The Linear $k-$Arboricity of Cartesian Product of Multipartite Balanced Complete Graphs*

Tianfeng Huang, Liancui Zuo† and Chunhong Shang

Abstract

A linear $k-$forest of an undirect graph G is a subgraph whose components are paths with length at most k. The linear $k-$arboricity of G, denoted by $la_k(G)$, is the minimum number of linear $k-$forests partitioning the edge set $E(G)$. In the present paper, we studied the linear $(n-1)-$arboricity of Cartesian product graph $(K_{n,n})^{[m]}$ and $\left(K_{n(l)}\right)^{[m]}$, and obtained the exact values of linear $(n-1)-$arboricity of $(K_{n,n})^{[m]}$ and $\left(K_{n(l)}\right)^{[m]}$ in some special cases.

Keywords: linear $k-$forest; linear $k-$arboricity; Cartesian product graphs; multipartite balanced complete graphs

1 Introduction

Throughout this paper, all graphs we considered are simple, finite and undirected. Let N represent the set of natural numbers. For any integers a and b with $a \le b$, we use the symbol $[a,b]$ to denote the set $\{n \in N \,|a \le n \le b\}$. For a real number x, $\lceil x \rceil$ represents the smallest integer no less than x and $\lfloor x \rfloor$ represents the largest integer no more than x.

A graph G is $l-$partite ($l \ge 2$) if it is possible to partition the vertex set $V(G)$ into l independent sets $V_1, V_2, \cdots, V_l$ (called partite sets) such that every edge of G joins the vertices in different sets. A complete $l-$partite graph G is a $l-$partite graph with partite sets $V_1, V_2, \cdots, V_l$ having the additional property that if $u \in V_i$ and $v \in V_j$ where $i \ne j$, then the edge $uv \in E(G)$. If $|V_i| = n_i$ for all $i \in \{1, 2, \cdots, l\}$, then this graph is denoted by $K_{n_1,n_2,\cdots,n_l}$. Moreover, if $n_1 = n_2 = \cdots = n_l = n$, then it is called a balanced complete $l-$partite graph and denoted by $K_{n(l)}$. For $l = 2$, such graphs are denoted by $K_{n,n}$ and called balanced complete bipartite graphs. We refer to [5] for other notation and terminology in the graph theory.

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \cdots, G_t$, then we say that $G_1, G_2, \cdots, G_t$ decompose G or G can be decomposed into $G_1, G_2, \cdots, G_t$. Furthermore, a linear $k-$forest is a forest whose components are paths of length at most k. The linear $k-$arboricity of a graph G, denoted by $la_k(G)$, is the least number of linear $k-$forests needed to decompose G.

Habib and Peroche defined linear $k-$arboricity of a graph in [6], which is a natural generalization of edge coloring. Clearly, a matching induce a linear 1-arboricity, and $la_1(G)$ is the edge chromatic number, or chromatic index $\chi'(G)$ of a graph. The ordinary linear arboricity $la(G)$ (or $la_\infty(G)$) is the case where every component of each forest is a path without length constraint. Furthermore, the linear $k-$arboricity is a refinement of the ordinary linear arboricity.

The Cartesian product of m graphs $G_1, G_2, \cdots, G_m$ is the graph $H = G_1 \Box G_2 \Box \cdots \Box G_m$, where $V(H) = \prod_{i=1}^{m} V(G_i)$ and two vertices $(u_1, u_2, \cdots, u_m)$ and $(v_1, v_2, \cdots, v_m)$ are adjacent if and only if $u_j v_j \in E(G_j)$ for some j and $u_i = v_i$ for all other $i \ne j$. If $G_i = G$ for all $i \in [1,m]$, we denote $G_1 \Box G_2 \Box \cdots \Box G_m$ by $G^{[m]}$. Then we can obtain that

$$|V(H)| = \prod_{i=1}^{m} |V(G_i)|,$$

$$|E(H)| = \sum_{j=1}^{m} \left[|E(G_j)| \prod_{i \ne j} |V(G_i)| \right]$$

and

$$d_H(u) = \sum_{j=1}^{m} d_{G_j}(u_j)$$

for any vertex $u = (u_1, u_2, \cdots, u_m)$.

About an upper bound on $la_k(G)$, Habib and Peroche proposed the following conjecture in 1982.

