert on the down-and-out call option price estimation, based on exact simulation algorithm. Meanwhile, initial parameters involved in SV and SVCJ models are $K;T;r;V_0;\kappa;\sigma;\theta;\rho;\sigma_s;\overline{\mu};\lambda_J;\mu_v;\rho_J$. In view of these different parameters will influence the price and the error in Monte Carlo method, this paper will analyze these influences.

It can be observed that along with the increase of simulation path number, price estimations of Monte Carlo, antithetic variable and conditional Monte Carlo all tend to converge. In comparison, variances of conditional Monte Carlo are the smallest under SV model, but variances of antithetic variable

are the smallest under SVCJ model in contrast with other two methods. This reflects the jumps will affect conditional Monte Carlo method.

TABLE 1.
INFLUENCES OF DIFFERENT SIMULATION PATH NUMBER M ON DOWN-AND-OUT CALL OPTION PRICE UNDER SV MODEL

Simulation method	Monte Carlo		Antithetic variable		Conditional Monte Carlo	
Path number	Price	Standard deviation	Price	Standard deviation	Price	Standard deviation
5000	7.2621	0.1345	7.1139	0.0691	7.2235	0.0483
10000	7.1710	0.0939	7.0729	0.0480	7.2160	0.0340
20000	7.1449	0.0670	7.1358	0.0344	7.2204	0.0241
30000	7.0805	0.0543	7.1018	0.0282	7.2092	0.0196
40000	7.1344	0.0476	7.1217	0.0246	7.2032	0.0170
50000	7.1299	0.0425	7.0903	0.0217	7.1883	0.0152

$S_0=100; K=100; T=1; r=0.05; V_0=0.015; \sigma=0.15; \theta=0.02; \kappa=4; \rho=-0.25; n=50$

TABLE 2.
INFLUENCES OF DIFFERENT SIMULATION PATH NUMBER M ON DOWN-AND-OUT CALL OPTION PRICE UNDER SVCJ MODEL

Simulation method	Monte Carlo		Antithetic variable		Conditional Monte Carlo	
Path number	Price	Standard deviation	Price	Standard deviation	Price	Standard deviation
5000	8.0821	0.1510	8.1990	0.0980	8.1376	0.1494
10000	8.2071	0.1079	8.1713	0.0700	8.1104	0.1053
20000	8.1429	0.0757	8.1193	0.0493	8.1233	0.0745
30000	8.0976	0.0617	8.0781	0.0403	8.1198	0.0608
40000	8.1184	0.0534	8.1086	0.0349	8.1342	0.0527
50000	8.0649	0.0476	8.0882	0.0313	8.1401	0.0471

V. Sensitivity Analysis of Barrier Option Greeks Solution

Option sensitivity refers to sensitive degree that option price shows to the variation of pricing parameters. Sensitivity analysis can help investors to select appropriate option portfolios and reduce risks. Approaches for solving Greeks with Monte Carlo method can be summed up as three types: finite-difference approximation, Pathwise differential estimation (PW: Pathwise Method) and likelihood ratio estimation (LR: Likelihood Ration Method), among which finite-difference approximation is easy to operate, but it may bring errors to estimated values [1]. Therefore, this paper will apply PW method and LR method to examine influences that the variation of parameters makes on down-and-out call option price. In detail, PW is based on the differential for revenue function; LR is based on the differential for density function of underlying asset price; both of them are able to solve unbiased estimators of Greeks; LR has a wider applicability, while PW can produce more precise results. Next, it will discuss formulas for Delta, Gamma and Rho under LR and PW methods.

A. LR Method

Consider a down-and-out call option, whose exercise price is K, execution time is T, and barrier price is H, its price should be $e^{-rT}(S_T-K)^{+}1\{\min(S_i)>H\}$, wherein S_T indicates the underlying asset price at time T.

Influenced by $1\{\min(S_i)>H\}$, revenue function of barrier option is discontinuous. Thus, PW method is inapplicable to solving the sensitivity index of barrier option, whilst LR method is still applicable.

