Periodic Solutions for Some Strongly Nonlinear Oscillators Applying He's Methods

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Abstract—**In this paper, He's energy balance method is used to determine frequency formulation relations of nonlinear oscillators with discontinuous term, and He's amplitude frequency formulation is used to obtain a periodic solution for a nonlinear oscillator with fractional potential. By direct calculation and numerical simulations, compared with the exact solutions show that the results obtained are of high accuracy.**

Index Terms—**He's energy balance method; He's amplitude frequency formulation; periodic solution; nonlinear oscillator; fractional potential.**

I. Introduction

In this paper, we shall study the existence of periodic solutions for the following generalized nonlinear oscillators

$$u'' + f(u)u = 0,\ u(0) = A,\ u'(0) = 0, \tag{1}$$

where $f(u) > 0$ is a known function of u.

In recent years, with the ever-increasing development of nonlinear science, various kinds of analytical methods and numerical methods have been used to handle the problem and other nonlinear problems, such as Exp-function method [1-3], variational iteration method [4,5], parameter-expansion method [6], and homotopy perturbation method [7-9], etc.

However, in case that there exists no small parameter in equation (1), the traditional perturbation methods cannot be applied directly to (1). Hereby, we shall apply He's energy balance method and He's frequency amplitude formulation [10,11] to solve the problem.

II. He's energy balance method

Firstly, we consider the following nonlinear oscillator with discontinuous term

$$u'' + au^3 + bu + cu|u| = 0,\ u(0) = A,\ u'(0) = 0. \tag{2}$$

Using the semi-inverse method [12], the variational principle of equation (2) can be easily obtained:

$$\begin{aligned} & J(u) \\ = & \int_{-\frac{T}{2}}^{0} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 - \frac{1}{3}cu^3 \Big] \mathrm{d}t \\ & + \int_{0}^{\frac{T}{2}} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \frac{1}{3}cu^3 \Big] \mathrm{d}t \\ = & \int_{-\frac{T}{2}}^{\frac{T}{2}} \Big[-\frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 \\ & + \mathrm{sgn}(u)\frac{1}{3}cu^3 \Big] \mathrm{d}t. \end{aligned} \tag{3}$$

The Hamiltonian of equation (2), therefore, can be written in the form

$$\begin{aligned} H &= \frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \mathrm{sgn}(u)\frac{1}{3}cu^3 \\ &= \frac{1}{4}aA^4 + \frac{1}{2}bA^2 + \mathrm{sgn}(A)\frac{1}{3}cA^3, \end{aligned}$$

that is

$$\begin{aligned} & \frac{1}{2}u'^2 + \frac{1}{4}au^4 + \frac{1}{2}bu^2 + \mathrm{sgn}(u)\frac{1}{3}cu^3 \\ & -\frac{1}{4}aA^4 - \frac{1}{2}bA^2 - \mathrm{sgn}(A)\frac{1}{3}cA^3 = 0. \end{aligned} \tag{4}$$

We use the following trial function to determine the angular frequency ω:

$$u = A\cos\omega t. \tag{5}$$

Substituting (5) into (4), we obtain the following residual

$$\begin{aligned} R(t) = & \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{4}aA^4\cos^4\omega t \\ & + \frac{1}{2}bA^2\cos^2\omega t + \mathrm{sgn}(A\cos\omega t)\frac{1}{3}cA^3\cos^3\omega t \\ & - \frac{1}{4}aA^4 - \frac{1}{2}bA^2 - \mathrm{sgn}(A)\frac{1}{3}cA^3. \end{aligned}$$

We set

$$\int_0^{\frac{T}{4}} R(t)\cos\omega t\,\mathrm{d}t = 0,\ T = \frac{2\pi}{\omega}$$

to determine the $\omega - A$ relationship, which reads

$$\omega^2 = \frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b,$$

then

$$\omega = \Big[\frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b\Big]^{\frac{1}{2}}. \tag{6}$$

We, therefore, obtain the following periodic solution

$$u(t) = A\cos\Big[\Big(\frac{13}{10}aA^2 + (\frac{16-3\pi}{8})cA + b\Big)^{\frac{1}{2}} t\Big].$$