Conjecture 1.1. [7] If G is a graph with maximum degree $\Delta(G)$ and $k \ge 2$, then

$$la_k(G) \le \begin{cases} \left\lceil \frac{\Delta(G)\cdot|V(G)|}{2\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil, & when\ \Delta(G) = |V(G)| - 1, \\ \left\lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil, & when\ \Delta(G) < |V(G)| - 1. \end{cases}$$

For $k = |V(G)| - 1$, it is Akiyama's conjecture.

*Manuscript received December 29, 2017. This work was supported by MECF of Tianjin with code 135302JW1713, NSFC with code 61572358, NSF of Tianjin with code 16JCYBJC23600, Program for Innovative Research Team in Universities of Tianjin with code TD13-5078, and Tianjin Training Programs of Innovation for Undergraduates with code 201710065092.

†Tianfeng Huang, Liancui Zuo and Chunhong Shang are with College of Mathematical Science, Tianjin Normal University, Tianjin, 300387, China. Email: 1247869368@qq.com; qxtfhuang@163.com

Conjecture 1.2. [8] $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

In recent years, many parameters and classes of graphs have been studied. For example, the restricted connectivity of Cartesian product graphs is obtained in [23]. In [24], some results of resistance distance and Kirchhoff index based on R-graph are obtained. And in [25], some results on 3-equitable labeling are gained.

So far, there have been many results on the verification of Conjecture 1.1 in the literature, especially for graphs with particular structures. In [6,10,11], the linear $k-$arboricity of trees are studied. In [12,13,14], the linear $k-$arboricity and the linear arboricity of some regular graphs are studied. In [15,16,17,18], the linear 2-arboricity of planar graphs are obtained and the linear $k-$arboricity of cubic graphs are obtained. In [2,3,4,9,19], the linear $k-$arboricity of the balanced complete multipartite graphs $K_{n(m)}$, $K_{n,n}$, K_n, and Cartesian product of some graphs are studied. In [20,21,22], the linear $k-$arboricity of some complete bipartite graphs is obtained.

2 Main results

As preparation, we need the following lemmas.

Lemma 2.1. [2] If $G = G_1 \cup G_2 \cup \cdots \cup G_n$, then $la_k(G) \leq la_k(G_1) + la_k(G_2) + \cdots + la_k(G_n)$.

Lemma 2.2. [9] If H is subgraph of G, then $la_k(G) \geq la_k(H)$.

Lemma 2.3. [9] For any graph G with maximum degree $\Delta(G)$, then

$$la_k(G) \geq \max\left\{ \lceil \Delta(G)/2 \rceil, \left\lceil |E(G)| \middle/ \left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor \right\rceil \right\}.$$

Assume that G and H are graphs. A spanning subgraph of G is called an $H-$factor if each component of F is isomorphic to H. If G is expressible as an edge-disjoint union of $H-$factors, then this union is called an $H-$factorization.

Lemma 2.4. [1] If a graph G has an $H-$factorization with t $H-$factors, then

$$la_k(G) \leq t \cdot la_k(H).$$

Lemma 2.5. [2] Let $G = G_1 \Box G_2 \Box \cdots \Box G_m$. Then G can be decomposed into the edge-disjoint union of a G_1-factor, a G_2-factor, $\cdots$, and a G_m-factor. Therefore we have

$$la_k(G_1 \Box G_2 \Box \cdots \Box G_m)$$

$$\leq la_k(G_1) + la_k(G_2) + \cdots + la_k(G_m)$$

Let

$$G = K_{n_1,n_1} \Box K_{n_2,n_2} \Box \cdots \Box K_{n_m,n_m},$$

then G can be decomposed into the edge-disjoint union of a $K_{n_1,n_1}-$factor, a $K_{n_2,n_2}-$factor, $\cdots$, and a $K_{n_m,n_m}-$factor, so $(K_{n,n})^{[m]}$ has a $K_{n,n}-$factorization that contains m $K_{n,n}-$factors.

Corollary 2.1. If m is even, $G = (K_{n,n})^{[m]}$ has a $(K_{n,n})^{[2]}-$factorization with $\frac{m}{2}$ $(K_{n,n})^{[2]}-$factors. If m is odd, then $G = (K_{n,n})^{[m]}$ can be decomposed into the edge-disjoint union of $\frac{m-1}{2}$ $(K_{n,n})^{[2]}-$factors and a $K_{n,n}-$factor.