Barrier option is path-dependent option, which implies that it needs to examine the density function of underlying asset prices at multiple moments, namely $g(x_1,\ldots,x_n)$, which can be decomposed to:

$$g(x_1,\ldots,x_n)=g_1(x_1|S_0)g_2(x_2|x_1)\ldots g_n(x_n|x_{n-1}),$$

wherein:

$$g_i(x)=\frac{1}{x_i\bar{\sigma}_i\sqrt{t_i-t_{i-1}}}\phi(d_i(x_i|x_{i-1}))$$

$$d_i(x_i|x_{i-1})=\frac{\ln(x_i/(x_{i-1}\xi_i))-\left(r-\frac{1}{2}\bar{\sigma}_i^2\right)(t_i-t_{i-1})}{\bar{\sigma}_i\sqrt{t_i-t_{i-1}}}$$

Delta is the derivative of option price as to underlying asset's initial price S_0, indicating the varying relation between option price and underlying asset's initial price; Gamma is the second derivative of option price as to underlying asset's initial price S_0, indicating the varying relation between Delta and underlying asset's initial price; Rho is the derivative of option price as to risk-free interest rate r, indicating the varying relation between option price and risk-free interest rate; under diverse parameters, differential of density function can be simplified as:

$$\frac{\partial g(x)}{\partial S_0}=\frac{\partial\left[g_1(x_1|S_0)g_2(x_2|x_1)\ldots g_n(x_n|x_{n-1})\right]}{\partial S_0}=\frac{\partial g_1(x_1|S_0)}{\partial S_0}=g_1(x_1|S_0)\frac{d_1(x_1|S_0)}{S_0\bar{\sigma}_1\sqrt{t_1-t_0}}.$$

$$\frac{\partial^2 g(x)}{\partial S_0^2}=\frac{\partial^2\left[g_1(x_1|S_0)g_2(x_2|x_1)\ldots g_n(x_n|x_{n-1})\right]}{\partial S_0^2}=\frac{\partial^2 g_1(x_1|S_0)}{\partial S_0^2}=g_1(x_1|S_0)\frac{d_1^2-d_1\bar{\sigma}_1\sqrt{t_1-t_0}-1}{S_0^2\bar{\sigma}_1^2(t_1-t_0)}.$$

$$\frac{\partial g(x)}{\partial r}=\frac{\partial\left[g_1(x_1|S_0)g_2(x_2|x_1)\ldots g_n(x_n|x_{n-1})\right]}{\partial r}$$

$$=\frac{\partial\left[e^{\log\left[g_1(x_1|S_0)g_2(x_2|x_1)\ldots g_n(x_n|x_{n-1})\right]}\right]}{\partial r}=g(x)\sum_{i=1}^{n}\frac{\partial\log\left[g_i(x_i|x_{i-1})\right]}{\partial r}.$$

Wherein:

$$\frac{\partial g_i(x_i|x_{i-1})}{\partial r}=-g_i(x_i|x_{i-1})\frac{d_i(x_i|x_{i-1})(t_i-t_{i-1})}{\bar{\sigma}_i\sqrt{t_i-t_{i-1}}}$$

Consequently, LR estimated value of barrier option's sensitive coefficient is:

Delta:

$$e^{-rT}(S_T-K)^{+}1\{\min(S_i)>H\}\times\left(\frac{d_1}{S_0\bar{\sigma}_1\sqrt{\Delta t_1}}\right)$$

Gamma:

$$e^{-rT}(S_T-K)^{+}1\{\min(S_i)>H\}\times\left(\frac{d_1^2-d_1\bar{\sigma}_1\sqrt{\Delta t_1}-1}{S_0^2\bar{\sigma}_1^2\Delta t_1}\right)$$

Rho:

$$e^{-rT}(S_T-K)^{+}1\{\min(S_i)>H\}\times\left(-T+\sum_{i=1}^{n}\frac{d_i\sqrt{\Delta t_i}}{\bar{\sigma}_i}\right)$$

B. Conditional Monte Carlo Greeks

Above estimated values can be directly used in Monte Carlo and antithetic variable methods to solve Greeks values, and it can obtain the analytical solution of down-and-out call option in conditional Monte Carlo under SV model as:

$$C=E\left[BS(S_0\xi,\bar{\sigma})-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\bar{\sigma}^2}r}BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right);\bar{\sigma}\right]$$

$$=E\left[S_0\xi\cdot N(d_1)-Ke^{-rT}\cdot N(d_2)-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\bar{\sigma}^2}r}\left(\frac{H^2}{S_0\xi}\cdot N(d_3)-Ke^{-rT}\cdot N(d_4)\right);\bar{\sigma}\right]$$

Wherein:

$$d_{1,2}=\frac{\ln\frac{S_0\xi}{K}+\left(r\pm\frac{1}{2}\bar{\sigma}^2\right)T}{\bar{\sigma}\sqrt{T}};$$

$$d_{3,4}=\frac{\ln\frac{H^2}{S_0\xi\cdot K}+\left(r\pm\frac{1}{2}\bar{\sigma}^2\right)T}{\bar{\sigma}\sqrt{T}};$$

$$\bar{\sigma}=\sqrt{\frac{(1-\rho^2)\int_0^T V_s ds}{T}};\ \xi=\exp\left(-\frac{\rho^2}{2}\int_0^T V_s ds+\rho\int_0^T\sqrt{V_s}dW_s^{(1)}\right)\cdot$$

Its sensitive parameter can be gained straightly by solving differential.