To illustrate the accuracy of the obtained results, we give an example as follows:

In case $a = 0, b = 0$, equation (1) becomes

$$u'' + cu|u| = 0,\ u(0) = A, u'(0) = 0,$$

its frequency reads $\omega = \big(\frac{16-3\pi}{8}\big)^{\frac{1}{2}} c^{\frac{1}{2}} A^{\frac{1}{2}}$, its exact frequency is $\omega_{ex} = 0.921318 c^{\frac{1}{2}} A^{\frac{1}{2}}$. Therefore, its accuracy reaches 0.0088. The above result is of high accuracy.

Next, we consider another nonlinear oscillator of the following form

$$u'' + au^{\frac{1}{m}} + bu^{2n+1} = 0,\ u(0) = A,\ u'(0) = 0. \tag{7}$$

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Using the semi-inverse method [12], the variational principle of equation (7) can be easily obtained:

$$J(u) = \int_0^T \Big[-\frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \Big] \,\mathrm{d}t.$$

The Hamiltonian of equation (7), therefore, can be written in the form

$$\begin{aligned} H &= \frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \\ &= \frac{am}{m+1}A^{\frac{m+1}{m}} + \frac{b}{2n+2}A^{2n+2}, \end{aligned}$$

that is

$$\begin{aligned} &\frac{1}{2}u'^2 + \frac{am}{m+1}u^{\frac{m+1}{m}} + \frac{b}{2n+2}u^{2n+2} \\ &-\frac{am}{m+1}A^{\frac{m+1}{m}} - \frac{b}{2n+2}A^{2n+2} = 0. \end{aligned} \tag{8}$$

We use the following trial function to determine the angular frequency ω:

$$u = A\cos\omega t. \tag{9}$$

Substituting (9) into (8), we obtain the following residual

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{am}{m+1}A^{\frac{m+1}{m}}\cos^{\frac{m+1}{m}}\omega t \\ &+\frac{b}{2n+2}A^{2n+2}\cos^{2n+2}\omega t - \frac{am}{m+1}A^{\frac{m+1}{m}} \\ &-\frac{b}{2n+2}A^{2n+2}, \end{aligned}$$

that is

$$\begin{aligned} R(t) &= A^2\omega^2\sin^2\omega t + \frac{2am}{m+1}A^{\frac{m+1}{m}}\cos^{\frac{m+1}{m}}\omega t \\ &+\frac{b}{n+1}A^{2n+2}\cos^{2n+2}\omega t - \frac{2am}{m+1}A^{\frac{m+1}{m}} \\ &-\frac{b}{n+1}A^{2n+2}. \end{aligned}$$

We set

$$\int_0^{\frac{T}{4}} R(t)\cos\omega t \,\mathrm{d}t = 0,\ T = \frac{2\pi}{\omega}$$

to determine the $\omega - A$ relationship, which reads

$$\begin{aligned} &\frac{1}{3}A^2\omega^2 - \frac{2am}{m+1}A^{\frac{m+1}{m}}\Big[1 - \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2m})}{\Gamma(2+\frac{1}{2m})}\Big] \\ &-\frac{bA^{2n+2}}{n+1}\Big[1 - \frac{(2n+2)!!}{(2n+3)!!}\Big] = 0, \end{aligned}$$

where $\Gamma(\cdot)$ is Γ function, and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$. Then

$$\omega = \frac{\sqrt{3}\alpha^{\frac{1}{2}}}{A}, \tag{10}$$

where

$$\begin{aligned} \alpha &= \frac{2am}{m+1}A^{\frac{m+1}{m}}\big(1 - \frac{\pi^{\frac{1}{2}}}{2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2m})}{\Gamma(2+\frac{1}{2m})}\big) \\ &+\frac{bA^{2n+2}}{n+1}\big(1 - \frac{(2n+2)!!}{(2n+3)!!}\big). \end{aligned}$$

We, therefore, obtain the following periodic solution

$$u(t) = A\cos\Big[\frac{\sqrt{3}\alpha^{\frac{1}{2}}}{A}t\Big].$$

To illustrate the accuracy of the obtained results, we give two examples as follows:

In case $m = 3, b = 0$, equation (7) becomes

$$u'' + au^{\frac{1}{3}} = 0,$$

its frequency reads $\omega = 1.0834b^{\frac{1}{2}}A^{-\frac{1}{3}}$, its exact frequency [13] is $\omega_{ex} = 1.0705a^{\frac{1}{2}}A^{-\frac{1}{3}}$. Therefore its accuracy reaches 0.0121.