Proof. $G = (K_{n,n})^{[m]}$can be decomposed into the edge-disjoint union of m $K_{n,n}-$factors by Lemma 2.5, and any two $K_{n,n}-$factors can form a $K^2_{n,n}-$factor. So Corollary holds. ☐

Lemma 2.6. [3] $la_k(K_{n,n}) = \lceil \frac{n}{2} \rceil + 1$ if $n-1 \leq k \leq 2n-2$.

Lemma 2.7. [2] Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ be two parts of $K_{n,n}$ for odd $n \geq 5$. Then the balanced complete bipartite graph $K_{n,n}$ can be decomposed into $\frac{n-1}{2}$ linear $(n-1)-$forests F_i and Q, where each F_i consists of two vertex-disjoint paths of length $(n-1)$ for $i \in [1, \frac{n-1}{2}]$, and

$$Q = \bigcup_{i=1}^{(n-1)/2} x_i y_i x_{n+1-i} y_{n+1-i} x_i \bigcup x_{\frac{n+1}{2}} y_{\frac{n+1}{2}}$$

is a vertex-disjoint union of $\frac{n-1}{2}$ cycles of length four and an isolated edge.

Lemma 2.8. [4] $la_{n-1}\left(K_{n(m)}\right) = \lceil \frac{mn}{2} \rceil$.

In the following, we studied the linear $(n-1)-$arboricity of Cartesian product graph $(K_{n,n})^{[m]}$ and $\left(K_{n(l)}\right)^{[m]}$.

Theorem 2.1. $la_{n-1}(K_{n,n} \Box K_{n,n}) = n+2$.

Proof. We can obtain that

$$|V(K_{n,n} \Box K_{n,n})| = 4n^2,$$

$$d_{K_{n,n} \Box K_{n,n}}(u) = 2n,$$

and

$$|E(K_{n,n} \Box K_{n,n})| = 4n^3.$$

Applying Lemma 2.3, we have

$$la_{n-1}(K_{n,n} \Box K_{n,n}) \geq n+2.$$

In the following, we will show that

$$la_{n-1}(K_{n,n} \Box K_{n,n}) \leq n+2.$$

Case 1. n is even.

It is obvious that

$$la_{n-1}(K_{n,n} \Box K_{n,n}) \leq 2la_{n-1}(K_{n,n})$$

$$\leq 2(n/2+1) = n+2$$

by Lemma 2.5 and Lemma 2.6.

Case 2. n is odd.

Subcase 2.1. $n=3$.

We will show that $la_2(K_{3,3}\Box K_{3,3})\le 5$ by direct construction in the following.

Let

$$K_{3,3}\Box K_{3,3}=K_{n_1,n_1}\Box K_{n_2,n_2},$$

where $n_1=n_2=3$.

Let

$$V(K_{n_1,n_1})=\{u_p|\,p\in[1,6]\},$$

and

$$X_1=\{u_1,u_2,u_3\}\,,Y_1=\{u_4,u_5,u_6\}$$

be two parts of K_{n_1,n_1}.

Let

$$V(K_{n_2,n_2})=\{v_p|\,p\in[1,6]\},$$

and

$$X_2=\{v_1,v_2,v_3\}\,,Y_2=\{v_4,v_5,v_6\}$$

be two parts of K_{n_2,n_2}.

Let

$$\begin{aligned}F_1=\{&(u_1,v_5)(u_1,v_1)(u_1,v_6),\\&(u_1,v_2)(u_1,v_4)(u_1,v_3),\\&(u_2,v_1)(u_2,v_5)(u_2,v_3),\\&(u_2,v_4)(u_2,v_2)(u_2,v_6),\\&(u_3,v_1)(u_3,v_6)(u_3,v_2),\\&(u_3,v_4)(u_3,v_3)(u_3,v_5),\\&(u_4,v_5)(u_4,v_1)(u_4,v_6),\\&(u_4,v_2)(u_4,v_4)(u_4,v_3),\\&(u_5,v_1)(u_5,v_5)(u_5,v_3),\\&(u_5,v_4)(u_5,v_2)(u_5,v_6),\\&(u_6,v_1)(u_6,v_6)(u_6,v_2),\\&(u_6,v_4)(u_6,v_3)(u_6,v_5)\},\end{aligned}$$