Delta:

$$\frac{\partial C}{\partial S_0}=E\left[\frac{\partial BS(S_0\xi,\bar{\sigma})}{\partial S_0}-\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{\xi}{H}BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\bar{\sigma}^2}r}\frac{\partial BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)}{\partial S_0};\bar{\sigma}\right]$$

$$=E\left[\xi\cdot N(d_1)-\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{\xi}{H}BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)+\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0}\cdot N(d_3);\bar{\sigma}\right]$$

$$=E\left[\xi\cdot N(d_1)-\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{\xi}{H}\left(\frac{H^2}{S_0\xi}\cdot N(d_3)-e^{-rT}K\cdot N(d_4)\right)+\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0}\cdot N(d_3);\bar{\sigma}\right]$$

$$\Rightarrow\frac{\partial C}{\partial S_0}=E\left[\xi\cdot N(d_1)+\frac{2}{\bar{\sigma}^2}r\cdot\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0}\cdot N(d_3)+\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{\xi}{H}e^{-rT}K\cdot N(d_4);\bar{\sigma}\right]$$

Gamma:

$$\frac{\partial^2 C}{\partial S_0^2}=E\left[\begin{array}{l}\frac{\partial^2 BS(S_0\xi,\bar{\sigma})}{\partial S_0^2}+\frac{2}{\bar{\sigma}^2}r\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{\xi}{H}\right)^2\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r-1}BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)\\-2\left(1-\frac{2}{\bar{\sigma}^2}r\right)\frac{\xi}{H}\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{\partial BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)}{\partial S_0}-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\bar{\sigma}^2}r}\frac{\partial^2 BS\left(\frac{H^2}{S_0\xi},\bar{\sigma}\right)}{\partial S_0^2};\bar{\sigma}\end{array}\right]$$

$$=E\left[\begin{array}{l}\xi\cdot N'(d_1)\frac{1}{S_0\bar{\sigma}\sqrt{T}}+\frac{2}{\bar{\sigma}^2}r\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{\xi}{H}\right)^2\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r-1}\left(\frac{H^2}{S_0\xi}\cdot N(d_3)-e^{-rT}K\cdot N(d_4)\right)\\+2\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0^2}\cdot N(d_3)-\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0^2}\left(2N(d_3)+N'(d_3)\frac{1}{\bar{\sigma}\sqrt{T}}\right);\bar{\sigma}\end{array}\right]$$

$$=E\left[\begin{array}{l}\xi\cdot N'(d_1)\frac{1}{S_0\bar{\sigma}\sqrt{T}}-\frac{2}{\bar{\sigma}^2}r\left(1-\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{\xi}{H}\right)^2\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r-1}e^{-rT}K\cdot N(d_4)\\-\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0^2}N'(d_3)\frac{1}{\bar{\sigma}\sqrt{T}}-\frac{2}{\bar{\sigma}^2}r\left(1+\frac{2}{\bar{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{-\frac{2}{\bar{\sigma}^2}r}\frac{H}{S_0^2}\cdot N(d_3);\bar{\sigma}\end{array}\right]$$

Rho:

$$\frac{\partial C}{\partial r}=E\left[\frac{\partial BS\left(S_0\xi,\overline{\sigma}\right)}{\partial r}+\frac{2}{\overline{\sigma}^2}\log\left(1-\frac{2}{\overline{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r}BS\left(\frac{H^2}{S_0\xi},\overline{\sigma}\right)-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r}\frac{\partial BS\left(\frac{H^2}{S_0\xi},\overline{\sigma}\right)}{\partial r};\overline{\sigma}\right]$$

$$=E\left[\begin{array}{l}KTe^{-rT}\cdot N(d_2)+\frac{2}{\overline{\sigma}^2}\log\left(1-\frac{2}{\overline{\sigma}^2}r\right)\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r}\left(\frac{H^2}{S_0\xi}N(d_3)-e^{-rT}KN(d_4)\right)\\-\left(\frac{S_0\xi}{H}\right)^{1-\frac{2}{\overline{\sigma}^2}r}KTe^{-rT}\cdot N(d_4);\overline{\sigma}\end{array}\right]$$

In the same way, it can gain the analytical solution of down-and-out call option Greeks in conditional Monte Carlo under SVCJ model.

VI. Numerical Simulation Results of Greeks

In this part, we will respectively make numerical results comparisons on option price estimations and Greeks analysis of down-and-out call option in Monte Carlo method, antithetic variable method and conditional Monte Carlo method, based on exact simulation algorithm.