In case $n = 1, a = 0$, equation (7) becomes

$$u'' + bu^3 = 0,$$

its frequency reads $\omega = 0.8367Ab^{\frac{1}{2}}$, its exact frequency [13] is $\omega_{ex} = 0.8472Ab^{\frac{1}{2}}$. Therefore its accuracy reaches 0.0124. Compared with the results in [14], our result is higher accuracy.

III. He's Frequency Amplitude Formulation

In this section, we shall consider the following nonlinear oscillator with fractional potential

$$\begin{aligned} &u'' + au + bu^{2n+1} + cu^{\frac{1}{2n+1}} = 0, \\ &u(0) = A,\ u'(0) = 0, \end{aligned} \tag{11}$$

where a, b, c are constants, and $n \in N^+$.

If we take $n = 1$ in equation (11), then equation (11) reduced to a class of nonlinear oscillator [14]

$$u'' + au + bu^3 + cu^{\frac{1}{3}} = 0.$$

If we take $a = 1, c = 0, n = 1$ in equation (11), then equation (11) reduced to the well-known Duffing equation

$$u'' + u + bu^3 = 0. \tag{12}$$

In order to use He's amplitude frequency formulation, we choose two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$, which are, respectively, the solutions of the following linear equations:

$$\begin{aligned} &u'' + \omega_1^2 u = 0,\quad \omega_1^2 = 1, \\ &u'' + \omega_2^2 u = 0,\quad \omega_2^2 = \omega^2, \end{aligned}$$

where ω is assumed to be the frequency of the nonlinear oscillator equation (11). Substituting $u_1(t)$ and $u_2(t)$ into equation (11), we obtain, respectively, the following residuals

$$\begin{aligned} R_1(t) &= -A\cos t + aA\cos t + bA^{2n+1}\cos^{2n+1} t \\ &+cA^{\frac{1}{2n+1}}\cos^{\frac{1}{2n+1}} t, \end{aligned} \tag{13}$$

$$\begin{aligned} R_2(t) &= -A\omega^2\cos\omega t + aA\cos\omega t \\ &+bA^{2n+1}\cos^{2n+1}\omega t \\ &+cA^{\frac{1}{2n+1}}\cos^{\frac{1}{2n+1}}\omega t. \end{aligned} \tag{14}$$

He's amplitude frequency formulation reads [10,11]

$$\omega^2 = \frac{\omega_1^2 R_2(t_2) - \omega_2^2 R_1(t_1)}{R_2(t_2) - R_1(t_1)}, \tag{15}$$

where t_1 and t_2 are location points. Generally, setting

$$t_1 = \frac{T_1}{12},\quad t_2 = \frac{T_2}{12},$$

where T_1 and T_2 are periods of the trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$, respectively, i.e. $T_1 = 2\pi$ and $T_2 = 2\pi/\omega$.

From (13),(14),(15), by direct calculates, yields

$$\omega^2 = a + bA^{2n}\big(\frac{3}{4}\big)^n + cA^{-\frac{2n}{2n+1}}\big(\frac{3}{4}\big)^{-\frac{n}{2n+1}},$$

then

$$\omega = \big(a + bA^{2n}\big(\frac{3}{4}\big)^n + cA^{-\frac{2n}{2n+1}}\big(\frac{3}{4}\big)^{-\frac{n}{2n+1}}\big)^{\frac{1}{2}}. \qquad (16)$$

We, therefore, obtain the following periodic solution

$$u(t) = A\cos\big[\big(a + bA^{2n}\big(\frac{3}{4}\big)^n + cA^{-\frac{2n}{2n+1}}\big(\frac{3}{4}\big)^{-\frac{n}{2n+1}}\big)^{\frac{1}{2}}t\big].$$