$$\begin{aligned}F_2=\{&(u_1,v_1)(u_5,v_1)(u_3,v_1),\\&(u_4,v_1)(u_2,v_1)(u_6,v_1),\\&(u_1,v_2)(u_6,v_2)(u_2,v_2),\\&(u_4,v_3)(u_3,v_2)(u_5,v_2),\\&(u_5,v_3)(u_1,v_3)(u_6,v_3),\\&(u_2,v_3)(u_4,v_3)(u_3,v_3),\\&(u_1,v_4)(u_5,v_4)(u_3,v_4),\\&(u_4,v_4)(u_2,v_4)(u_6,v_4),\\&(u_1,v_5)(u_6,v_5)(u_2,v_5),\\&(u_4,v_5)(u_3,v_5)(u_5,v_5),\\&(u_5,v_6)(u_1,v_6)(u_6,v_6),\\&(u_2,v_6)(u_4,v_6)(u_3,v_6)\},\end{aligned}$$

$$\begin{aligned}F_3=\{&(u_1,v_1)(u_1,v_4),(u_1,v_2)(u_1,v_5),\\&(u_1,v_3)(u_1,v_6),(u_2,v_1)(u_2,v_4),\\&(u_2,v_2)(u_2,v_5),(u_2,v_3)(u_2,v_6),\\&(u_3,v_1)(u_3,v_4),(u_3,v_2)(u_3,v_5),\\&(u_3,v_3)(u_3,v_6),(u_4,v_1)(u_4,v_4),\\&(u_4,v_2)(u_4,v_5),(u_4,v_3)(u_4,v_6),\\&(u_5,v_1)(u_5,v_4),(u_5,v_2)(u_5,v_5),\\&(u_5,v_3)(u_5,v_6),(u_6,v_1)(u_6,v_4),\\&(u_6,v_2)(u_6,v_5),(u_6,v_3)(u_6,v_6)\},\end{aligned}$$

$$\begin{aligned}F_4=\{&(u_1,v_1)(u_4,v_1),(u_2,v_1)(u_5,v_1),\\&(u_3,v_1)(u_6,v_1),(u_1,v_2)(u_4,v_2),\\&(u_2,v_2)(u_5,v_2),(u_3,v_2)(u_6,v_2),\\&(u_1,v_3)(u_4,v_3),(u_2,v_3)(u_5,v_3),\\&(u_3,v_3)(u_6,v_3),(u_1,v_4)(u_4,v_4),\\&(u_2,v_4)(u_5,v_4),(u_3,v_4)(u_6,v_4),\\&(u_1,v_5)(u_4,v_5),(u_2,v_5)(u_5,v_5),\\&(u_3,v_5)(u_6,v_5),(u_1,v_6)(u_4,v_6),\\&(u_2,v_6)(u_5,v_6),(u_3,v_6)(u_6,v_6)\},\end{aligned}$$

and

$$\begin{aligned}F_5=\{&(u_1,v_6)(u_1,v_2)(u_5,v_2),\\&(u_1,v_3)(u_1,v_5)(u_5,v_5),\\&(u_2,v_1)(u_2,v_6)(u_6,v_6),\\&(u_2,v_4)(u_2,v_3)(u_6,v_3),\\&(u_4,v_1)(u_3,v_1)(u_3,v_5),\\&(u_3,v_2)(u_3,v_4)(u_4,v_4),\\&(u_4,v_6)(u_4,v_2)(u_2,v_2),\\&(u_4,v_3)(u_4,v_5)(u_2,v_5),\\&(u_5,v_1)(u_5,v_6)(u_3,v_6),\\&(u_5,v_4)(u_5,v_3)(u_3,v_3),\\&(u_6,v_5)(u_6,v_1)(u_1,v_1),\\&(u_6,v_2)(u_6,v_4)(u_1,v_4)\}.\end{aligned}$$

Then, it is not difficult to verify that each F_i is a linear 2−forest for $i\in[1,5]$, and thus the result holds.

Subcase 2.2. $n\ge 5$.

Let

$$K_{n,n}\Box K_{n,n}=K_{n_1,n_1}\Box K_{n_2,n_2}$$

where $n_1=n_2=n$.

Let

$$V(K_{n_1,n_1})=\{u_p|\,p\in[1,2n]\},$$

and

$$X_1=\{u_1,u_2,\cdots,u_n\}\,,Y_1=\{u_{n+1},u_{n+2},\cdots,u_{2n}\}$$

be two parts of K_{n_1,n_1}.