TABLE 5. OPTION PRICE ESTIMATIONS AND GREEKS ANALYSIS IN THREE METHODS

Simulation method	Monte Carlo		Antithetic variable		Conditional Monte Carlo	
Model Category	SV model	SVCJ model	SV model	SVCJ model	SV model	SVCJ model
Price	7.1710	7.3447	7.0729	7.2204	7.2160	7.1851
Standard deviation	0.0939	0.0966	0.0480	0.0506	0.0340	0.0481
LR Delta	0.9270	0.9360	0.9027	0.8900	0.9130	0.9046
std(LR Delta)	0.0742	0.0753	0.0730	0.0740	0.0016	0.0016
LR Gamma	-0.0815	-0.1334	-0.0979	-0.1415	-0.0945	-0.1523
std(LR Gamma)	0.0680	0.0675	0.0657	0.0657	0.0000	0.0000
LR Rho	59.4428	58.3757	57.4007	56.1198	57.9380	58.6432
std(LR Rho)	2.7027	2.7636	2.7064	2.7740	0.0205	0.0037

In accordance with observation, among estimated variances and Greeks variances of down-and-out call option in Monte Carlo method, antithetic variable method and conditional Monte Carlo method under SV and SVCJ models, conditional Monte Carlo always generates smaller variances, which manifests that conditional Monte Carlo owns an effect of decreasing variance compared to the other two methods.

VII. Conclusion

Under the two stochastic volatility models of SV and SVCJ, and based on exact simulation algorithm for underlying asset process, this paper studied computational problems concerning acceleration simulation theories of down-and-out call option price and Greeks, and discussed acceleration effects of the two variance reduction techniques, namely conditional Monte Carlo and antithetic variable, under different circumstances.

It can be concluded from the simulated results that different initial parameters have certain influence on variance reduction effects of conditional Monte Carlo. Errors of option price estimation of SV model by conditional Monte Carlo are always less than errors generated from the other two methods, While jump part will influence the variance reduction effects under SVCJ model. Then conditional Monte Carlo behave best when estimating Greeks of option. In contrast to commonly used Euler discrete method and Monte Carlo method, we can obtain unbiased estimated values with less error.

The algorithm proposed in this paper can expediently solve computational problems of other more complicated products, such as problems involved in computing basket option under stochastic volatility models, etc.

Appendix

Next we give the proof of Theorem 1, the analytical solution of down-and-out call option price under Jump process with constant volatility.

Assume under risk neutral measure Q the asset price satisfies the following model:

$$dS_t=\left(r-\lambda\overline{\mu}\right)S_t dt+\sigma S_t W_t+S_t\left(J^s-1\right)dN_t.$$

Wherein N_t is a Poisson process of parameter λ_J, and it is independent from the Brownian Motion W_t, J^s represents underlying asset price's relative jump range. In particular, if it has a jump at the moment of t ,then $S_{t+}=S_{t-}J^s$. J^s accords with lognormal distribution, namely $\log J^s\sim N\left(\mu_s+\rho_J J^v,\sigma_s^2\right)$; therein parameters μ_s and $\overline{\mu}$ are correlated, and satisfy the expression,

$$\mu_s=\log\left(1+\overline{\mu}\right)-\frac{1}{2}\sigma_s^2$$

The price of down-and-out call option is:

$$C=E\left[e^{-rT}\left(S(T)-K\right)^{+}1\{\min(S_i)>H\}\right],$$

$$\tau=\inf\{0<t<T;S_t\le H\}$$

Calculating the underlying asset price:

$$S_T=S_0\left[\prod_{n=1}^{q_T}J^s\right]e^{\left(r-\lambda\overline{\mu}-\frac{1}{2}\overline{\sigma}^2\right)T+\overline{\sigma}\sqrt{T}Z_1}$$

Since $\log J^s\sim N\left(\mu_s,\sigma_s^2\right)\Rightarrow J^s=\mu_s+\sigma_s Z_2$,wherein Z_1 and Z_2 are independent.