To illustrate the accuracy of the obtained results, we give two examples as follows:

In case $n = 1, a = b = 0$, equation (11) becomes

$$u'' + cu^{\frac{1}{3}} = 0,$$

its frequency reads $\omega = c^{\frac{1}{2}}A^{-\frac{1}{3}}\big(\frac{3}{4}\big)^{-\frac{1}{6}} = 1.0491c^{\frac{1}{2}}A^{-\frac{1}{3}}$, its exact frequency [14] is $\omega_{ex} = 1.0705c^{\frac{1}{2}}A^{-\frac{1}{3}}$. Therefore its accuracy reaches 0.0204.

In case $n = 1, a = c = 0$, equation (11) becomes

$$u'' + bu^3 = 0,$$

its frequency reads $\omega = (\frac{3}{4})^{\frac{1}{2}}Ab^{\frac{1}{2}} = 0.866Ab^{\frac{1}{2}}$, its exact frequency is $\omega_{ex} = 0.8472Ab^{\frac{1}{2}}$. Therefore its accuracy reaches 0.0222.

IV. Numerical Simulations

In this section, we present some numerical results at different values. Figures 1, 2, 3 and 4 illustrate excellent agreement of the obtained result with the exact one.

V. Conclusion

In this work, the nonlinear oscillators are efficiently handled by He's energy balance method or He's frequency formulation. It has been proved to be a powerful mathematical tool for searching exact solutions for nonlinear oscillators. The analytical approximation obtained by this new methods are valid for the whole solution domain with high accuracy. Moreover, the methods used in this paper can be extended to solve many other types of nonlinear oscillators, see, for example, [15-17].

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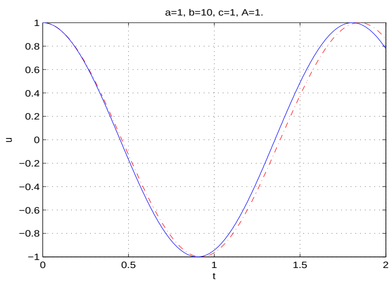
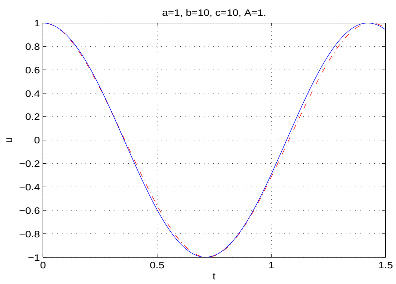
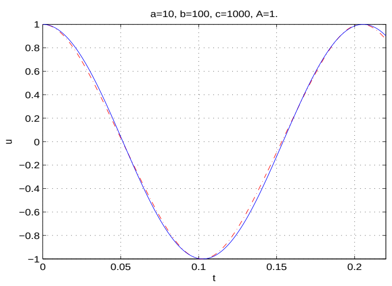
Figures:

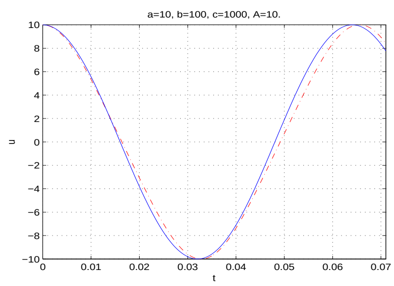
dashed line (- - -): exact solution;