Let

$$V(K_{n_2,n_2})=\{v_p|\,p\in[1,2n]\},$$

and

$$X_2=\{v_1,v_2,\cdots,v_n\}\,,Y_2=\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$$

be two parts of K_{n_2,n_2}.

Clearly, the vertex subset $\{\,(u_i,v_j)|\,j\in[1,2n]\}$ induces a balanced complete bipartite graph which is denoted by $K_{n,n}^{(i)}$ for $i\in[1,2n]$, and the vertex subset $\{\,(u_i,v_j)|\,i\in[1,2n]\}$ induces a balanced complete bipartite graph which is denoted by $K_{n,n}^{(j)}$ for $j\in[1,2n]$. It is obvious that $K_{n,n}\Box K_{n,n}$ can be decomposed into $2n$ disjoint balanced complete bipartite graphs $K_{n,n}^{(i)}$ for $i\in[1,2n]$ and $2n$ disjoint balanced complete bipartite graphs $K_{n,n}^{(j)}$ for $j\in[1,2n]$.

By Lemma 2.7, we can obtain that

$$K_{n,n}^{(i)}=\frac{n-1}{2}\cdot 2P_n\cup M_i$$

and

$$K_{n,n}^{(j)} = \frac{n-1}{2} \cdot 2P_n' \cup N_j$$

for $i \in [1, 2n]$, $j \in [1, 2n]$.

Here

$$M_i = \frac{n-1}{2} C_4 \cup P_2,$$

$$N_j = \frac{n-1}{2} C_4' \cup P_2',$$

and

$$C_4 = (u_i, v_k)(u_i, v_{k+n})(u_i, v_{n+1-k})(u_i, v_{2n+1-k})(u_i, v_k),$$

$$P_2 = \left(u_i, v_{\frac{n+1}{2}}\right)\left(u_i, v_{\frac{3n+1}{2}}\right),$$

$$C_4' = (u_k, v_j)(u_{k+n}, v_j)(u_{n+1-k}, v_j)(u_{2n+1-k}, v_j)(u_k, v_j),$$

$$P_2' = \left(u_{\frac{n+1}{2}}, v_j\right)\left(u_{\frac{3n+1}{2}}, v_j\right)$$

for $k \in \left[1, \frac{n-1}{2}\right]$.

Let

$$E_i = \{(u_i, v_1)(u_{i+n}, v_1), (u_i, v_2)(u_{i+n}, v_2), \cdots, (u_i, v_{2n})(u_{i+n}, v_{2n})\}.$$

Now all edges E_i, M_i and M_{i+n} form $\frac{n-1}{2}(K_2 \square C_4)$ and one C_4 for $i \in [1, n]$. Since each $K_2 \square C_4$ can be decomposed into two $2P_4$ (for example, we have

$$\begin{aligned} K_2 \square C_4 = &\{(u_1, v_1)(u_1, v_4)(u_2, v_4)(u_2, v_1), \\ &(u_1, v_3)(u_1, v_2)(u_2, v_2)(u_2, v_3)\} \\ \cup &\{(u_1, v_2)(u_1, v_1)(u_2, v_1)(u_2, v_2), \\ &(u_1, v_4)(u_1, v_3)(u_2, v_3)(u_2, v_4)\} \end{aligned}$$

where $V(K_2) = \{u_1, u_2\}$ and $V(C_4) = \{v_1, v_2, v_3, v_4\}$) and $C_4 = 2P_3$, we have two isomorphic edge-disjoint linear 3−forests $\frac{n-1}{2}(2P_4) \cup P_3$.

Let

$$E = E_1 \cup E_2 \cup \cdots \cup E_n,$$

$$M = M_1 \cup M_2 \cup \cdots \cup M_{2n},$$

$$N = N_1 \cup N_2 \cup \cdots \cup N_{2n}.$$

Then it is clear that $E \subseteq E(N)$, and

$$E(N) - E = \{(u_k, v_j)(u_{2n+1-k}, v_j), (u_{k+n}, v_j)(u_{n+1-k}, v_j) | k \in [1, \tfrac{n-1}{2}], j \in [1, 2n]\}.$$

Obviously, $E(N) - E$ can form a linear $(n-1)$-forest. Thus, we can use three colors to color $M \cup N$. Hence

$$la_{n-1}(K_{n,n} \square K_{n,n}) \leq \frac{n-1}{2} + \frac{n-1}{2} + 3 = n + 2$$

for odd $n \geq 5$.