When $q_T=n$, $\sum_{n=1}^{q_T}\ln J^s=n\mu_s+\sqrt{n}\sigma_s Z_2$, the underlying asset price is:

$$S_T = S_0 e^{\left(r-\lambda\bar{\mu}-\frac{1}{2}\sigma^2\right)T+n\mu_s+\sigma\sqrt{T}Z_1+\sqrt{n}\sigma_s Z_2} = S_0 e^{\left(r_n-\frac{1}{2}\sigma_n^2\right)T+\sigma_n\sqrt{T}Z_n}$$

$$r_n = r-\lambda\bar{\mu}+\frac{n}{T}\left(\mu_s+\frac{1}{2}\sigma^2\right), \sigma_n^2 = \sigma^2+\frac{n}{T}\sigma_s$$

Denote $V_T = \sqrt{T}Z_n$, V_T is the standard Brownian motion under risk neutral measure Q, denote $h_n = \dfrac{\sigma_n^2 - 2r_n}{2\sigma_n}$,

$Z_t^* = e^{h_n V_t - \frac{1}{2}h_n^2 t}$.

According to Girsanov theorem, $W_t^{R_n} = V_t - h_n t$ is the standard Brownian motion under measure R_n, wherein $\dfrac{dR_n}{dQ} = Z_T^*$.

Under measure R_n, when $q_T = n$, at time T, the underlying asset price is $S_T = S_0 e^{\sigma_n W_T^{R_n}}$, for $W_t^{R_n}$, denote $m_T^{W^{R_n}} = \inf_{0\le t\le T} W_t^{R_n}$, $\tau > T \Leftrightarrow m_T^S > H$, and the joint density of $W_t^{R_n}$, $m_T^{W^{R_n}}$ is:

$$h(x,y) = \begin{cases} 0 & x<y, or, y>0 \\ \sqrt{\dfrac{2}{\pi}}\cdot\dfrac{x-2y}{T^{3/2}}\cdot e^{-\frac{(x-2y)^2}{2T}} & y\le 0, y\le x \end{cases}.$$

Then the underlying asset price process accords with: $dS_t = (r-\lambda\bar{\mu})S_t dt + \bar{\sigma}S_t W_t + S_t(J^s-1)dN_t$, under this process, the price of down-and-out call option is:

$$C = \sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\cdot\left(\left[S_0 N(d_{n_1}) - Ke^{-r_n T}\cdot N(d_{n_3})\right] - \left[S_0\cdot e^{b+\frac{2br_n}{\sigma_n^2}}N(d_{n_2}) - Ke^{-r_n T}\cdot e^{\frac{2br_n}{\sigma_n^2}-b}\cdot N(d_{n_4})\right]\right)$$

$$= \sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\cdot\left(BS(S_0,\sigma_n) - \left(\frac{S_0}{H}\right)^{1-\frac{2}{\sigma_n^2}r_n} BS\left(\frac{H^2}{S_0},\sigma_n\right)\right)$$

wherein:

$$d_{n_1,n_3} = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r_n\pm\frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$d_{n_2,n_4} = \frac{\ln\left(\frac{H^2}{S_0 K}\right)+\left(r_n\pm\frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$r_n = r-\lambda\bar{\mu}+\frac{n}{T}\left(\mu_s+\frac{1}{2}\sigma^2\right);\ \sigma_n^2 = \sigma^2+\frac{n}{T}\sigma_s$$

Proof:

Under measure R_n, when $q_T = n$, $S_T = S_0\cdot e^{\sigma_n W_T^{R_n}}$, then:

$$S_T > K \Rightarrow S_0\cdot e^{\sigma_n W_T^{R_n}} > K \Rightarrow W_T^{R_n} > \frac{\ln\frac{K}{S_0}}{\sigma_n} \Rightarrow W_T^{R_n} > \frac{a}{\sigma_n}, a = \ln\frac{K}{S_0}$$

$$m_T^S > H \Rightarrow \min_{0\le t\le T}\{S_t\} > H \Rightarrow \min_{0\le t\le T}\left\{S_0\cdot e^{\sigma_n W_t^{R_n}}\right\} > H \Rightarrow \min_{0\le t\le T}\{W_t^{R_n}\} > \frac{\ln\frac{H}{S_0}}{\sigma_n} \Rightarrow m_T^{W^{R_n}} > \frac{b}{\sigma_n}, b = \ln\frac{H}{S_0}$$

Thus under risk neutral measure Q, we first calculate the expectation of down-and-out call option, then transform the expectation under measure R_n:

$$C = e^{-rT}E^Q\left[(S_T-K)^+\cdot 1\{m_T^S > H\}\right] = e^{-rT}\sum_{n=0}^{\infty}E^Q\left[(S_T-K)^+\cdot 1\{m_T^S > H\}\middle| q_T = n\right]P(q_T = n)$$

$$= e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}E^{R_n}\left[\left(\frac{S_T-K}{Z^*}\right)^+\cdot 1\left\{m_T^{W^{R_n}} > \frac{b}{\sigma_n}\right\}\middle| q_T = n\right]$$