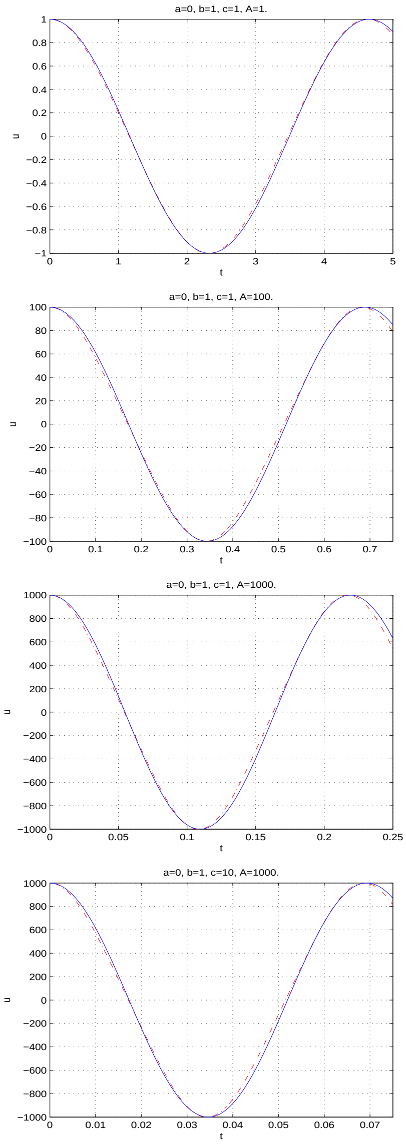
continuous line (—): approximate solution.