Therefore, we have obtained that $la_{n-1}(K_{n,n} \square K_{n,n}) = n + 2$. □

Theorem 2.2. $\left\lceil \frac{mn^2}{2(n-1)} \right\rceil \leq la_{n-1}(K_{n,n})^{[m]} \leq \left\lceil \frac{mn}{2} \right\rceil + m$.

Proof. It is not difficult to verify that

$$|V(K_{n,n} \square K_{n,n} \square \cdots \square K_{n,n})| = (2n)^m,$$

$$d_{K_{n,n} \square K_{n,n} \square \cdots \square K_{n,n}}(u) = mn$$

for any vertex $u = (u_1, u_2, \cdots, u_m)$, and

$$|E(K_{n,n} \square K_{n,n} \square \cdots \square K_{n,n})| = m \cdot n^2 \cdot (2n)^{m-1}.$$

Applying Lemma 2.3, we have

$$la_{n-1}(K_{n,n})^{[m]} \geq \left\lceil \frac{mn^2}{2(n-1)} \right\rceil = \left\lceil \frac{m(n+1)}{2} + \frac{m}{2(n-1)} \right\rceil.$$

We will show that

$$la_{n-1}(K_{n,n})^{[m]} \leq \left\lceil \frac{mn}{2} \right\rceil + m$$

according to the parity of n.

Case 1. n is even.

By Lemma 2.4 and Lemma 2.6, we obtain that

$$la_{n-1}(K_{n,n})^{[m]} \leq m \cdot la_{n-1}(K_{n,n}) = m \cdot \left(\frac{n}{2} + 1\right) = \frac{mn}{2} + m.$$

Case 2. n is odd.

If m is even, then by Lemma 2.4, Corollary 2.1 and Theorem 2.1, we have

$$la_{n-1}(K_{n,n})^{[m]} \leq \frac{m}{2} \cdot la_{n-1}(K_{n,n} \square K_{n,n}) = \frac{m}{2} \cdot (n+2) = \frac{mn}{2} + m.$$

If m is odd, then by Lemma 2.4, 2.6, Corollary 2.1 and Theorem 2.1, we obtain that

$$\begin{aligned} &la_{n-1}(K_{n,n})^{[m]} \leq \\ &\tfrac{m-1}{2} \cdot la_{n-1}(K_{n,n} \square K_{n,n}) + la_{n-1}(K_{n,n}) \\ &= \tfrac{m-1}{2} \cdot (n+2) + \left\lceil \tfrac{n}{2} \right\rceil + 1 = \left\lceil \tfrac{mn}{2} \right\rceil + m. \end{aligned}$$

In a word, we have

$$\left\lceil \frac{mn^2}{2(n-1)} \right\rceil \leq la_{n-1}(K_{n,n})^{[m]} \leq \left\lceil \frac{mn}{2} \right\rceil + m.$$

□

Corollary 2.2. For odd $n \geq 5$ and $m \leq n-1$, we have $la_{n-1}(K_{n,n})^{[m]} = \frac{m(n+1)}{2} + 1$.

Proof. For odd $n \geq 5$ and $m \leq n-1$, we have

$$\begin{aligned} &la_{n-1}(K_{n,n})^{[m]} \\ &\geq \tfrac{m(n+1)}{2} + \left\lceil \tfrac{m}{2(n-1)} \right\rceil \\ &= \tfrac{m(n+1)}{2} + 1 \end{aligned}$$

by Lemma 2.3.

Similar to subcase 2.2 proof process of Theorem 2.1, we can obtain that

$$la_{n-1}(K_{n,n})^{[m]} \leq \frac{n-1}{2} \cdot m + 2 + m - 1 = \frac{m(n+1)}{2} + 1.$$ □

Corollary 2.3. For odd $n \geq 5$, when odd $m \geq n$ or even $m = k(n-1)$ with $k > 1$, we have

$$\frac{m(n+1)}{2} + \left\lceil \frac{m}{2(n-1)} \right\rceil \leq la_{n-1}(K_{n,n})^{[m]} \leq \frac{m(n+1)}{2} + \left\lceil \frac{m}{n-1} \right\rceil.$$

Proof. By Lemma 2.3, we can know that

$$la_{n-1}(K_{n,n})^{[m]} \geq \frac{m(n+1)}{2} + \left\lceil \frac{m}{2(n-1)} \right\rceil$$

for odd n. Assume that $n \geq 5$ is odd.