$$= e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}E^{R_n}\left[(S_T-K)\cdot e^{-h_n W_T^{R_n}-\frac{1}{2}h_n^2 T}\cdot 1\left\{W_T^{R_n} > \frac{a}{\sigma_n}\right\}\cdot 1\left\{m_T^{W^{R_n}} > \frac{b}{\sigma_n}\right\}\right]$$

$$= e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\cdot e^{-\frac{1}{2}h_n^2 T}\left(\begin{aligned}&E^{R_n}\left[S_T\cdot e^{-h_n W_T^{R_n}}\cdot 1\left\{W_T^{R_n} > \frac{a}{\sigma_n}\right\}\cdot 1\left\{m_T^{W^{R_n}} > \frac{b}{\sigma_n}\right\}\right]\\ &-E^{R_n}\left[K\cdot e^{-h_n W_T^{R_n}}\cdot 1\left\{W_T^{R_n} > \frac{a}{\sigma_n}\right\}\cdot 1\left\{m_T^{W^{R_n}} > \frac{b}{\sigma_n}\right\}\right]\end{aligned}\right)$$

$$= e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\cdot e^{-\frac{1}{2}h_n^2 T}(A-B)$$

wherein:

$$A = E^{R_n}\left[S_T\cdot e^{-h_n W_T^{R_n}}\cdot 1\left\{W_T^{R_n} > \frac{a}{\sigma_n}\right\}\cdot 1\left\{m_T^{W^{R_n}} > \frac{b}{\sigma_n}\right\}\right]$$

$$= \int_{\frac{a}{\sigma_n}}^{\infty}\int_{\frac{b}{\sigma_n}}^{0} S_0\cdot e^{\sigma_n x}\cdot e^{-h_n x}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{x-2y}{T^{3/2}}\cdot e^{-\frac{(x-2y)^2}{2T}}dydx$$

$$= \int_{\frac{a}{\sigma_n}}^{\infty} S_0\cdot e^{\sigma_n x-h_n x}\cdot\left(\int_{\frac{b}{\sigma_n}}^{0}\sqrt{\frac{2}{\pi}}\cdot\frac{x-2y}{T^{3/2}}\cdot e^{-\frac{(x-2y)^2}{2T}}dy\right)dx$$

$$= \int_{\frac{a}{\sigma_n}}^{\infty} S_0\cdot e^{\sigma_n x-h_n x}\cdot\frac{1}{\sqrt{2\pi T}}\left(e^{-\frac{x^2}{2T}} - e^{-\frac{\left(x-\frac{2b}{\sigma_n}\right)^2}{2T}}\right)dx$$

$$= \frac{S_0}{\sqrt{2\pi T}}\left(\int_{\frac{a}{\sigma_n}}^{\infty} e^{\sigma_n x-h_n x-\frac{x^2}{2T}}dx - \int_{\frac{a}{\sigma_n}}^{\infty} e^{\sigma_n x-h_n x-\frac{\left(x-\frac{2b}{\sigma_n}\right)^2}{2T}}dx\right) = A_1 - A_2$$

$$A_1 = \frac{S_0}{\sqrt{2\pi T}}\int_{\frac{a}{\sigma_n}}^{\infty} e^{(\sigma_n-h_n)x-\frac{x^2}{2T}}dx = \int_{\frac{a}{\sigma_n}}^{\infty} e^{\frac{-1}{2T}\left(x^2-2(\sigma_n-h_n)Tx\right)}dx$$

$$= S_0 e^{\frac{(\sigma_n-h_n)^2 T}{2}}\int_{\frac{\frac{a}{\sigma_n}-(\sigma_n-h_n)T}{\sqrt{T}}}^{\infty} e^{-y^2}dy$$

$$= S_0 e^{\frac{(\sigma_n-h_n)^2 T}{2}}\cdot N\left(-\frac{\frac{a}{\sigma_n}-(\sigma_n-h_n)T}{\sqrt{T}}\right)$$

$$= S_0 e^{\frac{(\sigma_n-h_n)^2 T}{2}}N(d_{n_1}),\quad d_{n_1} = \frac{-\frac{a}{\sigma_n}+(\sigma_n-h_n)T}{\sqrt{T}}$$

$$A_2=\frac{S_0}{\sqrt{2\pi T}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{(\sigma_n-h_n)x-\frac{\left(x-\frac{2b}{\sigma_n}\right)^2}{2T}}dx=\int_{\frac{a}{\sigma_n}}^{\infty}e^{\frac{-1}{2T}\left(\left(x-\frac{2b}{\sigma_n}\right)^2-2(\sigma_n-h_n)Tx\right)}dx$$