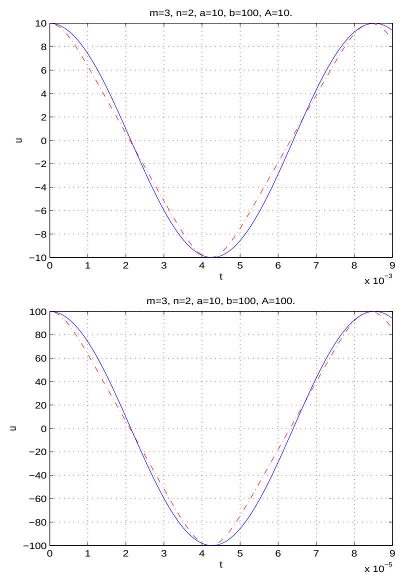


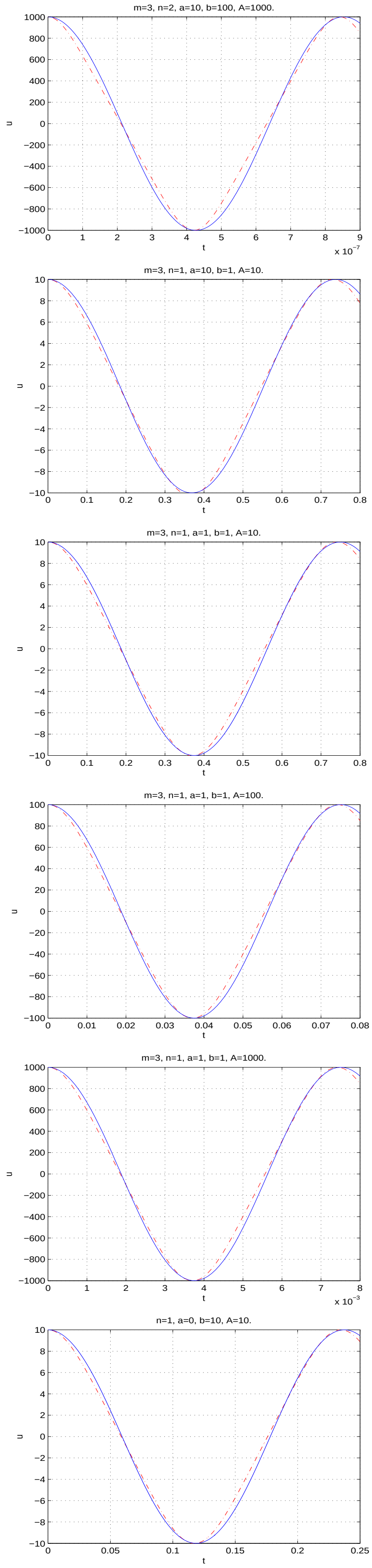




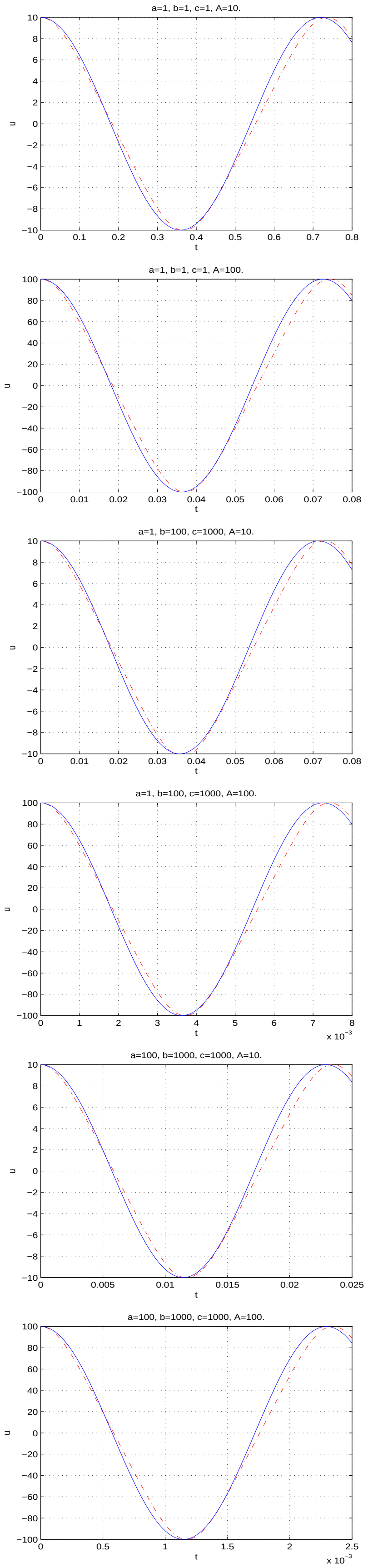


Figures 1. Comparison of exact solution of equation(2) with approximate solution $u = A\cos\omega t$ at different values of a, b, c and A, where ω is defined by equation (6).

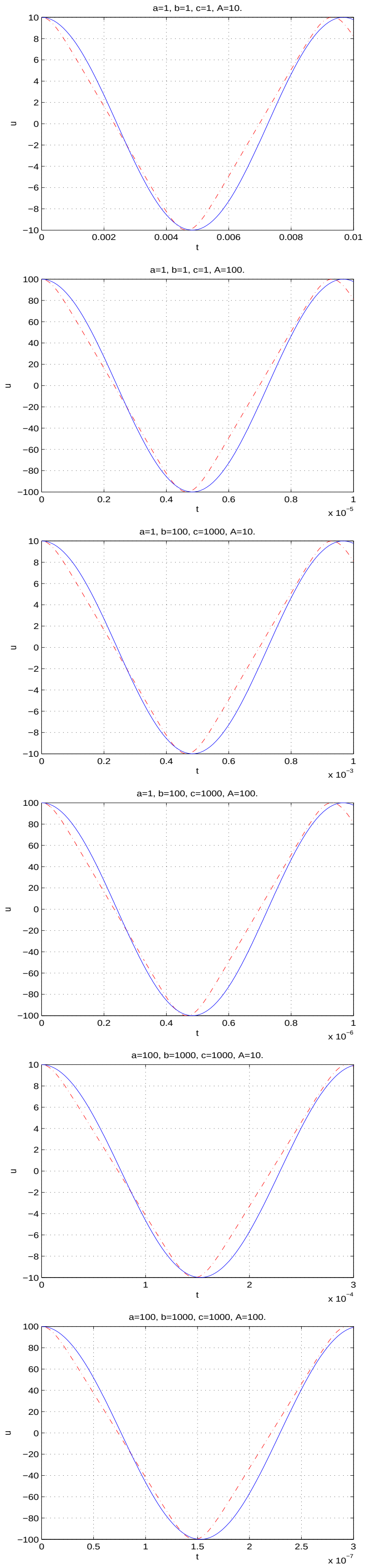




Figures 2. Comparison of exact solution of equation (7) with approximate solution $u = A\cos\omega t$ at different values of a, b, m, n and A, where ω is defined by equation (10).



Figures 3. In case n=1, comparison of exact solution of equation(11) with approximate solution $u = A\cos\omega t$ under different values of a, b, c and A, where ω is defined by equation (16).



Figures 4. In case n=3, comparison of exact solution of equation(11) with approximate solution $u = A\cos\omega t$ under different values of a, b, c and A,where ω is defined by equation (16).