Case 1. $m = k(n-1)$ is even with $k > 1$.

Then by Lemma 2.5 and Corollary 2.2, we have

$$la_{n-1}(K_{n,n})^{[m]} \leq k \cdot la_{n-1}(K_{n,n})^{[n-1]} \leq k \cdot \frac{n^2+1}{2} = \frac{m(n+1)}{2} + \left\lceil \frac{m}{n-1} \right\rceil.$$

Case 2. m is odd and $m \geq n$.

Let $m = k(n-1) + r, r \neq 0$. Then r is odd. By Lemma 2.5 and Corollary 2.2, we have

$$la_{n-1}(K_{n,n})^{[m]} \leq k \cdot la_{n-1}(K_{n,n})^{[n-1]} + la_{n-1}(K_{n,n})^{[r]}$$

$$\leq k \cdot \frac{n^2+1}{2} + \frac{(n+1)r}{2} + 1 = \frac{m(n+1)}{2} + \left\lceil \frac{m}{n-1} \right\rceil.$$ □

Theorem 2.3. We have

$$la_{n-1}(K_{n(l)} \square K_{n(l)}) = nl$$

when at least one of n and l is even, and

$$nl \leq la_{n-1}(K_{n(l)} \square K_{n(l)}) \leq nl + 1$$

otherwise.

Proof. On the one hand, we can obtain that

$$|V(K_{n(l)} \square K_{n(l)})| = (nl)^2, d_{K_{n(l)} \square K_{n(l)}}(u) = 2l(n-1)$$

for any vertex $u = (u_1, u_2)$, and

$$|E(K_{n(l)} \square K_{n(l)})| = l^3 n^2 (n-1).$$

Applying Lemma 2.3, we have

$$la_{n-1}(K_{n(l)} \square K_{n(l)}) \geq nl.$$

On the other hand, by Lemma 2.5 and Lemma 2.8, we obtain that

$$la_{n-1}(K_{n(l)} \square K_{n(l)}) \leq 2la_{n-1}(K_{n(l)}) = 2 \cdot \left\lceil \frac{nl}{2} \right\rceil.$$

Furthermore, we have

$$la_{n-1}(K_{n(l)} \square K_{n(l)}) \leq nl$$

when n is even or l is even, and

$$la_{n-1}(K_{n(l)} \square K_{n(l)}) \leq nl + 1$$

otherwise. Thus the result holds. □

Theorem 2.4. $\lceil \frac{mnl}{2} \rceil \leq la_{n-1}(K_{n(l)})^{[m]} \leq m \cdot \lceil \frac{nl}{2} \rceil$.

Proof. It is not difficult to verify that

$$|V(K_{n(l)} \square K_{n(l)} \square \cdots \square K_{n(l)})| = (nl)^m,$$

$$d_{K_{n(l)} \square K_{n(l)} \square \cdots \square K_{n(l)}}(u) = ml(n-1)$$

for any vertex $u = (u_1, u_2)$, and

$$|E(K_{n(l)} \square K_{n(l)} \square \cdots \square K_{n(l)})| = \frac{mn^m l^{m+1}(n-1)}{2}.$$

Applying Lemma 2.3, we have

$$la_{n-1}(K_{n(l)})^{[m]} \geq \left\lceil \frac{mnl}{2} \right\rceil.$$

By Lemma 2.5 and Lemma 2.8, we obtain that

$$la_{n-1}(K_{n(l)})^{[m]} \leq m \cdot la_{n-1}(K_{n(l)}) = m \cdot \left\lceil \frac{nl}{2} \right\rceil.$$

Hence, we have

$$\left\lceil \frac{mnl}{2} \right\rceil \leq la_{n-1}(K_{n(l)})^{[m]} \leq m \cdot \left\lceil \frac{nl}{2} \right\rceil.$$

Particularly, we obtain that

$$la_{n-1}(K_{n(l)})^{[m]} = \frac{mnl}{2}$$

when at least one of n and l is even, and

$$\left\lceil \frac{mnl}{2} \right\rceil \leq la_{n-1}(K_{n(l)})^{[m]} \leq \frac{mnl}{2} + \frac{m}{2}$$

otherwise. So Theorem holds. □

References

[1] C. H. Yen, H. L. Fu, Linear 2-arboricity of the complete graph, Taiwanese J. Math., 14 (1) (2010), 273-286.