$$=\frac{S_0}{\sqrt{2\pi T}}e^{\frac{(\sigma_n-h_n)^2T}{2}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{\frac{-1}{2T}\left[\left(\left(x-\frac{2b}{\sigma_n}\right)^2-2(\sigma_n-h_n)T\left(x-\frac{2b}{\sigma_n}\right)\right)\right]+(\sigma_n-h_n)\frac{2b}{\sigma_n}}dx$$

$$=\frac{S_0}{\sqrt{2\pi T}}e^{\frac{(\sigma_n-h_n)^2T}{2}+(\sigma_n-h_n)\frac{2b}{\sigma_n}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{-\frac{\left(x-\frac{2b}{\sigma_n}-(\sigma_n-h_n)T\right)^2}{2T}}dx$$

$$=S_0e^{\frac{(\sigma_n-h_n)^2T}{2}+(\sigma_n-h_n)\frac{2b}{\sigma_n}}\int_{\frac{\frac{a}{\sigma_n}-\frac{2b}{\sigma_n}-(\sigma_n-h_n)T}{\sqrt{T}}}^{\infty}e^{-y^2}dy$$

$$=S_0e^{\frac{(\sigma_n-h_n)^2T}{2}+(\sigma_n-h_n)\frac{2b}{\sigma_n}}\cdot N\left(-\frac{\frac{a}{\sigma_n}-\frac{2b}{\sigma_n}-(\sigma_n-h_n)T}{\sqrt{T}}\right)$$

$$=S_0e^{\frac{(\sigma_n-h_n)^2T}{2}+(\sigma_n-h_n)\frac{2b}{\sigma_n}}N\left(d_{n_2}\right),\ d_{n_2}=\frac{-\frac{a}{\sigma_n}+\frac{2b}{\sigma_n}+(\sigma_n-h_n)T}{\sqrt{T}}$$

$$B=E^{R_n}\left[K\cdot e^{-h_nW_T^{R_n}}\cdot 1\left\{W_T^{R_n}>\frac{a}{\sigma_n}\right\}\cdot 1\left\{m_T^{W^{R_n}}>\frac{b}{\sigma_n}\right\}\right]$$

$$=\int_{\frac{a}{\sigma_n}}^{\infty}\int_{\frac{b}{\sigma_n}}^{0}K\cdot e^{-h_nx}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{x-2y}{T^{3/2}}\cdot e^{-\frac{(x-2y)^2}{2T}}dydx$$

$$=\frac{K}{\sqrt{2\pi T}}\left(\int_{\frac{a}{\sigma_n}}^{\infty}e^{-h_nx-\frac{x^2}{2T}}dx-\int_{\frac{a}{\sigma_n}}^{\infty}e^{-h_nx-\frac{\left(x-\frac{2b}{\sigma_n}\right)^2}{2T}}dx\right)$$

$$=B_1-B_2$$

$$B_1=\frac{K}{\sqrt{2\pi T}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{-h_nx-\frac{x^2}{2T}}dx=\int_{\frac{a}{\sigma_n}}^{\infty}e^{\frac{-1}{2T}\left(x^2+2h_nTx\right)}dx$$

$$=\frac{K}{\sqrt{2\pi T}}e^{\frac{h_n^2T}{2}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{-\frac{(x+h_nT)^2}{2T}}dx=K\cdot e^{\frac{h_n^2T}{2}}\int_{\frac{\frac{a}{\sigma_n}+h_nT}{\sqrt{T}}}^{\infty}e^{-y^2}dy$$

$$=K\cdot e^{\frac{h_n^2T}{2}}\cdot N\left(-\frac{\frac{a}{\sigma_n}+h_nT}{\sqrt{T}}\right)$$

$$=K\cdot e^{\frac{h_n^2T}{2}}\cdot N\left(d_{n_3}\right),\ d_{n_3}=\frac{-\frac{a}{\sigma_n}-h_nT}{\sqrt{T}}$$

$$B_2=\frac{K}{\sqrt{2\pi T}}\int_{\frac{a}{\sigma_n}}^{\infty}e^{-h_nx-\frac{\left(x-\frac{2b}{\sigma_n}\right)^2}{2T}}dx=\int_{\frac{a}{\sigma_n}}^{\infty}e^{\frac{-1}{2T}\left(\left(x-\frac{2b}{\sigma_n}\right)^2+2h_nTx\right)}dx$$

$$=K\cdot e^{\frac{h_n^2T}{2}-\frac{2h_nb}{\sigma_n}}\cdot N\left(-\frac{\frac{a}{\sigma_n}-\frac{2b}{\sigma_n}+h_nT}{\sqrt{T}}\right)$$

$$=K\cdot e^{\frac{h_n^2T}{2}-\frac{2h_nb}{\sigma_n}}\cdot N\left(d_{n_4}\right),\ d_{n_4}=\frac{-\frac{a}{\sigma_n}+\frac{2b}{\sigma_n}-h_nT}{\sqrt{T}}$$