[2] L. Zuo, S. He, and B. Xue, The linear $(n-1)$-arboricity of Cartesian product graphs, Appl. Analy. And Disc. Math., 9 (2015),13-28.

[3] B. L. Chen, K. C. Huang, On the linear k-arboricity of K_n and $K_{n,n}$, Discrete Math., 254 (2002), 51-61.

[4] B. Xue, L. Zuo, On the linear $(n-1)$-arboricity of $K_{n(m)}$, Discrete Applied Math., 158 (2010),1546-1550.

[5] D. B. West, Introduction to Graph Theory, 2nd ed. Prentice. Hall, Upper Saddle River, New Jersey, 2001.

[6] M. Habib, B. Peroche, La k-arboricity lineaire des arbres. Ann. Discrete Math.,17 (1983), 307 -317.

[7] M. Habib, B. Peroche, Some problems about linear arboricity. Discrete Math.,41 (1982),219- 220.

[8] J. Akiyama, Three developing topics in graph theory,Doctoral Dissertation. University of Toyo,1980.

[9] H. L. Fu, K. C. Huang, C. H. Yen, The linear 3-arboricity of $K_{n,n}$ and K_n, Discrete Math., 308 (2008)3816-3823.

[10] G. J. Chang, Algorithmic aspects of linear k-arboricity, Taiwanese J. Math., 3 (1999), 73-81.

[11] G. J. Chang, B. L. Chen, H. L. FU, K. C. Huang, Linear k-arboricity on trees.Discrete Appl. Math., 103 (2000), 281-287.

[12] R. E. L. Aldred, N. C. Wormald, More on the linear k-arboricity of regular graphs, Australas J. Combin., 18 (1998), 97-104.

[13] N. Alon, V. J. Teague, N. C. Wormald, Linear arboricity and linear k-arboricity of regular graphs, Graphs and Combin., 17 (2001), 11-16.

[14] H. Enomoto, B. Peroche, The linear arboricity of some regular graphs, J. Graph Theory 8 (1984), 309-324.

[15] K. W. Lih, L. D. Tong, W. F. Wang, The linear 2-arboricity of planar graphs. Graphs and Combin., 19 (2003), 241-248.

[16] J. C. Bermond, J. L. Fouquet, M. Habib, B. Peroche, On linear k-arboricity.Discrete Math., 52 (1984),123-132.

[17] B. Jackson, N. C. Wormald, On the linear k-arboricity of cubic graphs.Discrete Math., 162 (1996), 293-297.

[18] C. Thomassen, Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5. J. Combin. Theory, Ser B., 75 (1999),100-109.

[19] C. H. Yen, H. L. Fu, Linear 3-arboricity of the balanced complete multipartite graph, J. Combin. Math. Comput. 60 (2007), 33-46.

[20] S. He, L. Zuo, The linear 6-arboricity of the complete bipartite graph $K_{m,n}$, Discrete Mathematics, Algorithms and applications,Vol.5, NO.4 (2013) 1350029(ten pages).

[21] S. He, L. Zuo, The linear 6-arboricity of the complete bipartite graph $K_{m,n}$, Advances in Mathematics (China), 44 (1)2015,47-54.

[22] L. Zuo, B. Xue, S. He,The linear 2- and linear 4-arboricity of the complete bipartite graph $K_{m,n}$, Inter. J. Combin., Vol. 2013,ID501701.

[23] L. Chen, J. Meng, Y. Tian, F. Xia, Restricted connectivity of Cartesian product graphs, IAENG International Journal of Applied Mathematics, 46:1, pp.58-63, 2016.

[24] Qun Liu, Some Results of Resistance Distance and Kirchhoff Index Based on R-Graph, IAENG International Journal of Applied Mathematics, 46:3, pp. 346-352, 2016.

[25] G. V. Ghodasra, S. G. Sonchhatra, Further results on 3-equitable labeling, IAENG International Journal of Applied Mathematics, 45:1, pp.1-15, 2015.