After simplifying, C has the following expression.

$$C=e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}\left(\lambda T\right)^n}{n!}\cdot e^{-\frac{1}{2}h_n^2T}\left(A-B\right)$$

$$=e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}\left(\lambda T\right)^n}{n!}\cdot e^{-\frac{1}{2}h_n^2T}\left(A_1-A_2-B_1+B_2\right)$$

$$=e^{-rT}\sum_{n=0}^{\infty}\frac{e^{-\lambda T}\left(\lambda T\right)^n}{n!}\cdot e^{-\frac{1}{2}h_n^2T}\begin{pmatrix}S_0e^{\frac{(\sigma_n-h_n)^2T}{2}}\left[N\left(d_{n_1}\right)-e^{(\sigma_n-h_n)\frac{2b}{\sigma_n}}N\left(d_{n_2}\right)\right]\\-K\cdot e^{\frac{h_n^2T}{2}}\cdot\left[N\left(d_{n_3}\right)+e^{-\frac{2h_nb}{\sigma_n}}\cdot N\left(d_{n_4}\right)\right]\end{pmatrix}$$

$$=\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}\left(\lambda' T\right)^n}{n!}\cdot\begin{pmatrix}S_0e^{\frac{\sigma_n^2T-2h_n\sigma_nT-2r_nT}{2}}\left[N\left(d_{n_1}\right)-e^{(\sigma_n-h_n)\frac{2b}{\sigma_n}}N\left(d_{n_2}\right)\right]\\-K\cdot e^{-r_nT}\cdot\left[N\left(d_{n_3}\right)+e^{-\frac{2h_nb}{\sigma_n}}\cdot N\left(d_{n_4}\right)\right]\end{pmatrix}$$

$$d_{n_1}=\frac{-\frac{a}{\sigma_n}+(\sigma_n-h_n)T}{\sqrt{T}};\ d_{n_2}=\frac{-\frac{a}{\sigma_n}+\frac{2b}{\sigma_n}+(\sigma_n-h_n)T}{\sqrt{T}};$$

$$d_{n_3}=\frac{-\frac{a}{\sigma_n}-h_nT}{\sqrt{T}};d_{n_4}=\frac{-\frac{a}{\sigma_n}+\frac{2b}{\sigma_n}-h_nT}{\sqrt{T}}.$$

Plugging

$$h_n=\frac{\sigma_n^2-2r_n}{2\sigma_n}=\frac{1}{2}\sigma_n-\frac{r_n}{\sigma_n};\ a=\ln\frac{K}{S_0};\ b=\ln\frac{H}{S_0}.$$

$$C=\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}\left(\lambda' T\right)^n}{n!}\cdot\begin{pmatrix}S_0e^{\frac{\sigma_n^2T-2h_n\sigma_nT-2r_nT}{2}}\left[N\left(d_{n_1}\right)-e^{(\sigma_n-h_n)\frac{2b}{\sigma_n}}N\left(d_{n_2}\right)\right]\\-K\cdot e^{-r_nT}\cdot\left[N\left(d_{n_3}\right)+e^{-\frac{2h_nb}{\sigma_n}}\cdot N\left(d_{n_4}\right)\right]\end{pmatrix}$$

$$=\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}\left(\lambda' T\right)^n}{n!}\cdot\begin{pmatrix}S_0\left[N\left(d_{n_1}\right)-e^{b+\frac{2br_n}{\sigma_n^2}}N\left(d_{n_2}\right)\right]\\-Ke^{-r_nT}\cdot\left[N\left(d_{n_3}\right)+e^{\frac{2br_n}{\sigma_n^2}-b}\cdot N\left(d_{n_4}\right)\right]\end{pmatrix}$$

$$=\sum_{n=0}^{\infty}\frac{e^{-\lambda' T}\left(\lambda' T\right)^n}{n!}\cdot\left(BS\left(S_0,\sigma_n\right)-\left(\frac{S_0}{H}\right)^{1-\frac{2}{\sigma_n^2}r_n}BS\left(\frac{H^2}{S_0},\sigma_n\right)\right)$$

$$d_{n_1,\ n_3}=\frac{\ln\left(\frac{S_0}{K}\right)+\left(r_n\pm\frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}};$$

$$d_{n_2,\ n_4}=\frac{\ln\left(\frac{H^2}{S_0K}\right)+\left(r_n\pm\frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}}.$$

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